ENUMERATING THE PRIME KNOTS AND LINKS BY A CANONICAL ORDER

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ABSTRACT

The first author defined a canonical well-order in the set of links. After reviewing the definition, we explain how to enumerate the prime links by this order. At the end of this paper, we show a table of the first 156 prime links under this order.

Keywords: Knot, Lattice point, Length, Link, Table

1. Introduction

We consider unoriented links, so that we say that two links L and L' in S^3 are *equivalent* and we denote it by $L = L'$ if there is a homeomorphism $h : (S^3, L) \to (S^3, L')$. Although there are earlier studies on tabulation of knots and links, we mention here only three recent methods of tabulation.

J. H. Conway's method: J. H. Conway observed that every link is obtained from a *basic polyhedron*, a planar graph with a certain property, by replacing the vertices with *algebraic tangles* which are tangles constructed by a certain rule. By this idea, he made a table containing prime links with up to 10 or more crossings (See [1]). This method is suitable for understanding some global features of knots and links.

C. H. Dowker-M. B. Thistlethwaite's method: C. H. Dowker and M. B. Thistlethwaite assigned a sequence of integers to every knot diagram (See [3]). Although there are lots of sequences of integers representing the same knot, they made a table of prime knots with up to 13 or more crossings by combining this method with a computer use. A similar method can be used for a tabulation of links with 2 or more components (See [2]).

Y. Nakagawa's method: R. W. Ghrist showed that every oriented link can be realized as a periodic orbit on a template (See [4]). Using this result, Y. Nakagawa defines an injection from the set of knots to a set of positive integers and made a table of oriented knots from this viewpoint (See [7]). It is not clear to apply a similar method to the set of links with 2 or more components.

Apart from these methods, a method of enumerating the set of links and the set of closed connected orientable 3-manifolds was suggested in [5]. The idea is to introduce a canonical well-order in the set of links which also induces a well-order in the set of closed connected orientable 3-manifolds. In fact, in [5] by this method, the first 28 prime links and the first 26 closed connected orientable 3-manifolds are classified without any computer aid. The purpose of this paper is to explain how to enumerate the set of prime links by this method and to enlarge the table of the first 28 prime links without any computer aid. In Section 2, we review the definition of the well-order described in [5]. In Section 3, we explain how to enumerate the set of prime links. At the end, we show a table of the first 156 prime links.

2. Defnition of a well-order in the set of links

Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^n$ the product of n copies of $\mathbf{Z}$. We put

$$\mathbf{X} = \coprod_{n=1}^{\infty} \mathbf{Z}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbf{Z},\ n = 1, 2, \ldots\}.$$

We call elements of $\mathbf{X}$ *lattice points*. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put $\ell(\mathbf{x}) = n$ and call it the *length* of $\mathbf{x}$. Let $|\mathbf{x}|$ and $|\mathbf{x}|_N$ be the lattice points determined from $\mathbf{x}$ by the following formulas:

$|\mathbf{x}| = (|x_1|, |x_2|, \ldots, |x_n|)$ and
$|\mathbf{x}|_N = (|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ where $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$ and $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

We define a well-order in $\mathbf{X}$ as follows (See [5]):

Definition 2.1. We define a well-order in $\mathbf{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 \cdots$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we define $\mathbf{x} < \mathbf{y}$ if we have one of the following conditions (1)-(4):

(1) $\ell(\mathbf{x}) < \ell(\mathbf{y})$.
(2) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$ by the lexicographic order (on the natural number order).
(4) $|\mathbf{x}| = |\mathbf{y}|$ and $\mathbf{x} < \mathbf{y}$ by the lexicographic order on the well-order of $\mathbf{Z}$ defined above.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put

$$\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|.$$

Let $\beta(\mathbf{x})$ be the $(\max|\mathbf{x}|+1)$-string braid determined from $\mathbf{x}$ by the identity

$$\beta(\mathbf{x}) = \sigma_{|x_1|}^{\mathrm{sign}(x_1)}\sigma_{|x_2|}^{\mathrm{sign}(x_2)}\cdots\sigma_{|x_n|}^{\mathrm{sign}(x_n)},$$

where we define $\sigma_{|0|}^{\mathrm{sign}(0)} = 1$. We note that $\max|\mathbf{x}|+1$ is the minimum string number of the braid indicated by the right-hand side of the identity. Let $\mathrm{cl}\beta(\mathbf{x})$ be the closure of the braid $\beta(\mathbf{x})$. Let $\mathbf{L}$ be the set of the links. Then we have a map

$$\mathrm{cl}\beta : \mathbf{X} \to \mathbf{L}$$

sending $\mathbf{x}$ to $\mathrm{cl}\beta(\mathbf{x})$. By Alexander's braiding theorem, the map $\mathrm{cl}\beta$ is surjective. For $L \in \mathbf{L}$, we define a map

$$\sigma : \mathbf{L} \to \mathbf{X}$$

by $\sigma(L) = \min\{\mathbf{x} \in \mathbf{X} \,|\, \mathrm{cl}\beta(\mathbf{x}) = L\}$. Then σ is a right inverse of the map $\mathrm{cl}\beta$ and hence is injective. Now we have a well-order in $\mathbf{L}$ by the following definition:

Definition 2.2. For $L, L' \in \mathbf{L}$, we define $L < L'$ if $\sigma(L) < \sigma(L')$.

For a link $L \in \mathbf{L}$, we call $\ell(\sigma(L))$ the *length* of L.

3. A method of a tabulation of prime links

Let $\mathbf{L}^{\mathrm{p}}$ be the set of prime links. We use the injection σ for our method of a tabulation of $\mathbf{L}^{\mathrm{p}}$. For $k \in \mathbf{Z}$, let k^n and $-k^n$ be the lattice points determined by

$$k^n = (\underbrace{k, k, \ldots, k}_{n}) \quad \text{and} \quad -k^n = (-k)^n,$$

respectively. Let Δ be the subset of $\mathbf{X}$ consisting of 0, 1^m where $m \geqq 2$ and $(x_1, x_2, \ldots, x_n)$ where $n \geqq 2$, $x_1 = 1$, $1 \leqq |x_i| \leqq \frac{n}{2}$, $|x_n| \geqq 2$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, \max|\mathbf{x}|\}$. Then we have

$$\sharp\{\mathbf{x} \in \Delta \,|\, \ell(\mathbf{x}) = n\} < \infty$$

for every $n \geqq 1$, where $\sharp A$ denotes the cardinality of a set A. By this finiteness and the definition of the well-order in $\mathbf{X}$, we have

$$\sharp\{\mathbf{y} \in \Delta | \mathbf{y} < \mathbf{x}\} < \infty$$

for every $\mathbf{x} \in \Delta$. The following lemma is proved in [5]:

Lemma 3.1. $\sigma(\mathbf{L}^{\mathrm{p}}) \subset \Delta$.

From this lemma, the following plan on a tabulation of prime links comes to mind: First, we enumerate the lattice points of Δ under our order. Second, we construct the sequence of the links obtained by replacing $\mathbf{x}$ with $\mathrm{cl}\beta(\mathbf{x})$. Finally, we obtain a desired table of prime links from the sequence by removing the non-prime links and the links which have already appeared. However, to carry out this plan without any computer aid, we have a difficulty coming from the reason that Δ has lots of extra lattice points, that is, lattice points $\mathbf{x}$ such that the link $\mathrm{cl}\beta(\mathbf{x})$ is not prime or has already appeared. To save our energy, it is reasonable to prepare a subset $\Delta^* \subset \Delta$ containing the set $\sigma(\mathbf{L}^{\mathrm{p}})$ and to use Δ^* instead of Δ. To obtain such a subset Δ^*, we need some preliminaries. For $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_m) \in \Delta$, let $\mathbf{x}^T$, $-\mathbf{x}$, $(\mathbf{x},\ \mathbf{y})$ and $\delta(\mathbf{x})$ be the lattice points determined by the following formulas:

$$
\begin{aligned}
\mathbf{x}^T &= (x_n, \ldots, x_2, x_1),\\
-\mathbf{x} &= (-x_1, -x_2, \ldots, -x_n),\\
(\mathbf{x},\ \mathbf{y}) &= (x_1, \ldots, x_n, y_1, \ldots, y_m),\\
\delta(\mathbf{x}) &= (x'_1, x'_2, \ldots, x'_n),\\
\text{where}\quad x'_i &= \begin{cases} \mathrm{sign}(x_i)(\max|\mathbf{x}| + 1 - x_i) & (x_i \neq 0)\\ 0 & (x_i = 0).\end{cases}
\end{aligned}
$$

A point of our argument is to define some transformations on lattice points.

Definition 3.2. Let $\mathbf{x},\ \mathbf{y} \in \mathbf{X},\ k,\ l,\ m \in \mathbf{Z}$ with $m > 0$ and $\varepsilon = \pm 1$. An *elementary transformation* on lattice points is one of the following operations and their inverses.

(1) $(\mathbf{x},\ k,\ l) \to (\mathbf{x},\ l,\ k)$, where $|k| > |l| + 1$ or $|l| > |k| + 1$.
(2) $(\mathbf{x},\ \varepsilon k^m,\ k+1,\ k) \to (\mathbf{x},\ k+1,\ k,\ \varepsilon(k+1)^m)$, where $k(k+1) \neq 0$.
(3) $(\mathbf{x},\ k,\ \varepsilon(k+1)^m,\ -k) \to (\mathbf{x},\ -(k+1),\ \varepsilon k^m,\ k+1)$, where $k(k+1) \neq 0$.
(4) $(\mathbf{x},\ \mathbf{y}) \to (\mathbf{y},\ \mathbf{x})$
(5) $\mathbf{x} \to \mathbf{x}^T$
(6) $\mathbf{x} \to -\mathbf{x}$
(7) $\mathbf{x} \to \delta(\mathbf{x})$
(8) $(1^m,\ \mathbf{x},\ \varepsilon,\ \mathbf{y}) \to (1^m,\ \mathbf{y},\ \varepsilon,\ \mathbf{x})$, where $\min|\mathbf{x}| \geqq 2$ and $\min|\mathbf{y}| \geqq 2$.

Then we have the following two lemmas (See [5, 6] for the proofs).

Lemma 3.3. If $\mathbf{x}$ is transformed into $\mathbf{y}$ by an elementary transformation, then we have $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ (modulo split unions of trivial links for (7)).

Lemma 3.4. Let $\mathbf{x},\ \mathbf{y},\ \mathbf{z} \in \mathbf{X},\ k \in \mathbf{Z},\ 0 < m \in \mathbf{Z}$. If $\mathbf{x}$ is transformed into a smaller lattice point than $\mathbf{x}$ or one of the following lattice points by elementary transformations, then we have $\mathbf{x} \notin \sigma(\mathbf{L}^{\mathrm{p}})$.

(1) $(\mathbf{y}, k^m)$, where $\mathbf{y}$ does not contain $\pm k$.
(2) $(\mathbf{y}, k, -k)$.
(3) $(\mathbf{y}, \mathbf{z})$, where $\max|\mathbf{y}| < \min|\mathbf{z}|$ or $\max|\mathbf{z}| < \min|\mathbf{y}|$.

Definition 3.5. We define the set Δ^* obtained from Δ by removing the lattice points which satisfy the condition of Lemma 3.4.

Using Δ^* instead of Δ, we enumerate the lattice points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ of Δ^* under our order and construct the sequence of the links $\mathrm{cl}\beta(\mathbf{x}), \mathrm{cl}\beta(\mathbf{y}), \mathrm{cl}\beta(\mathbf{z}), \ldots$ and remove the non-prime links and the links which have already appeared from the sequence and then we obtain a desired table of prime links. We could carry out this process without any computer aid for the lattice points of length up to 9 and obtain the table, showing at the end of this paper. This plan for the lattice points of length 10 is in preparation except the case of lattice points representing knots. In the following three remarks, we observe some points of carrying out our tabulation.

Remark 3.6 (A note on making the list of Δ^*). Let $\Delta_n^* = \{\mathbf{x} \in \Delta^* \mid \ell(\mathbf{x}) = n\}$ for every $n > 0$. By a step-by-step method, we can make the list of Δ_n^* as follows:

(1) From Δ, we determine the ordered set $A_n = \{|\mathbf{x}|_N : x \in \Delta_n^*\}$.
(2) From A_n, we determine the ordered set $B_n = \{|\mathbf{x}| : x \in \Delta_n^*\}$.
(3) From B_n, we determine the ordered set Δ_n^*.

In each step, we use Lemma 3.4. Somtimes, a technical transformation is necessary. Since we might not find it out, our list might have a few lattice points of $\Delta \setminus \Delta^*$. However, there is no problem on such a list except an economical problem, since our list includes the set $\sigma(\mathbf{L}^{\mathrm{p}})$.

Remark 3.7 (A note on making the links of Δ^*). The link $\mathrm{cl}\beta(\mathbf{x})$ has a natural orientation coming from the braid $\beta(\mathbf{x})$. If $\mathbf{x}$ is transformed into $\mathbf{y}$ by an elementary transformation, then we have $\mathrm{cl}\beta(\mathbf{x}) = \pm\mathrm{cl}\beta(\mathbf{y})$ as oriented links (See [5] for the proof). When one considers links with two or more components, it should be noticed that in general there are two lattice points $\mathbf{x},\ \mathbf{y} \in \mathbf{X}$ such that

$$\begin{aligned} \mathrm{cl}\beta(\mathbf{x}) &= \mathrm{cl}\beta(\mathbf{y}) \text{ as unoriented links, and} \\ \mathrm{cl}\beta(\mathbf{x}) &\neq \pm\mathrm{cl}\beta(\mathbf{y}) \text{ as oriented links.} \end{aligned}$$

Thus, in the case of links with two or more components, we have more extra lattice points in Δ^*. *Removing this type of extra lattice points from* Δ^* *remains as an open question.*

Remark 3.8 (A note on determining links). To make the table of prime links with up to 9 lengths, we compute the Goeritz invariant and the absolute values of the linking numbers on the link $\mathrm{cl}\beta(\mathbf{x})$ for every $\mathbf{x} \in \Delta^*$ with $\ell(\mathbf{x}) \leqq 9$. Then we look for links with up to 9 crossings with the same data, for example from D. Rolfsen's table. We always find them because the link $\mathrm{cl}\beta(\mathbf{x})$ has at most 9 crossings. For those links with the same data as $\mathrm{cl}\beta(\mathbf{x})$, we can determine which link is equivalent to $\mathrm{cl}\beta(\mathbf{x})$ by calculating some other link invariants. Actually, Conway's notations of links and simple geometric arguments are often useful in saving time.

In the table shown at the end of this paper, we denote by μ, G, and $|lk|$ the number of the components of the link, the Goeritz invariant of the link and the absolute values of the linking numbers of all pairs of the components of the link, respectively. The following remark implies that every prime link is canonically embedded in the set $\mathbf{Q}_+$ of positive rational numbers (See [6]).

Remark 3.9. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta$, we define a map $\zeta : \Delta \to \mathbf{Q}_+$ by

$$\zeta(\mathbf{x}) = n + \frac{x_2}{(n+1)^{n-1}} + \cdots + \frac{x_n}{n+1}.$$

Then we have the following properties (1) and (2).

(1) ζ is injective.

(2) If $\zeta(\mathbf{x})$ is given, we can reconstruct $\mathbf{x}$.

Combining this embedding ζ with the injection σ, we can reconstruct $L \in \mathbf{L}^{\mathrm{p}}$ from the rational number $\zeta\sigma(L)$.

In the argument above, we discussed a tabulation of $\mathbf{L}^{\mathrm{p}}$. Since the subset $\sigma(\mathbf{L}) \subset \mathbf{X}$ is not in Δ, we need different treatments of Definition 3.2 and Lemma 3.4 to tabulate the set $\mathbf{L}$ itself. This is possible by using the injection $\sigma : \mathbf{L} \to \mathbf{X}$. By a further modification of our argument, we can also see that every link is canonically embedded in $\mathbf{Q}_+$.

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$\ell(x)$	$\|x\|_N$	$\|x\|$	x	μ	G	$\|lk\|$	L
1	0	0	0	1	1		O
2	1^2	1^2	1^2	2	2	1	2_1^2
3	1^3	1^3	1^3	1	3		3_1
4	1^4	1^4	1^4	2	4	2	4_1^2
	$(1^2,2^2)$	$(1,2,1,2)$	$(1,-2,1,-2)$	1	5		4_1
5	1^5	1^5	1^5	1	5		5_1
	$(1^3,2^2)$	$(1^2,2,1,2)$	$(1^2,-2,1,-2)$	2	8	0	5_1^2
6	1^6	1^6	1^6	2	6	3	6_1^2
	$(1^4,2^2)$	$(1^3,2,1,2)$	$(1^3,2,-1,2)$	1	7		5_2
			$(1^3,-2,1,-2)$	1	11		6_2
		$(1^2,2,1^2,2)$	$(1^2,2,1^2,2)$	3	2, 2	1, 1, 1	6_3^3
			$(1^2,-2,1^2,-2)$	3	6, 2	1, 1, 1	6_1^3
	$(1^3,2^3)$	$(1^2,2,1,2^2)$	$(1^2,-2,1,-2^2)$	1	13		6_3
		$(1,2,1,2,1,2)$	$(1,-2,1,-2,1,-2)$	3	4, 4	0, 0, 0	6_2^3
	$(1^2,2^2,3^2)$	$(1,2,1,3,2,3)$	$(1,-2,1,3,-2,3)$	2	12	2	6_3^2
7	1^7	1^7	1^7	1	7		7_1
	$(1^5,2^2)$	$(1^4,2,1,2)$	$(1^4,2,-1,2)$	2	10	3	6_2^2
			$(1^4,-2,1,-2)$	2	14	1	7_1^2
		$(1^3,2,1^2,2)$	$(1^3,2,1^2,2)$	2	4	2	7_7^2
			$(1^3,2,-1^2,2)$	2	8	0	7_8^2
			$(1^3,-2,1^2,-2)$	2	16	0	7_4^2
	$(1^4,2^3)$	$(1^3,2,1,2^2)$	$(1^3,-2,1,-2^2)$	2	18	1	7_2^2
		$(1^2,2,1^2,2^2)$	$(1^2,-2,1^2,-2^2)$	2	20	2	7_5^2
		$(1^2,2,1,2,1,2)$	$(1^2,-2,1,-2,1,-2)$	2	24	0	7_6^2
	$(1^3,2^2,3^2)$	$(1^2,2,1,3,2,3)$	$(1^2,2,-1,-3,2,-3)$	1	9		6_1
			$(1^2,-2,1,3,-2,3)$	1	19		7_6
	$(1^2,2^3,3^2)$	$(1,2,1,2,3,2,3)$	$(1,-2,1,-2,3,-2,3)$	1	21		7_7
		$(1,2,1,3,2^2,3)$	$(1,-2,1,3,-2^2,3)$	3	10, 2	1, 1, 1	7_1^3
8	1^8	1^8	1^8	2	8	4	8_1^2
	$(1^6,2^2)$	$(1^5,2,1,2)$	$(1^5,2,-1,2)$	1	13		7_3
			$(1^5,-2,1,-2)$	1	17		8_2
		$(1^4,2,1^2,2)$	$(1^4,2,1^2,2)$	3	2, 2	1, 1, 2	8_7^3
			$(1^4,2,-1^2,2)$	3	6, 2	1, 1, 2	8_8^3
			$(1^4,-2,1^2,-2)$	3	10, 2	1, 1, 2	8_1^3
		$(1^3,2,1^3,2)$	$(1^3,2,1^3,2)$	1	3		8_{19}
			$(1^3,2,-1^3,2)$	1	9		8_{20}
			$(1^3,-2,1^3,-2)$	1	21		8_5
	$(1^5,2^3)$	$(1^4,2,1,2^2)$	$(1^4,2,-1,2^2)$	1	17		7_5
			$(1^4,-2,1,-2^2)$	1	23		8_7

$\ell(x)$	$\lvert x\rvert_N$	$\lvert x\rvert$	x	μ	G	$\lvert lk\rvert$	L
8	$(1^5,2^3)$	$(1^3,2,1^2,2^2)$	$(1^3,2,-1^2,2^2)$	1	15		8_{21}
			$(1^3,-2,1^2,-2^2)$	1	27		8_{10}
		$(1^3,2,1,2,1,2)$	$(1^3,2,-1,2,-1,2)$	3	4, 4	0, 0, 2	8^3_9
			$(1^3,-2,1,-2,1,-2)$	3	16, 2	0, 0, 1	8^3_5
		$(1^2,2,1^2,2,1,2)$	$(1^2,-2,1^2,-2,1,-2)$	1	35		8_{16}
	$(1^4,2^4)$	$(1^3,2,1,2^3)$	$(1^3,-2,1,-2^3)$	1	25		8_9
		$(1^3,2^2,1,2^2)$	$(1^3,-2^2,1,-2^2)$	3	14, 2	1, 1, 2	8^3_2
		$(1^2,2,1,2,1,2^2)$	$(1^2,-2,1,-2,1,-2^2)$	1	37		8_{17}
		$(1^2,2,1,2^2,1,2)$	$(1^2,-2,1,-2^2,1,-2)$	3	6, 6	0, 1, 1	8^3_6
		$(1^2,2^2,1^2,2^2)$	$(1^2,2^2,1^2,2^2)$	3	4	0, 2, 2	8^3_{10}
			$(1^2,-2^2,1^2,-2^2)$	3	8, 4	0, 2, 2	8^3_4
		$(1,2,1,2,1,2,1,2)$	$(1,-2,1,-2,1,-2,1,-2)$	1	15, 3		8_{18}
	$(1^4,2^2,3^2)$	$(1^3,2,1,3,2,3)$	$(1^3,2,-1,-3,2,-3)$	2	16	0	7^2_3
			$(1^3,-2,1,3,-2,3)$	2	26	3	8^2_5
		$(1^2,2,1^2,3,2,3)$	$(1^2,2,1^2,-3,2,-3)$	2	12	2	8^2_{16}
			$(1^2,2,-1^2,-3,2,-3)$	2	8	0	8^2_{15}
			$(1^2,-2,1^2,3,-2,3)$	2	28	2	8^2_9
	$(1^3,2^3,3^2)$	$(1^2,2,1,2,3,2,3)$	$(1^2,-2,1,-2,3,-2,3)$	2	34	1	8^2_8
		$(1^2,2,1,3,2^2,3)$	$(1^2,-2,1,3,-2^2,3)$	2	32	0	8^2_{12}
		$(1,2,1,2,1,3,2,3)$	$(1,-2,1,-2,1,3,-2,3)$	2	40	0	8^2_{13}
	$(1^3,2^2,3^3)$	$(1^2,2,1,3,2,3^2)$	$(1^2,-2,1,3,-2,3^2)$	2	30	1	8^2_7
	$(1^2,2^4,3^2)$	$(1,2,1,2,3,2^2,3)$	$(1,-2,1,-2,3,-2^2,3)$	2	32	0	8^2_{10}
		$(1,2,1,3,2^3,3)$	$(1,-2,1,3,-2^3,3)$	2	28	2	8^2_{11}
		$(1,2^2,1,3,2^2,3)$	$(1,2^2,1,3,2^2,3)$	4	2, 2		8^4_3
			$(1,2^2,1,3,-2^2,3)$	4	4, 2, 2		8^4_2
			$(1,-2^2,1,3,-2^2,3)$	4	8, 2, 2		8^4_1
		$(1,2,3,2,1,2,3,2)$	$(1,-2,3,-2,1,-2,3,-2)$	2	12, 3	2	8^2_{14}
	$(1^2,2^2,3^2,4^2)$	$(1,2,1,3,2,4,3,4)$	$(1,-2,1,3,-2,-4,3,-4)$	1	29		8_{12}
9	1^9	1^9	1^9	1	9		9_1
	$(1^7,2^2)$	$(1^6,2,1,2)$	$(1^6,2,-1,2)$	2	16	4	8^2_2
			$(1^6,-2,1,-2)$	2	20	2	9^2_1
		$(1^5,2,1^2,2)$	$(1^5,2,1^2,2)$	2	4	2	9^2_{43}
			$(1^5,2,-1^2,2)$	2	16	0	9^2_{44}
			$(1^5,-2,1^2,-2)$	2	24	0	9^2_{13}
		$(1^4,2,1^3,2)$	$(1^4,2,1^3,2)$	2	2	3	9^2_{49}
			$(1^4,2,-1^3,2)$	2	14	3	9^2_{51}
			$(1^4,-2,1^3,-2)$	2	26	1	9^2_{19}
			$(1^4,-2,-1^3,-2)$	2	10	1	9^2_{50}
	$(1^6,2^3)$	$(1^5,2,1,2^2)$	$(1^5,2,-1,2^2)$	2	22	3	8^2_3
			$(1^5,-2,1,-2^2)$	2	28	2	9^2_2

$\ell(x)$	$\|x\|_N$	$\|x\|$	x	μ	G	$\|lk\|$	L
9	$(1^6,2^3)$	$(1^4,2,1^2,2^2)$	$(1^4,2,-1^2,2^2)$	2	22	1	9^2_{52}
			$(1^4,-2,1^2,-2^2)$	2	34	3	9^2_{20}
		$(1^4,2,1,2,1,2)$	$(1^4,2,-1,2,-1,2)$	2	24	0	9^2_{55}
			$(1^4,-2,1,-2,1,-2)$	2	40	0	9^2_{31}
		$(1^3,2,1^3,2^2)$	$(1^3,2,1^3,2^2)$	2	3	4	9^2_{53}
			$(1^3,2,-1^3,2^2)$	2	6, 3	1	9^2_{54}
			$(1^3,-2,1^3,2^2)$	2	24	4	8^2_4
			$(1^3,-2,1^3,-2^2)$	2	12, 3	2	9^2_{23}
		$(1^3,2,1^2,2,1,2)$	$(1^3,2,-1^2,2,-1,2)$	2	20	2	9^2_{57}
			$(1^3,-2,1^2,-2,1,-2)$	2	46	1	9^2_{35}
		$(1^2,2,1^2,2,1^2,2)$	$(1^2,-2,1^2,-2,1^2,-2)$	2	10, 5	3	9^2_{40}
	$(1^5,2^4)$	$(1^4,2,1,2^3)$	$(1^4,-2,1,-2^3)$	2	32	0	9^2_5
		$(1^4,2^2,1,2^2)$	$(1^4,-2^2,1,-2^2)$	2	36	2	9^2_{14}
		$(1^3,2,1^2,2^3)$	$(1^3,-2,1^2,-2^3)$	2	38	1	9^2_{21}
		$(1^3,2,1,2,1,2^2)$	$(1^3,-2,1,-2,1,-2^2)$	2	50	1	9^2_{34}
		$(1^3,2,1,2^2,1,2)$	$(1^3,-2,1,-2^2,1,-2)$	2	48	0	9^2_{37}
		$(1^3,2^2,1^2,2^2)$	$(1^3,2^2,1^2,2^2)$	2	4	2	9^2_{59}
			$(1^3,-2^2,1^2,-2^2)$	2	44	2	9^2_{29}
		$(1^2,2,1^2,2,1,2^2)$	$(1^2,-2,1^2,-2,1,-2^2)$	2	54	1	9^2_{39}
		$(1^2,2,1,2,1^2,2^2)$	$(1^2,2,-1,2,1^2,2^2)$	2	5	4	9^2_{61}
			$(1^2,-2,1,-2,1^2,-2^2)$	2	56	0	9^2_{41}
		$(1^2,2,1,2,1,2,1,2)$	$(1^2,-2,1,-2,1,-2,1,-2)$	2	66	1	9^2_{42}
	$(1^5,2^2,3^2)$	$(1^4,2,1,3,2,3)$	$(1^4,2,-1,-3,2,-3)$	1	23		8_6
			$(1^4,-2,1,3,-2,3)$	1	33		9_{11}
		$(1^3,2,1^2,3,2,3)$	$(1^3,2,1^2,-3,2,-3)$	1	13		9_{43}
			$(1^3,2,-1^2,-3,2,-3)$	1	17		9_{44}
			$(1^3,-2,1^2,3,-2,3)$	1	37		9_{36}
			$(1^3,-2,-1^2,3,-2,3)$	1	7		9_{42}
	$(1^4,2^3,3^2)$	$(1^3,2,1,2,3,2,3)$	$(1^3,2,-1,2,3,-2,3)$	1	11		7_2
			$(1^3,2,-1,2,-3,2,-3)$	1	31		8_{14}
			$(1^3,-2,1,-2,3,-2,3)$	1	47		9_{26}
			$(1^3,-2,1,-2,-3,2,-3)$	1	19		8_4
		$(1^3,2,1,3,2^2,3)$	$(1^3,2,-1,-3,2^2,-3)$	3	14, 2	1, 1, 1	8^3_3
			$(1^3,-2,1,3,-2^2,3)$	3	22, 2	1, 1, 2	9^3_6
		$(1^2,2,1^2,2,3,2,3)$	$(1^2,2,1^2,2,-3,2,-3)$	3	6, 2	0, 1, 1	9^3_{13}
			$(1^2,2,-1^2,2,-3,2,-3)$	3	10, 2	0, 1, 1	9^3_{14}
			$(1^2,-2,1^2,-2,3,-2,3)$	3	26, 2	0, 1, 1	9^3_2
		$(1^2,2,1^2,3,2^2,3)$	$(1^2,2,1^2,-3,2^2,-3)$	3	4, 4	0, 0, 2	9^3_{19}
			$(1^2,2,-1^2,-3,2^2,-3)$	3	4, 4	0, 0, 0	9^3_{18}
			$(1^2,-2,1^2,3,-2^2,3)$	3	12, 4	0, 0, 2	9^3_8

$\ell(x)$	$\lvert x\rvert_N$	$\lvert x\rvert$	x	μ	G	$\lvert lk\rvert$	L
9	$(1^4,2^3,3^2)$	$(1^2,2,1,2,1,3,2,3)$	$(1^2,2,-1,2,1,3,-2,3)$	1	23		9_{45}
			$(1^2,-2,1,-2,1,3,-2,3)$	1	59		9_{32}
		$(1^2,2,1,3,2,1,3,2)$	$(1^2,-2,1,3,-2,1,3,-2)$	3	30, 2	1, 1, 2	9^3_{11}
	$(1^4,2^2,3^3)$	$(1^3,2,1,3,2,3^2)$	$(1^3,2,-1,-3,2,-3^2)$	1	25		8_8
			$(1^3,-2,1,3,-2,3^2)$	1	41		9_{20}
		$(1^2,2,1^2,3,2,3^2)$	$(1^2,-2,1^2,3,-2,3^2)$	3	22, 2	0, 1, 1	9^3_1
	$(1^3,2^4,3^2)$	$(1^2,2,1,2^2,3,2,3)$	$(1^2,2,-1,2^2,3,-2,3)$	1	15		7_4
			$(1^2,2,-1,2^2,-3,2,-3)$	1	27		8_{11}
			$(1^2,-2,1,-2^2,3,-2,3)$	1	49		9_{27}
			$(1^2,-2,1,-2^2,-3,2,-3)$	1	29		8_{13}
		$(1^2,2,1,3,2^3,3)$	$(1^2,-2,1,3,2^3,3)$	1	33		8_{15}
			$(1^2,-2,1,3,-2^3,3)$	1	45		9_{24}
		$(1^2,2^2,1,2,3,2,3)$	$(1^2,-2^2,1,-2,3,-2,3)$	1	53		9_{30}
		$(1^2,2^2,1,3,2^2,3)$	$(1^2,2^2,1,3,2^2,3)$	3	2, 2	1, 1, 1	9^3_{17}
			$(1^2,2^2,1,3,-2^2,3)$	3	14, 2	1, 1, 1	9^3_{16}
			$(1^2,2^2,1,-3,2^2,-3)$	3	10, 2	1, 1, 1	9^3_{15}
			$(1^2,-2^2,1,3,-2^2,3)$	3	26, 2	1, 1, 1	9^3_4
	$(1^3,2^4,3^2)$	$(1,2,1,2,1,2,3,2,3)$	$(1,-2,1,-2,1,-2,3,-2,3)$	3	32, 2	0, 0, 1	9^3_{10}
		$(1,2,1,2,1,3,2^2,3)$	$(1,-2,1,-2,1,3,2^2,3)$	3	8, 4	0, 0, 2	9^3_{20}
			$(1,-2,1,-2,1,3,-2^2,3)$	3	8, 8	0, 0, 0	9^3_{12}
			$(1,-2,1,-2,1,-3,2^2,-3)$	3	8	0, 0, 0	9^3_{21}
		$(1,2,1,2^2,1,3,2,3)$	$(1,-2,1,-2^2,1,3,-2,3)$	1	61		9_{33}
		$(1,2,1,2,3,2,1,2,3)$	$(1,2,-1,2,3,-2,1,-2,3)$	1	3, 3		9_{46}
			$(1,-2,1,-2,3,-2,1,-2,3)$	1	69		9_{34}
			$(1,-2,1,-2,-3,-2,1,-2,-3)$	1	9, 3		9_{47}
	$(1^3,2^3,3^3)$	$(1^2,2,1,2,3,2,3^2)$	$(1^2,-2,1,-2,3,-2,3^2)$	1	55		9_{31}
		$(1^2,2,1,3,2^2,3^2)$	$(1^2,-2,1,3,-2^2,3^2)$	1	51		9_{28}
		$(1,2,1,3,2,1,3,2,3)$	$(1,-2,1,3,-2,1,3,-2,3)$	1	15, 5		9_{40}
	$(1^3,2^2,3^2,4^2)$	$(1^2,2,1,3,2,4,3,4)$	$(1^2,-2,1,3,-2,-4,3,-4)$	2	46	1	9^2_{11}
	$(1^2,2^5,3^2)$	$(1,2,1,2^3,3,2,3)$	$(1,-2,1,-2^3,3,-2,3)$	1	39		9_{17}
		$(1,2,1,2,3,2^3,3)$	$(1,-2,1,-2,3,-2^3,3)$	1	43		9_{22}
		$(1,2,1,3,2^4,3)$	$(1,-2,1,3,-2^4,3)$	3	18, 2	1, 1, 2	9^3_5
		$(1,2^2,1,2,3,2^2,3)$	$(1,-2^2,1,-2,3,-2^2,3)$	3	12, 4	0, 0, 0	9^3_9
		$(1,2^2,3,2,1,2,3,2)$	$(1,-2^2,3,-2,1,-2,3,-2)$	1	51		9_{29}
	$(1^2,2^3,3^2,4^2)$	$(1,2,1,2,3,2,4,3,4)$	$(1,-2,1,-2,3,-2,-4,3,-4)$	2	50	1	9^2_{12}
			$(1,-2,1,-2,-3,2,4,-3,4)$	2	20	2	8^2_6
		$(1,2,1,3,2^2,4,3,4)$	$(1,-2,1,3,-2^2,-4,3,-4)$	2	48	0	9^2_{25}