Immersed 2-knots with essential singularity

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Abstract

It is shown that there are infinitely many immersed 2-knots with more than any previously given number of double point singularities which are not equivalent to the connected sum of any immersed 2-knot and any unknotted immersed sphere.

Keywords: Immersed 2-knot, Immersed surface-link, Essential singularity, Unknotted immersed sphere, Marked graph diagram, Symmetric ideal.

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1. Introduction

An immersed surface-link is a generically immersed closed oriented surface in the 4-space $\mathbb{R}^4$. When the surface has only one component, it is also called an immersed surface-knot. When the surface consists of 2-spheres, it is also called an immersed sphere-link or simply an immersed 2-link. When the immersion is an embedding, it is also called a surface-link. Two (immersed) surface-links $\mathcal{L}$ and $\mathcal{L}'$ are equivalent if there is an orientation-preserving auto-homeomorphism $h$ of $\mathbb{R}^4$ sending $\mathcal{L}$ to $\mathcal{L}'$ orientation-preservingly. An immersed 2-link is studied in [9] in relation to a cross-sectional link. A normal

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form of an immersed surface-link introduced by S. Kamada and K. Kawamura in [5] is used to define a marked graph diagram of an immersed surface-link in [6]. In this paper, with an example obtained from a surface-knot described by a marked graph diagram, it is shown as the main theorem (Theorem 3.6) that for any positive integer \( n \), there are infinitely many immersed 2-knots with at least \( n \) double point singularities every of which is essential double point singularities, that is, infinitely many immersed 2-knots with at least \( n \) double point singularities which are not equivalent to the connected sum of any immersed 2-knot and any unknotted immersed sphere.

This paper is organized as follows: Section 2 is devoted to a review of a marked graph diagram of an immersed surface-link. In particular, an unknotted immersed sphere is defined there. In Section 3, the main theorem is proved.

2. Marked graph representation of immersed surface-links

In this section, we review (oriented) marked graph diagrams representing immersed surface-links described in [6]. A marked graph is a 4-valent graph in \( \mathbb{R}^3 \) each of whose vertices is a vertex with a marker looks like \( \begin{array}{c} \phantom{0} \\ \downarrow \end{array} \). Two marked graphs are said to be equivalent if they are ambient isotopic in \( \mathbb{R}^3 \) with keeping the rectangular neighborhoods of markers. As usual, a marked graph in \( \mathbb{R}^3 \) can be described by a link diagram on \( \mathbb{R}^2 \) with some 4-valent vertices equipped with markers, called a marked graph diagram. An orientation of a marked graph \( G \) in \( \mathbb{R}^3 \) is a choice of an orientation for each edge of \( G \). An orientation of a marked graph \( G \) is said to be consistent if every vertex in \( G \) looks like \( \begin{array}{c} \phantom{0} \\ \downarrow \end{array} \). A marked graph \( G \) in \( \mathbb{R}^3 \) is said to be orientable if \( G \) admits a consistent orientation. Otherwise, it is said to be non-orientable. By an oriented marked graph we mean an orientable marked graph in \( \mathbb{R}^3 \) with a fixed consistent orientation. Two oriented marked graphs are said to be equivalent if they are ambient isotopic in \( \mathbb{R}^3 \) with keeping the rectangular neighborhood, marker and consistent orientation. For \( t \in \mathbb{R} \), we denote by \( \mathbb{R}^3_t \) the hyperplane of \( \mathbb{R}^4 \) whose fourth coordinate is equal to \( t \in \mathbb{R} \), i.e., \( \mathbb{R}^3_t = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\} \). An immersed surface-link \( \mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \) can be described in terms of its cross-sections \( \mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3_t, \ t \in \mathbb{R} \) (cf. [3]). It is shown [5] that any immersed surface-link \( \mathcal{L} \), there is an immersed surface-link \( \mathcal{L}' \subset \mathbb{R}^3 [-2, 2] \) satisfying the following
conditions:

(1) The intersections $L'_{1}$ and $L'_{-1}$ are H-trivial links;

(2) All saddle points of $L'$ are in $\mathbb{R}^3[0]$;

(3) All maximal points of $L'$ are in $\mathbb{R}^3[2]$;

(4) All minimal points of $L'$ are in $\mathbb{R}^3[-2]$;

(5) The intersections $L' \cap (\mathbb{R}^3[1, 2])$ and $L' \cap (\mathbb{R}^3[-2, -1])$ are disjoint unions of a disjoint system of trivial knot cones and a disjoint system of Hopf link cones.

We call $L'$ a normal form of $L$. Let $L$ be an immersed surface-link in $\mathbb{R}^4$, and $L'$ a normal form of $L$. Then $L'_0$ is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Fig. 1. We choose an orientation for each edge of $L'_0$ that coincides with the induced orientation on the boundary of $L' \cap \mathbb{R}^3 \times (-\infty, 0]$ from the orientation of $L'$. The resulting oriented marked graph $G$ is called an oriented marked graph of $L$. As usual, $G$ is described by a link diagram $D$ with rigid marked vertices. Such a diagram $D$ is called an oriented marked graph diagram or an oriented ch-diagram (cf. [13]) of $L$.

Let $D$ be an oriented marked graph diagram. We obtain two links $L_-(D)$ and $L_+(D)$ from $D$ by replacing each marked vertex $\phantom{\text{Marked Vertex}}$ with $\phantom{\text{Marked Vertex}}$ (and $\phantom{\text{Marked Vertex}}$, respectively. Links $L_-(D)$ and $L_+(D)$ are also called the negative
resolution and the positive resolution of $D$, respectively. By replacing a neighborhood of each marked vertex $v_i$ ($1 \leq i \leq n$) with an oriented band $B_i$ as illustrated in Fig. 2. Denote the disjoint union $B_1 \sqcup \cdots \sqcup B_n$ of bands by $B(D)$. A link $L$ is H-trivial if $L$ is a split union of trivial knots and Hopf links. A marked graph diagram $D$ is said to be H-admissible if both resolutions $L_-(D)$ and $L_+(D)$ are H-trivial classical link diagrams.

From now on, we recall how to construct an immersed surface-link $\mathcal{L}$ in $\mathbb{R}^4$ from a given H-admissible oriented marked graph diagram (cf. [5, 6]). Let $D$ be an H-admissible oriented marked graph diagram. We define a surface-link $\mathcal{F}(D) \subset \mathbb{R}^3 \times [-1, 1]$, called the proper surface associated with $D$, by
Let \( A \) be a graph diagram of the oriented immersed surface-link \( \mathcal{L} \) with orientations induced from the orientation of \( D \). It is known that a marked graph diagram \( D \) has a consistent orientation, the resolutions \( L_+(D) \) and \( L_-(D) \) have the orientations induced from the orientation of \( D \). We choose an orientation for the proper surface \( \mathcal{F}(D) \) so that the induced orientation of the cross-section \( L_+(D) = \mathcal{F}(D)_t = \mathcal{F}(D) \cap \mathbb{R}^3_t \) at \( t = 1 \) matches the orientation of \( L_+(D) \).

Let \([a, b]\) be a closed interval with \( a < b \). For a link \( L \), let \( \tilde{L} * [a, b] \) (or \( \tilde{L} * [a, b] \)) be a cone with \( L[a] \) (or \( L[b] \)) as the base and a point in \( \mathbb{R}^3[a] \) (or \( \mathbb{R}^3[a] \)), respectively. Let \( H = (O_1 \cup \cdots \cup O_m) \cup (P_1 \cup \cdots \cup P_n) \) be an H-trivial link in \( \mathbb{R}^3 \), where \( O_i \) is a trivial knot and \( P_j \) is a Hopf link for \( i = 1, \ldots, m \), \( j = 1, \ldots, n \).

- Let \( H_\lambda[a, b] \) be a disjoint union of a disjoint system of trivial knot cones \( \tilde{O}_i * [a, b] \) (\( i = 1, \ldots, m \)) and a disjoint system of Hopf link cones \( \tilde{P}_j * [a, b] \) (\( j = 1, \ldots, n \)) in \( \mathbb{R}^3[a, b] \).

- Let \( H_\vee[a, b] \) be a disjoint union of a disjoint system of trivial knot cones \( \tilde{O}_i * [a, b] \) (\( i = 1, \ldots, m \)) and a disjoint system of Hopf link cones \( \tilde{P}_j * [a, b] \) (\( j = 1, \ldots, n \)) in \( \mathbb{R}^3[a, b] \).

By capping off \( \mathcal{F}(D) \) with \( L_+(D)_\lambda[1, 2] \) and \( L_-(D)_\vee[-2, -1] \), we obtain an oriented immersed surface-link \( \mathcal{S}(D) \) in \( \mathbb{R}^4 \). We call the oriented immersed surface-link \( \mathcal{S}(D) \) the oriented immersed surface-link associated with \( D \). It is straightforward from the construction of \( \mathcal{S}(D) \) that \( D \) is an oriented marked graph diagram of the oriented immersed surface-link \( \mathcal{S}(D) \).

**Definition 2.1** (cf. [5]). A positive (or negative) standard singular 2-knot, denoted by \( \mathcal{S}(+) \) (or \( \mathcal{S}(-) \)) is the immersed 2-knot of the marked graph diagram \( D \) (or \( D' \)) in Fig. 4, respectively. An unknotted immersed sphere is defined to be the connected sum \( m\mathcal{S}(+) \# n\mathcal{S}(-) \) for any non-negative integers \( m, n \) with \( m + n > 0 \).

A double point singularity \( p \) of an immersed 2-knot \( \mathcal{S} \) is inessential if \( \mathcal{S} \) is the connected sum of an immersed 2-knot and an unknotted immersed sphere such that \( p \) belongs to the unknotted immersed sphere. Otherwise, \( p \) is essential.
3. Confirming immersed 2-knots with essential singularity

In this section, the main theorem will be shown with an example of infinitely many immersed 2-knots with essential singularity. For an immersed 2-knot $K$, let $E(K) = \text{Cl}(S^4 \setminus N(K))$. Let $\tilde{E}(K)$ be the infinite cyclic covering of $E(K)$. Then the homology $H(K) = H_1(\tilde{E}(K))$ is a finitely generated $\Lambda$-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. This module is called the first Alexander module of $K$ (cf. [11]). Let

$$DH(K) = \{x \in H(K) | \exists \{\lambda_i\}_{1 \leq i \leq m} : \text{coprime \,(} m \geq 2) \text{ with } \lambda_i x = 0, \forall i\},$$

called the annihilator $\Lambda$-submodule, which is known to be equal to the integral torsion part of the Alexander module $H(K)$ (cf. [7, Section 3]). Let $\epsilon(K)$ be the first elementary ideal of $DH(K)$. A $\Lambda$-ideal is symmetric if the ideal is unchanged by replacing $t$ by $t^{-1}$. Let $DH(K)^* = \text{hom}(DH(K), \mathbb{Q}/\mathbb{Z})$ have the induced $\Lambda$-module structure, called the dual $\Lambda$-module of $DH(K)$. The following lemma is used in our argument.

**Lemma 3.1.** If $K$ is a 2-knot such that the dual $\Lambda$-module $DH(K)^*$ is $\Lambda$-isomorphic to $DH(K)$, then the first elementary ideal $\epsilon(K)$ is symmetric.

This lemma is direct from the $t$-isometric non-singular symmetric pairing

$$\ell : DH(K) \times DH(K) \to \mathbb{Q}/\mathbb{Z},$$

called the Farber-Levine pairing (see [2, 7, 12]), because this pairing induces a $t$-anti isomorphism $DH(K) \cong DH(K)^*$, so that the assumption on $DH(K)$ implies that there is a $t$-anti $\Lambda$-isomorphism from $DH(K)$ to itself. For example, if the module $DH(K)$ is given by $\Lambda/(2t - 1, m)$ for a non-zero
integer \( m \), then \( DH(K)^* \) is \( \Lambda \)-isomorphic to \( DH(K) \) and by Lemma 3.1, the ideal \( \epsilon(K) \) is symmetric. To see that \( DH(K)^* \) is \( \Lambda \)-isomorphic to \( DH(K) \), take a \( \Lambda \)-exact sequence

\[
0 \rightarrow \Lambda \xrightarrow{f_3} \Lambda^2 \xrightarrow{f_1} \Lambda \rightarrow DH(K) \rightarrow 0,
\]

where the \( \Lambda \)-homomorphisms \( f_i \) \((i = 1, 2)\) are given by

\[
f_1(e_1) = (2t - 1)e, \quad f_1(e_2) = me \quad \text{and} \quad f_2(e) = -me + (2t - 1)e_2
\]

for the standard bases \( e \in \Lambda \) and \( e_i \in \Lambda^2 \((i = 1, 2)\). Then \( DH(K)^* \) is \( \Lambda \)-isomorphic to \( Ext^2_\Lambda(DH(K), \Lambda) \) by Levine [12] (cf. [7, Section 3]) and \( Ext^2_\Lambda(DH(K), \Lambda) \) is \( \Lambda \)-isomorphic to the cokernel of the \( \Lambda \)-dual homomorphism \( f_2^\# : \Lambda^2 \rightarrow \Lambda \) of \( f_2 \). Thus, it is shown that \( DH(K)^* \) is \( \Lambda \)-isomorphic to \( \Lambda/(2t - 1, m) = DH(K) \).

For any marked graph diagram \( D \) of \( K \), the fundamental group \( \pi(K) \) of \( K \) is generated by the connected components of \( D \), namely, the connected components obtained from \( D \) by cutting the under-crossing points and the relations \( s_3 = s_2^{-1}s_1s_2 \) for all crossings as in \((a)\) or \((b)\) in Fig. 5.

\[
\begin{array}{c}
s_1 \\
\downarrow \\
s_2 \\
\downarrow \hspace{0.5cm} c \\
\downarrow \\
s_3 \\
\downarrow \\
(a)
\end{array} \quad \begin{array}{c}
s_1 \\
\downarrow \\
s_2 \\
\downarrow \hspace{0.5cm} c \\
\downarrow \\
s_3 \\
\downarrow \\
(b)
\end{array}
\]

Figure 5: Labels at a crossing

A computation of the Alexander module \( H(K) \) and the ideal \( \epsilon(K) \) is shown in a concrete example as follows:

**Example 3.2.** Let \( T \) be the ribbon torus-knot of \( D \) in Fig. 6. The fundamental group \( \pi(T) \) is isomorphic to the group \( \langle x_1, x_2 | r_1, r_2 \rangle \), where

\[
r_1 : x_2^{-1}x_1x_2 = x_1^{-1}x_2x_1, \quad r_2 : (x_2x_1^{-1})^3x_2(x_2x_1^{-1})^{-3} = x_1.
\]

Then the following \( \Lambda \)-semi-exact sequence

\[
\Lambda[r_1^*, r_2^*] \xrightarrow{d_2} \Lambda[x_1^*, x_2^*] \xrightarrow{d_1} \Lambda \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0
\]
of the group presentation of $\pi(T)$ is obtained by using the fundamental formula of the Fox differential calculus in [1], where $\Lambda[r^*_1, r^*_2]$ and $\Lambda[x^*_1, x^*_2]$ are free $\Lambda$-modules with bases $r^*_i$ ($i = 1, 2$) and $x^*_j$ ($j = 1, 2$), respectively, and the $\Lambda$-homomorphisms $\varepsilon, d_1$ and $d_2$ are given as follows:

$$\varepsilon(t) = 1, \quad d_1(x^*_j) = t - 1 (j = 1, 2), \quad d_2(r^*_i) = \sum_{j=1}^{u} \frac{\partial r^*_i}{\partial x^*_j} x^*_j (i = 1, 2)$$

for the Fox differential calculus $\frac{\partial r^*_i}{\partial x^*_j}$ regarded as an element of $\Lambda$ by letting $x^*_j$ to $t$. The Alexander module $H(T)$ is identified with the quotient $\Lambda$-module $\text{Ker}(d_1)/\text{Im}(d_2)$ (see [8, Theorem 7.1.5]). The Alexander matrix $M_T = (m_{ij})$ defined by $m_{ij} = \frac{\partial r^*_i}{\partial x^*_j}$ is a presentation matrix of the $\Lambda$-homomorphism $d_2$ and calculated as follows:

$$M_T = \begin{bmatrix}
-2t^{-1} + t^{-2} & 2t^{-1} - t^{-2} \\
3 - 4t^{-1} & -3 + 4t^{-1}
\end{bmatrix}.$$  

Hence we have

$$H(T) \cong \Lambda/(2t - 1, 3t - 4),$$

which is equal to $DH(T)$. Thus, the first elementary ideal $\epsilon(T)$ of $T$ is

$$\epsilon(T) = <2t - 1, 3t - 4> = <2t - 1, 3t - 4, 3(2t - 1) - 2(3t - 4)> = <2t - 1, 5>.$$  

The surface-link $T$ represented by the marked graph diagram $D$ is ambient isotopic to the surface-link $T'$ represented by the motion picture in Fig. 7. Let $s'$ be the circle $l_1 \cup l_2 \cup \{(a, b, c, t)|1 < t < 2\} \cup \{(d, e, f, t)|1 < t < 2\}$ in $T'$. The circle $s'$ bounds a disk $d'$ in $\mathbb{R}^4$ such that the interior $\text{int}d'$ of $d'$ meets $T'$ with 10 crossings and $\text{Int}(\text{int}d', T') = 0$, where $\text{Int}$ denotes the intersection number. Since $T$ and $T'$ are ambient isotopic, there is a disk $d$ such that $\partial d \subset T$ and $\text{int}d$ meets $T$ with 10 crossings and $\text{Int}(\text{int}d, T) = 0$. Let $d \times I$ be a thickening of $d$. Let $K$ be the immersed 2-knot obtained from $T$ by replacing the annulus $T \cap (d \times I)$ by $d \times \partial I$. Then $K$ is the immersed 2-knot with 20 double point singularities. Since the first elementary ideal $\epsilon(K)$ of $K$ is the same as that of $T$, $\epsilon(K) = <2t - 1, 5>.$

The following lemma is useful in a computation for a symmetric ideal.
Lemma 3.3. The following statements are equivalent:

1. The ideal $< 2t - 1, m >$ is symmetric.
2. An integer $m$ is $\pm 2^r$ or $\pm 2^r 3$ for any integer $r \geq 0$.

Proof. First, it is easy to show that $< 2t - 1, 0 > = < 2t - 1 >$ is not symmetric. The ideal $< 2t - 1, \pm 3 > = < -t - 1, \pm 3 >$ is symmetric. It is observed that

$$< 2t - 1, ab > = < t - 2, ab > \Rightarrow < 2t - 1, a > = < t - 2, a > \quad (3.1)$$

for all non-zero integers $a, b$. Thus, $< 2t - 1, \pm 1 >$ is symmetric. Let $m$ be
even, that is, \( m = 2n \) for some integer \( n \). Then

\[
< 2t - 1, m > = < 2t - 1, 2n >
= < 2t - 1, 2n, n(2t - 1) - 2nt >
= < 2t - 1, n >.
\]

By mathematical induction, if \( m = 2^rn \) for \( r \geq 0 \) and some odd integer \( n \), then

\[
< 2t - 1, m > = < 2t - 1, n >.
\]

Let \( p \) be a prime with \( |p| \geq 5 \). Since \( \mathbb{Z}_p[t, t^{-1}] \) is a principal ideal domain, \( < 2t - 1, p > \neq < t - 2, p > \). By the contraposition of (3.1), for any non-zero
integer $m$ divided by a prime $p \geq 5$, $< 2t - 1, m > \neq < t - 2, m >$. Suppose that $< 2t - 1, 9 >$ is symmetric, i.e., $< 2t - 1, 9 > =< t - 2, 9 >$. Then

$$< t - 2, 9 > =< t - 2, 9, 2t - 1 >$$

$$= < t - 2, 9, 2t - 1 - 2(t - 2) >$$

$$= < t - 2, 3 > =< t - 5, 3 >,$$

$$< 2t - 1, 9 > =< t - 5, 9 >. (\because 2^{-1} \equiv 5 \pmod{9}).$$

Thus $< t - 5, 3 > =< t - 5, 9 >$. Then there are $a(t)$, $b(t) \in \mathbb{Z}[t, t^{-1}]$ such that $3 = a(t)(t - 5) + b(t)$9. For $b(t)$, there are $b'(t) \in \mathbb{Z}[t, t^{-1}]$ and $c \in \mathbb{Z}$ such that $b(t) = b'(t)(t - 5) + c$. Thus

$$3 = a(t)(t - 5) + (b'(t)(t - 5) + c)9.$$

Then $(a(t) + 9b'(t))(t - 5) = 3 - 9c \in \mathbb{Z} \setminus \{0\}$. This is a contradiction. Hence $< 2t - 1, 9 >$ is not symmetric. \hfill \Box

**Lemma 3.4.** There are infinitely many immersed 2-knots with at least one essential double point singularity whose ideals are mutually distinct.

**Proof.** Let $T_n$ be the ribbon torus-knot of $D_n$ in Fig. 8 $(n \geq 1)$. Let $K_n$ be the immersed 2-knot obtained from $T_n$ analogously to the method in Example 3.2. By the same calculation as in Example 3.2, we have $DH(K_n) = H(K_n) \cong \Lambda/(2t - 1, n)$. Suppose that the immersed 2-knot $K^*$ is equivalent to the connected sum of a 2-knot $K$ and an unknotted immersed sphere $S_0$. By Lemma 3.1, the first elementary ideal $\epsilon(K)$ is symmetric for any 2-knot $K$. Then the identity $\epsilon(K^*) = \epsilon(K)$ is obtained since $\epsilon(S(+)) = \epsilon(S(-)) =< 1 >$, so that the ideal $\epsilon(K^*)$ is symmetric. On the other hand, by Lemma 3.3, $< 2t - 1, m >$ is not symmetric except that $m$ is 0, $\pm 2r$ or $\pm 2r3$ $(r \geq 0)$. Therefore, the immersed 2-knot $K_n$ obtained from $D_n$ is an immersed 2-knot with at least one essential singularity except that $n$ is $2r + 2$ or $2r3$ $(r \geq 0)$. Infiniteness of the immersed 2-knots under consideration is seen from infiniteness of the ideals $< 2t - 1, m >$ for all $m$. \hfill \Box

Let $J$ be one of the immersed 2-knots $K_n(n = 1, 2, 3, \ldots)$ such that the first elementary ideal $\epsilon(J)$ is asymmetric. Then the following corollary is obtained.

**Corollary 3.5.** The connected sum $J \# U$ of $J$ and any immersed 2-knot $U$ such that the group orders $|DH(J)|$ and $|DH(U)|$ are coprime is an immersed 2-knot with at least one essential double point singularity.
Proof. Suppose that the immersed 2-knot $J#U$ is a connected sum of a 2-knot $K$ and an unknotted immersed sphere $S_0$. Since $DH(K) = DH(J#U) = DH(J) \otimes DH(U)$ and $|DH(J)|$ and $|DH(U)|$ are coprime, the Farber-Levine pairing $\ell : DH(K) \times DH(K) \to \mathbb{Q}/\mathbb{Z}$ induces the nonsingular $t$-isometric symmetric pairing on the direct summand $DH(J) = \Lambda/(2t - 1, m)$ for some $m$, so that as in the proof of Lemma 3.4, the ideal $\epsilon(J) = < 2t - 1, m >$ must be symmetric, which is a contradiction. \hfill \Box

Finally, the ideal $(2t - 1, 5)$ is known to be the first elementary ideal of a ribbon torus-knot in [4].

By using an immersed 2-knot in Lemma 3.4, the following main theorem is proved.

**Theorem 3.6.** Let $K = nK_m$ be the connected sum of $n$ copies of an immersed 2-knot $K_m$ with at least one essential double point singularity whose first elementary ideal is $< 2t - 1, m >$ for any integer $m \geq 5$ without
factors 2 and 3. Then $K$ gives infinitely many immersed 2-knots with at least $n$ double point singularities every of which is essential.

**Proof.** Assume that there is an immersed 2-knot $K'$ with only $d(< n)$ essential double point singularities such that $K = K' \# S_0$, where $S_0$ is an unknotted singular 2-knot. We know that $DH_1(S_0) = 0$. Thus

$$DH(K') \cong DH(K') \oplus DH(S_0) \cong DH(K) \cong \oplus \left( \Lambda/(2t - 1, m) \right).$$

Therefore

$$e(DH(K)) = e(DH(K')) = n,$$  \hspace{1cm} (3.2)

where $e(H)$ is the minimum number of $\Lambda$-generators of a finitely generated $\Lambda$-module $H$.

Now, for simplicity, we denote $E(K')$ or $\tilde{E}(K')$ by $E$ or $\tilde{E}$, respectively. By Wang exact sequence, there is an exact sequence

$$\cdots \rightarrow H_d(\tilde{E}) \xrightarrow{t^{-1}} H_d(\tilde{E}) \xrightarrow{\nu} H_d(E) \xrightarrow{\delta_d} H_{d-1}(\tilde{E}) \rightarrow \cdots.$$

We have $H_1(E) = H_0(\tilde{E}) = H_0(E) = \mathbb{Z}$, so that we obtain

$$\cdots \rightarrow H_1(\tilde{E}) \xrightarrow{t^{-1}} H_1(\tilde{E}) \xrightarrow{0} \mathbb{Z} \xrightarrow{\delta_1} \mathbb{Z} \rightarrow 0.$$

Then $t - 1 : H_1(\tilde{E}) \rightarrow H_1(\tilde{E})$ is onto. Since $H_1(\tilde{E})$ is a finitely generated $\Lambda$-module, the map $t - 1$ is an isomorphism by Noetherian property.

Suppose that $H_1(\tilde{E}; \mathbb{Q}) = H_1(\tilde{E}) \otimes \mathbb{Q} \cong \Lambda_\mathbb{Q} \oplus M$, where $\Lambda_\mathbb{Q} = \mathbb{Q}[t, t^{-1}]$ which is a principal ideal domain, $k$ is a non-negative integer, and $M$ is the $\Lambda_\mathbb{Q}$-torsion part. We have an isomorphism $t - 1 : H_1(\tilde{E}; \mathbb{Q}) \rightarrow H_1(\tilde{E}; \mathbb{Q})$. By the $\Lambda_\mathbb{Q}$-exact sequence

$$0 \rightarrow \Lambda_\mathbb{Q} \xrightarrow{t^{-1}} \Lambda_\mathbb{Q} \rightarrow \Lambda_\mathbb{Q}/(t - 1) \cong \mathbb{Q} \rightarrow 0,$$

the map $t - 1 : \Lambda_\mathbb{Q} \rightarrow \Lambda_\mathbb{Q}$ cannot be an epimorphism. Therefore, $k = 0$ and $H_1(\tilde{E}; \mathbb{Q})$ is a $\Lambda_\mathbb{Q}$-torsion module, which is a $\Lambda$-torsion module. The homology $H_2(\tilde{E}; \mathbb{Q}/\mathbb{Z})$ is a $\mathbb{Z}$-torsion $\Lambda$-module. From the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$, we obtain a long $\Lambda$-exact sequence

$$\cdots \rightarrow H_2(\tilde{E}; \mathbb{Q}/\mathbb{Z}) \xrightarrow{h_2} H_1(\tilde{E}; \mathbb{Z}) \xrightarrow{f_1} H_1(\tilde{E}; \mathbb{Q}) \xrightarrow{g_1} H_1(\tilde{E}; \mathbb{Q}/\mathbb{Z}) \rightarrow \cdots.$$
Let \( x \in H_1(\tilde{E}) = H_1(\tilde{E}; \mathbb{Z}) \). Since \( H_1(\tilde{E}; \mathbb{Q}) \) is a \( \Lambda \)-torsion module, there is a non-zero element \( \lambda \in \Lambda \) such that \( \lambda f_1(x) = 0 \). Then \( \lambda x \in \text{Ker}(f_1) = \text{Im}(h_2) \).

There is an element \( y \in H_2(\tilde{E}; \mathbb{Q}/\mathbb{Z}) \) such that \( \lambda x = h_2(y) \). Since \( H_2(\tilde{E}; \mathbb{Q}/\mathbb{Z}) \) is a \( \mathbb{Z} \)-torsion \( \Lambda \)-module, there is a non-zero \( k \in \mathbb{Z} \) such that \( ky = 0 \). Then \( 0 \neq k\lambda \in \Lambda \) and \( k\lambda x = kh_2(y) = 0 \). Therefore \( H_1(\tilde{E}) \) is a \( \Lambda \)-torsion module.

For a finitely generated \( \Lambda \)-module \( H \), we define \( TH = \{ x \in H | \lambda x = 0 \text{ for a non-zero } \lambda \in \Lambda \} \), \( BH = H/TH \), and \( TDH = TH/DH \). Since \( H_1(\tilde{E}) = TH_1(\tilde{E}) \), the \( \Lambda \)-torsion-free part \( BH_1(\tilde{E}) = 0 \). By the second duality theorem in \([7]\), there are \( t \)-anti \( \Lambda \)-epimorphisms

\[
\theta : DH_2(\tilde{E}) \to E^1BH_1(\tilde{E}, \partial \tilde{E}) = E^1BH_1(\tilde{E}) = 0 \quad \text{and} \\
\theta' : DH_0(\tilde{E}, \partial \tilde{E}) = DH_0(\tilde{E}) = 0 \to E^1BH_3(\tilde{E}),
\]

where \( E^kH = \text{Ext}^k_\Lambda(H, \Lambda) \) for any non-negative integer \( k \) and any \( \Lambda \)-module \( H \), and there is a \( t \)-isometric non-singular \( \Lambda \)-pairing

\[
\ell : \text{Ker}(\theta) \times \text{Ker}(\theta') = DH_2(\tilde{E}) \times 0 \to \mathbb{Q}/\mathbb{Z}.
\]

Thus, \( DH_2(\tilde{E}) = 0 \). By the first duality theorem in \([7]\), there is a \( t \)-Hermitian non-singular pairing

\[
L : TDH_2(\tilde{E}) \times TDH_1(\tilde{E}, \partial \tilde{E}) \to \mathbb{Q}(\Lambda)/\Lambda.
\]

Since there is a \( \Lambda \)-epimorphism from \( H_1(\tilde{E}) = H_1(\tilde{E}, \partial \tilde{E}) \) to \( TDH_1(\tilde{E}, \partial \tilde{E}) \) and \( t-1 : H_1(\tilde{E}) \to H_1(\tilde{E}) \) is a \( \Lambda \)-isomorphism, the map \( t-1 : TDH_1(\tilde{E}, \partial \tilde{E}) \to TDH_1(\tilde{E}, \partial \tilde{E}) \) is a \( \Lambda \)-epimorphism. The fact that \( TDH_1(\tilde{E}, \partial \tilde{E}) \) is a finitely generated \( \Lambda \)-module implies that \( t-1 : TDH_1(\tilde{E}, \partial \tilde{E}) \to TDH_1(\tilde{E}, \partial \tilde{E}) \) is a \( \Lambda \)-isomorphism. Thus we have a \( \Lambda \)-isomorphism \( t-1 : T_2H_2(\tilde{E}) \to T_2H_2(\tilde{E}) \).

Since \( DH_2(\tilde{E}) = 0 \), the map \( t-1 : T_2H_2(\tilde{E}) \to T_2H_2(\tilde{E}) \) is a \( \Lambda \)-isomorphism. For \( x \in T_2H(\tilde{E}) \), there is an element \( x' \in T_2H(\tilde{E}) \) such that \( x = (t-1)x' \). Then \( p_*(x) = (1-1)p_*(x') = 0 \). The module \( T_2H(\tilde{E}) \) is a submodule of the kernel of \( p_* : H_2(\tilde{E}) \to H_2(E) \). So, we obtain the short \( \Lambda \)-exact sequence

\[
0 \to BH_2(\tilde{E}) \xrightarrow{t-1} BH_2(\tilde{E}) \xrightarrow{p_*} H_2(E) \cong \mathbb{Z}^d \to 0.
\]

We obtain the long exact sequence

\[
E^0(\mathbb{Z}^d) \to E^0BH_2(\tilde{E}) \to E^0BH_2(\tilde{E}) \to E^1(\mathbb{Z}^d) \to E^1BH_2(\tilde{E}) \to \cdots.
\]
Since $E^0H$ is a $\Lambda$-free module for a finitely generated $\Lambda$-module $H$, we have $E^0BH_2(\tilde{E}) \cong \Lambda^k$ for some non-negative integer $k$. So, the long exact sequence is as follows:

$$0 \to \Lambda^k \xrightarrow{t^{-1}} \Lambda^k \to (\Lambda/(t-1))^d \to G \to \cdots,$$

where $G = E^1BH_2(\tilde{E})$ is a finite $\Lambda$-module. Then we have

$$0 \to (\Lambda/(t-1))^k \to (\Lambda/(t-1))^d \to G \to \cdots.$$

Thus, $E^0BH_2(\tilde{E}) \cong \Lambda^d$.

By the second duality theorem in [7], there are a $t$-anti $\Lambda$-epimorphism $\theta : DH_1(\tilde{E}, \partial \tilde{E}) = DH_1(\tilde{E}) \to E^1BH_2(\tilde{E})$ and a $t$-isometric symmetric non-singular pairing $\phi : D \times D \to \mathbb{Q}/\mathbb{Z}$, where $D = \text{Ker}(\theta)$. For every prime $p$ and every positive integer $i$, let $\tilde{D}_p^i = \{x \in D | p^ix = 0\}$ and $\bar{D}_p^i = \tilde{D}_p^i/(\tilde{D}_p^{i-1} + p\tilde{D}_p^{i+1})$. The $t$-isometric symmetric non-singular pairing $\phi$ induces a $t$-isometric symmetric non-singular pairing $\tilde{\phi}_p^i : \tilde{D}_p^i \times \bar{D}_p^i \to \mathbb{Q}/\mathbb{Z}$ for all prime $p$ and all $i$ (see [10]). Suppose $D \neq 0$. Then there are $p \geq 5$ and $i$ with $\bar{D}_p^i \neq 0$, so that $\bar{D}_p^i \cong (\Lambda/(p, 2t-1))^r_i$ for some $r_i > 0$. The $t$-isometric symmetric non-singular pairing $\tilde{\phi}_p^i$ induces a $t$-anti automorphism of $\tilde{D}_p^i$, so that all the elementary ideals of $\tilde{D}_p^i$ are symmetric. This means that the ideal $(p, 2t-1)$ is symmetric, for it is the $(r_i - 1)$th elementary ideal of $\tilde{D}_p^i$.

This contradicts Lemma 3.3. Thus, $D = \text{Ker}(\theta) = 0$. Therefore $DH_1(\tilde{E})$ and $E^1BH_2(\tilde{E})$ are $t$-anti isomorphic. Then $DH_1(\tilde{E}) \cong E^2DH_1(\tilde{E})$ and $E^2E^1BH_2(\tilde{E})$ are $t$-anti isomorphic.

By Lemma 3.6 of [7], there is an exact sequence

$$0 \to BH_2(\tilde{E}) \to E^0E^0BH_2(\tilde{E}) \cong \Lambda^d \to E^2E^1BH_2(\tilde{E}) \to 0.$$

This means that $DH_1(\tilde{E}) \cong E^2E^1BH_2(\tilde{E})$ is generated by $d$ elements over $\Lambda$. Combining with (3.2), we obtain $n = e(DH_1(\tilde{E})) \leq d$, which is a contradiction. Thus, there is no immersed 2-knot $K'$ such that $K = K'\#S_0$. Infiniteness of the immersed 2-knots under consideration is seen from infiniteness of the ideals $<2t-1, m>$ for all $m$. This completes the proof of Theorem 3.6.

□

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