ENUMERATING THE EXTERIORS OF PRIME LINKS BY A CANONICAL ORDER

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ABSTRACT

The first author defined a well-order in the set of links by embedding it into a canonical well-ordered set of (integral) lattice points and we enumerated the first 443 prime links arising from the lattice points of lengths up to 10 under this order. In this paper we show a table of the first 142 exteriors of prime links arising from the lattice points of lengths up to 9.

Keywords: Exterior, Lattice point, Length, Link, Table

1. Introduction

We consider unoriented links unless otherwise stated, so that we say that two links L and L' in S^3 are *equivalent* and we denote it by $L = L'$ if there is a homeomorphism $h : (S^3, L) \to (S^3, L')$.

A method of enumerating the set of links and the set of closed connected orientable 3-manifolds was suggested in [1]. The idea is to introduce a well-order in the set of links by embedding it into a canonical well-ordered set of (integral) lattice points. This well-order also induces a well-order in the set of closed connected orientable 3-manifolds. By using this method, the first 28 prime links and the first 26 closed connected orientable 3-manifolds arising from the lattice points of lengths up to 7 are tabulated without any computer aid in [1]. We enlarged the table of the first 28 prime links into that of the first 443 prime links arising from the lattice points of lengths up to 10 in [3]. Our goal of this study is to enumerate 3-manifolds arising from the lattice points of lengths up to 10. In this paper we show a table of the first

142 exteriors of prime links arising from the lattice points of lengths up to 9, which is in a step of the goal.

In Section 2, we review the definition of the well-order described in [1]. In Section 3, we explain how to enumerate the set of exteriors of prime links. At the end, we show a table of the first 156 prime links and a table of the first 142 exteriors arising from the lattice points of lengths up to 9.

2. Definition of a well-order in the set of links

Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^n$ the product of n copies of $\mathbf{Z}$. We put

$$\mathbf{X} = \coprod_{n=1}^{\infty} \mathbf{Z}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbf{Z},\ n = 1, 2, \ldots\}.$$

We call elements of $\mathbf{X}$ *lattice points*. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put $\ell(\mathbf{x}) = n$ and call it the *length* of $\mathbf{x}$. Let $|\mathbf{x}|$ and $|\mathbf{x}|_N$ be the lattice points determined from $\mathbf{x}$ by the following formulas:

$|\mathbf{x}| = (|x_1|, |x_2|, \ldots, |x_n|)$ and
$|\mathbf{x}|_N = (|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ where $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$ and $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

We define a well-order in $\mathbf{X}$ as follows (See [1]):

Definition 2.1. We define a well-order in $\mathbf{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 \cdots$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we define $\mathbf{x} < \mathbf{y}$ if we have one of the following conditions (1)-(4):

(1) $\ell(\mathbf{x}) < \ell(\mathbf{y})$.
(2) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$ by the lexicographic order (on the natural number order).
(4) $|\mathbf{x}| = |\mathbf{y}|$ and $\mathbf{x} < \mathbf{y}$ by the lexicographic order on the well-order of $\mathbf{Z}$ defined above.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put

$$\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|.$$

Let $\beta(\mathbf{x})$ be the $(\max|\mathbf{x}| + 1)$-string braid determined from $\mathbf{x}$ by the identity

$$\beta(\mathbf{x}) = \sigma_{|x_1|}^{\mathrm{sign}(x_1)} \sigma_{|x_2|}^{\mathrm{sign}(x_2)} \cdots \sigma_{|x_n|}^{\mathrm{sign}(x_n)},$$

where we define $\sigma_{|0|}^{\mathrm{sign}(0)} = 1$. We note that $\max|\mathbf{x}| + 1$ is the minimum string number of the braid indicated by the right-hand side of the identity. Let $\mathrm{cl}\beta(\mathbf{x})$ be the closure of the braid $\beta(\mathbf{x})$. Let $\mathbf{L}$ be the set of links. Then we have a map

$$\mathrm{cl}\beta : \mathbf{X} \to \mathbf{L}$$

sending $\mathbf{x}$ to $\mathrm{cl}\beta(\mathbf{x})$. By Alexander's braiding theorem, the map $\mathrm{cl}\beta$ is surjective. For $L \in \mathbf{L}$, we define a map

$$\sigma : \mathbf{L} \to \mathbf{X}$$

by $\sigma(L) = \min\{\mathbf{x} \in \mathbf{X} \,|\, \mathrm{cl}\beta(\mathbf{x}) = L\}$. Then σ is a right inverse of the map $\mathrm{cl}\beta$ and hence is injective. Now we have a well-order in $\mathbf{L}$ by the following definition:

Definition 2.2. For $L, L' \in \mathbf{L}$, we define $L < L'$ if $\sigma(L) < \sigma(L')$.

For a link $L \in \mathbf{L}$, we call $\ell(\sigma(L))$ the *length* of L.

Let $\mathbf{L}^{\mathrm{p}}$ be the set of prime links, where we consider that the 2-component trivial link is not prime. We use the injection σ for our method of a tabulation of $\mathbf{L}^{\mathrm{p}}$. For $k \in \mathbf{Z}$, let k^n and $-k^n$ be the lattice points determined by

$$k^n = (\underbrace{k, k, \ldots, k}_{n}) \quad \text{and} \quad -k^n = (-k)^n,$$

respectively.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbf{X}$, let $(\mathbf{x},\ \mathbf{y})$ be the lattice point determined by the following formula:

$$(\mathbf{x},\ \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m).$$

With these terminologies, we show a table of prime links with up to length 9 at the end of this paper.

In the table, we denote by μ, G, and $|lk|$ the number of the components of the link, the Goeritz invariant of the link and the absolute values of the linking numbers of all pairs of the components of the link, respectively.

3. A method of a tabulation of prime link exteriors

Definition 3.1. For r-component links L and L' in S^3, their Alexandre polynomials $\Delta_L(t_1, \ldots, t_r)$ and $\Delta_{L'}(t_1, \ldots, t_r)$ are equivalent if there is an isomorphism $\varphi : \langle t_1, \ldots, t_r | t_i t_j = t_j t_i (i, j = 1, \ldots, r)\rangle \to \langle t_1, \ldots, t_r | t_i t_j = t_j t_i (i, j = 1, \ldots, r)\rangle$ satisfying

$$\Delta_{L'}(t_1, \ldots, t_r) = \pm t_1^{a_1} \cdots t_r^{a_r} \Delta_L(\varphi(t_1), \ldots, \varphi(t_r))$$

for some integers $a_i,\ i = 1, \ldots, r$.

For a link L in S^3, let $E(L) = cl(S^3 - N(L))$ be its exterior, where $N(L)$ is a regular neighborhood of L. Then we see the following lemma:

Lemma 3.2. For links L and L' in S^3, if there is a homeomorphism $E(L) \to E(L')$, then their Alexander polynomials are equivalent.

By using the above lemma, we divide the prime links into several groups, each of which consists of the links with the equivalent Alexander polynomials. For two or more component prime links with up to length 9, there are 12 groups consisting of two or more elememts:

(1) $4^2_1 < 7^2_7 < 9^2_{43} < 9^2_{59}$	(2) $5^2_1 < 7^2_8 < 8^2_{15}$	(3) $6^3_3 < 8^3_7$
(4) $6^3_1 < 8^3_8 < 9^3_{13} < 9^3_{17}$	(5) $6^3_2 < 8^3_9 < 9^3_{19} < 9^3_{18}$	(6) $6^2_3 < 8^2_{16}$
(7) $8^2_{12} < 8^2_{10}$	(8) $7^2_4 < 9^2_{44}$	(9) $6^2_1 < 9^2_{49}$
(10) $7^2_6 < 9^2_{55}$	(11) $7^2_2 < 9^2_{54}$	(12) $7^3_1 < 9^3_{14}$.

For (2), (3), (4), (6), (8), (9) and (12), their exteriors are homeomorphic to each others by twist homeomorphisms along a trivial component.

For (1), $E(4^2_1)$, $E(7^2_7)$ and $E(9^2_{43})$ are homeomorphic by twist homeomorphisms along a trivial component. These spaces are Seifert manifolds since 4^2_1 is a torus link. On the other hand, 9^2_{59} is decomposed into two nontrivial tangles and so $E(9^2_{59})$ is not a Seifert manifold. We conclude that $E(4^2_1) \cong E(7^2_7) \cong E(9^2_{43}) \ncong E(9^2_{59})$.

For (5), $E(6^3_2)$ are homeomorphic to $E(9^3_{18})$ by a twist homeomorphism along a trivial component and $E(8^3_9)$ are homeomorphic to $E(9^3_{19})$ as well. For 6^3_2, linking numbers of all pairs of the components are $0, 0, 0$, and those for 8^3_9 are $0, 0, 2$ and we have $E(6^3_2) \ncong E(8^3_9)$. We conclude that $E(6^3_2) \cong E(9^3_{18}) \ncong E(8^3_9) \cong E(9^3_{19})$

For (7), let $8^2_{12} = K_1 \cup K_2$ and $8^2_{10} = K'_1 \cup K'_2$, where K_1 and K'_1 are the trivial component of 8^2_{12} and 8^2_{10} respectively. Note that $\mathrm{lk}(K_1, K_2) = \mathrm{lk}(K'_1, K'_2) = 0$. Here $\mathrm{lk}(K_1, K_2)$ means the linking number of K_1 and K_2. Suppose that there is a homeomorphism $h : E(8^2_{12}) \to E(8^2_{10})$. From their Alexander polynomials, we see that $h(\partial N(K_1)) = \partial N(K'_1)$. Let $K'_{2,n}$ be a knot obtaind by twisting K'_2 along K'_1 n times. Since $\mathrm{lk}(K_1, K_2) = \mathrm{lk}(K'_1, K'_2) = 0$, $E(K_2)$ should be homeomorphic to $E(K'_{2,n})$ for some integer n and then $K_2 = K'_{2,n}$. However this is impossible. So we have $E(8^2_{12}) \ncong E(8^2_{10})$.

For (10), the same method as (7) will lead us to the result that $E(7^2_6) \ncong E(9^2_{55})$.

For (11), Suppose that there is a homeomorphism $h : E(7^2_2) \to E(9^2_{54})$. From their Alexander polynomials, $h_*(t_1) = t_i^{\pm 1}$ and $h_*(t_2) = t_j^{\pm 1}$ where $h_* : H_1(E(7^2_2)) \to H_1(E(9^2_{54}))$ is an isomorphism induced by h, t_1 and t_2 are meridians of the homology groups, and $\{i, j\} = \{1, 2\}$. Since the linking numbers for 7^2_2 and 9^2_{54} are both 1, h sends the meridians to the meridians geometrically and we can extend $h : E(7^2_2) \to E(9^2_{54})$ into $h : S^3 \to S^3$. This means $7^2_2 = 9^2_{54}$, which is impossible. So we have $E(7^2_2) \ncong E(9^2_{54})$.

We obtain a table of the exteriors of the prime links with up to length 9, by omitting the links $7^2_7, 9^2_{43}, 7^2_8, 8^2_{15}, 8^3_7, 8^3_8, 9^3_{13}, 9^3_{17}, 9^3_{19}, 9^3_{18}, 8^2_{16}, 9^2_{44}, 9^2_{49}, 9^3_{14}$ from the link table with up to length 9 because their exteriors have already appeared before.

In the table shown at the end of this paper, we denote by μ, G, and $|lk|$ the number of the components of the link, the Goeritz invariant of the link and the absolute values of the linking numbers of all pairs of the components of the link, respectively and for the column of $E(L)$ we put $\times$ in the above links. Therefore, if we omit the links with the mark $\times$, we have a table of link exteriors.

References

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$\ell(x)$	$\lvert x\rvert_N$	x	μ	G	$\lvert lk\rvert$	L	$E(L)$
1	0	0	1	1		O	
2	1^2	1^2	2	2	1	2_1^2	
3	1^3	1^3	1	3		3_1	
4	1^4	1^4	2	4	2	4_1^2	
	$(1^2,2^2)$	$(1,-2,1,-2)$	1	5		4_1	
5	1^5	1^5	1	5		5_1	
	$(1^3,2^2)$	$(1^2,-2,1,-2)$	2	8	0	5_1^2	
6	1^6	1^6	2	6	3	6_1^2	
	$(1^4,2^2)$	$(1^3,2,-1,2)$	1	7		5_2	
		$(1^3,-2,1,-2)$	1	11		6_2	
		$(1^2,2,1^2,2)$	3	$2,2$	$1,1,1$	6_3^3	
		$(1^2,-2,1^2,-2)$	3	$6,2$	$1,1,1$	6_1^3	
	$(1^3,2^3)$	$(1^2,-2,1,-2^2)$	1	13		6_3	
		$(1,-2,1,-2,1,-2)$	3	$4,4$	$0,0,0$	6_2^3	
	$(1^2,2^2,3^2)$	$(1,-2,1,3,-2,3)$	2	12	2	6_3^2	
7	1^7	1^7	1	7		7_1	
	$(1^5,2^2)$	$(1^4,2,-1,2)$	2	10	3	6_2^2	
		$(1^4,-2,1,-2)$	2	14	1	7_1^2	
		$(1^3,2,1^2,2)$	2	4	2	7_7^2	$\times$
		$(1^3,2,-1^2,2)$	2	8	0	7_8^2	$\times$
		$(1^3,-2,1^2,-2)$	2	16	0	7_4^2	
	$(1^4,2^3)$	$(1^3,-2,1,-2^2)$	2	18	1	7_2^2	
		$(1^2,-2,1^2,-2^2)$	2	20	2	7_5^2	
		$(1^2,-2,1,-2,1,-2)$	2	24	0	7_6^2	
	$(1^3,2^2,3^2)$	$(1^2,2,-1,-3,2,-3)$	1	9		6_1	
		$(1^2,-2,1,3,-2,3)$	1	19		7_6	
	$(1^2,2^3,3^2)$	$(1,-2,1,-2,3,-2,3)$	1	21		7_7	
		$(1,-2,1,3,-2^2,3)$	3	$10,2$	$1,1,1$	7_1^3	
8	1^8	1^8	2	8	4	8_1^2	
	$(1^6,2^2)$	$(1^5,2,-1,2)$	1	13		7_3	
		$(1^5,-2,1,-2)$	1	17		8_2	
		$(1^4,2,1^2,2)$	3	$2,2$	$1,1,2$	8_7^3	$\times$
		$(1^4,2,-1^2,2)$	3	$6,2$	$1,1,2$	8_8^3	$\times$
		$(1^4,-2,1^2,-2)$	3	$10,2$	$1,1,2$	8_1^3	
		$(1^3,2,1^3,2)$	1	3		8_{19}	
		$(1^3,2,-1^3,2)$	1	9		8_{20}	
		$(1^3,-2,1^3,-2)$	1	21		8_5	
	$(1^5,2^3)$	$(1^4,2,-1,2^2)$	1	17		7_5	
		$(1^4,-2,1,-2^2)$	1	23		8_7	

$\ell(x)$	$\|x\|_N$	x	μ	G	$\|lk\|$	L	$E(L)$
8	$(1^5,2^3)$	$(1^3,2,-1^2,2^2)$	1	15		8_{21}	
		$(1^3,-2,1^2,-2^2)$	1	27		8_{10}	
		$(1^3,2,-1,2,-1,2)$	3	4, 4	0, 0, 2	8^3_9	
		$(1^3,-2,1,-2,1,-2)$	3	16, 2	0, 0, 1	8^3_5	
		$(1^2,-2,1^2,-2,1,-2)$	1	35		8_{16}	
	$(1^4,2^4)$	$(1^3,-2,1,-2^3)$	1	25		8_9	
		$(1^3,-2^2,1,-2^2)$	3	14, 2	1, 1, 2	8^3_2	
		$(1^2,-2,1,-2,1,-2^2)$	1	37		8_{17}	
		$(1^2,-2,1,-2^2,1,-2)$	3	6, 6	0, 1, 1	8^3_6	
		$(1^2,2^2,1^2,2^2)$	3	4	0, 2, 2	8^3_{10}	
		$(1^2,-2^2,1^2,-2^2)$	3	8, 4	0, 2, 2	8^3_4	
		$(1,-2,1,-2,1,-2,1,-2)$	1	15, 3		8_{18}	
	$(1^4,2^2,3^2)$	$(1^3,2,-1,-3,2,-3)$	2	16	0	7^2_3	
		$(1^3,-2,1,3,-2,3)$	2	26	3	8^2_5	
		$(1^2,2,1^2,-3,2,-3)$	2	12	2	8^2_{16}	×
		$(1^2,2,-1^2,-3,2,-3)$	2	8	0	8^2_{15}	×
		$(1^2,-2,1^2,3,-2,3)$	2	28	2	8^2_9	
	$(1^3,2^3,3^2)$	$(1^2,-2,1,-2,3,-2,3)$	2	34	1	8^2_8	
		$(1^2,-2,1,3,-2^2,3)$	2	32	0	8^2_{12}	
		$(1,-2,1,-2,1,3,-2,3)$	2	40	0	8^2_{13}	
	$(1^3,2^2,3^3)$	$(1^2,-2,1,3,-2,3^2)$	2	30	1	8^2_7	
	$(1^2,2^4,3^2)$	$(1,-2,1,-2,3,-2^2,3)$	2	32	0	8^2_{10}	
		$(1,-2,1,3,-2^3,3)$	2	28	2	8^2_{11}	
		$(1,2^2,1,3,2^2,3)$	4	2, 2		8^4_3	
		$(1,2^2,1,3,-2^2,3)$	4	4, 2, 2		8^4_2	
		$(1,-2^2,1,3,-2^2,3)$	4	8, 2, 2		8^4_1	
		$(1,-2,3,-2,1,-2,3,-2)$	2	12, 3	2	8^2_{14}	
	$(1^2,2^2,3^2,4^2)$	$(1,-2,1,3,-2,-4,3,-4)$	1	29		8_{12}	
9	1^9	1^9	1	9		9_1	
	$(1^7,2^2)$	$(1^6,2,-1,2)$	2	16	4	8^2_2	
		$(1^6,-2,1,-2)$	2	20	2	9^2_1	
		$(1^5,2,1^2,2)$	2	4	2	9^2_{43}	×
		$(1^5,2,-1^2,2)$	2	16	0	9^2_{44}	×
		$(1^5,-2,1^2,-2)$	2	24	0	9^2_{13}	
		$(1^4,2,1^3,2)$	2	2	3	9^2_{49}	×
		$(1^4,2,-1^3,2)$	2	14	3	9^2_{51}	
		$(1^4,-2,1^3,-2)$	2	26	1	9^2_{19}	
		$(1^4,-2,-1^3,-2)$	2	10	1	9^2_{50}	
	$(1^6,2^3)$	$(1^5,2,-1,2^2)$	2	22	3	8^2_3	
		$(1^5,-2,1,-2^2)$	2	28	2	9^2_2	

$\ell(x)$	$\lvert x\rvert_N$	x	μ	G	$\lvert lk\rvert$	L	$E(L)$
9	$(1^6,2^3)$	$(1^4,2,-1^2,2^2)$	2	22	1	9^2_{52}	
		$(1^4,-2,1^2,-2^2)$	2	34	3	9^2_{20}	
		$(1^4,2,-1,2,-1,2)$	2	24	0	9^2_{55}	
		$(1^4,-2,1,-2,1,-2)$	2	40	0	9^2_{31}	
		$(1^3,2,1^3,2^2)$	2	3	4	9^2_{53}	
		$(1^3,2,-1^3,2^2)$	2	$6,3$	1	9^2_{54}	
		$(1^3,-2,1^3,2^2)$	2	24	4	8^2_4	
		$(1^3,-2,1^3,-2^2)$	2	$12,3$	2	9^2_{23}	
		$(1^3,2,-1^2,2,-1,2)$	2	20	2	9^2_{57}	
		$(1^3,-2,1^2,-2,1,-2)$	2	46	1	9^2_{35}	
		$(1^2,-2,1^2,-2,1^2,-2)$	2	$10,5$	3	9^2_{40}	
	$(1^5,2^4)$	$(1^4,-2,1,-2^3)$	2	32	0	9^2_5	
		$(1^4,-2^2,1,-2^2)$	2	36	2	9^2_{14}	
		$(1^3,-2,1^2,-2^3)$	2	38	1	9^2_{21}	
		$(1^3,-2,1,-2,1,-2^2)$	2	50	1	9^2_{34}	
		$(1^3,-2,1,-2^2,1,-2)$	2	48	0	9^2_{37}	
		$(1^3,2^2,1^2,2^2)$	2	4	2	9^2_{59}	
		$(1^3,-2^2,1^2,-2^2)$	2	44	2	9^2_{29}	
		$(1^2,-2,1^2,-2,1,-2^2)$	2	54	1	9^2_{39}	
		$(1^2,2,-1,2,1^2,2^2)$	2	5	4	9^2_{61}	
		$(1^2,-2,1,-2,1^2,-2^2)$	2	56	0	9^2_{41}	
		$(1^2,-2,1,-2,1,-2,1,-2)$	2	66	1	9^2_{42}	
	$(1^5,2^2,3^2)$	$(1^4,2,-1,-3,2,-3)$	1	23		8_6	
		$(1^4,-2,1,3,-2,3)$	1	33		9_{11}	
		$(1^3,2,1^2,-3,2,-3)$	1	13		9_{43}	
		$(1^3,2,-1^2,-3,2,-3)$	1	17		9_{44}	
		$(1^3,-2,1^2,3,-2,3)$	1	37		9_{36}	
		$(1^3,-2,-1^2,3,-2,3)$	1	7		9_{42}	
	$(1^4,2^3,3^2)$	$(1^3,2,-1,2,3,-2,3)$	1	11		7_2	
		$(1^3,2,-1,2,-3,2,-3)$	1	31		8_{14}	
		$(1^3,-2,1,-2,3,-2,3)$	1	47		9_{26}	
		$(1^3,-2,1,-2,-3,2,-3)$	1	19		8_4	
		$(1^3,2,-1,-3,2^2,-3)$	3	$14,2$	$1,1,1$	8^3_3	
		$(1^3,-2,1,3,-2^2,3)$	3	$22,2$	$1,1,2$	9^3_6	
		$(1^2,2,1^2,2,-3,2,-3)$	3	$6,2$	$0,1,1$	9^3_{13}	$\times$
		$(1^2,2,-1^2,2,-3,2,-3)$	3	$10,2$	$0,1,1$	9^3_{14}	$\times$
		$(1^2,-2,1^2,-2,3,-2,3)$	3	$26,2$	$0,1,1$	9^3_2	
		$(1^2,2,1^2,-3,2^2,-3)$	3	$4,4$	$0,0,2$	9^3_{19}	$\times$
		$(1^2,2,-1^2,-3,2^2,-3)$	3	$4,4$	$0,0,0$	9^3_{18}	$\times$
		$(1^2,-2,1^2,3,-2^2,3)$	3	$12,4$	$0,0,2$	9^3_8	

$\ell(x)$	$\|x\|_N$	x	μ	G	$\|lk\|$	L	$E(L)$
9	$(1^4,2^3,3^2)$	$(1^2,2,-1,2,1,3,-2,3)$	1	23		9_{45}	
		$(1^2,-2,1,-2,1,3,-2,3)$	1	59		9_{32}	
		$(1^2,-2,1,3,-2,1,3,-2)$	3	30, 2	1, 1, 2	9^3_{11}	
	$(1^4,2^2,3^3)$	$(1^3,2,-1,-3,2,-3^2)$	1	25		8_8	
		$(1^3,-2,1,3,-2,3^2)$	1	41		9_{20}	
		$(1^2,-2,1^2,3,-2,3^2)$	3	22, 2	0, 1, 1	9^3_1	
	$(1^3,2^4,3^2)$	$(1^2,2,-1,2^2,3,-2,3)$	1	15		7_4	
		$(1^2,2,-1,2^2,-3,2,-3)$	1	27		8_{11}	
		$(1^2,-2,1,-2^2,3,-2,3)$	1	49		9_{27}	
		$(1^2,-2,1,-2^2,-3,2,-3)$	1	29		8_{13}	
		$(1^2,-2,1,3,2^3,3)$	1	33		8_{15}	
		$(1^2,-2,1,3,-2^3,3)$	1	45		9_{24}	
		$(1^2,-2^2,1,-2,3,-2,3)$	1	53		9_{30}	
		$(1^2,2^2,1,3,2^2,3)$	3	2, 2	1, 1, 1	9^3_{17}	$\times$
		$(1^2,2^2,1,3,-2^2,3)$	3	14, 2	1, 1, 1	9^3_{16}	
		$(1^2,2^2,1,-3,2^2,-3)$	3	10, 2	1, 1, 1	9^3_{15}	
		$(1^2,-2^2,1,3,-2^2,3)$	3	26, 2	1, 1, 1	9^3_4	
	$(1^3,2^4,3^2)$	$(1,-2,1,-2,1,-2,3,-2,3)$	3	32, 2	0, 0, 1	9^3_{10}	
		$(1,-2,1,-2,1,3,2^2,3)$	3	8, 4	0, 0, 2	9^3_{20}	
		$(1,-2,1,-2,1,3,-2^2,3)$	3	8, 8	0, 0, 0	9^3_{12}	
		$(1,-2,1,-2,1,-3,2^2,-3)$	3	8	0, 0, 0	9^3_{21}	
		$(1,-2,1,-2^2,1,3,-2,3)$	1	61		9_{33}	
		$(1,2,-1,2,3,-2,1,-2,3)$	1	3, 3		9_{46}	
		$(1,-2,1,-2,3,-2,1,-2,3)$	1	69		9_{34}	
		$(1,-2,1,-2,-3,-2,1,-2,-3)$	1	9, 3		9_{47}	
	$(1^3,2^3,3^3)$	$(1^2,-2,1,-2,3,-2,3^2)$	1	55		9_{31}	
		$(1^2,-2,1,3,-2^2,3^2)$	1	51		9_{28}	
		$(1,-2,1,3,-2,1,3,-2,3)$	1	15, 5		9_{40}	
	$(1^3,2^2,3^2,4^2)$	$(1^2,-2,1,3,-2,-4,3,-4)$	2	46	1	9^2_{11}	
	$(1^2,2^5,3^2)$	$(1,-2,1,-2^3,3,-2,3)$	1	39		9_{17}	
		$(1,-2,1,-2,3,-2^3,3)$	1	43		9_{22}	
		$(1,-2,1,3,-2^4,3)$	3	18, 2	1, 1, 2	9^3_5	
		$(1,-2^2,1,-2,3,-2^2,3)$	3	12, 4	0, 0, 0	9^3_9	
		$(1,-2^2,3,-2,1,-2,3,-2)$	1	51		9_{29}	
	$(1^2,2^3,3^2,4^2)$	$(1,-2,1,-2,3,-2,-4,3,-4)$	2	50	1	9^2_{12}	
		$(1,-2,1,-2,-3,2,4,-3,4)$	2	20	2	8^2_6	
		$(1,-2,1,3,-2^2,-4,3,-4)$	2	48	0	9^2_{25}	