Immersed 2-knots with essential singularity

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Abstract

It is shown that there are infinitely many immersed 2-knots with more than any previously given number of double point singularities which are not equivalent to the connected sum of any immersed 2-knot and any unknotted immersed sphere.

Keywords:
Immersed 2-knot, Immersed surface-link, Essential singurality, Unknotted immersed sphere, Marked graph diagram, Symmetric ideal.
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1. Introduction

An *immersed surface-link* is a generically immersed closed oriented surface in the 4-space $\mathbb{R}^4$. When the surface has only one component, it is also called an *immersed surface-knot*. When the surface consists of 2-spheres, it is also called an *immersed sphere-link* or simply an *immersed 2-link*. When the immersion is an embedding, it is also called a *surface-link*. Two (immersed) surface-links $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if there is an orientation-preserving auto-homeomorphism h of $\mathbb{R}^4$ sending $\mathcal{L}$ to $\mathcal{L}'$ orientation-preservingly. An immersed 2-link is studied in [9] in relation to a cross-sectional link. A normal

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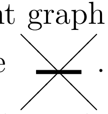
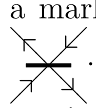
form of an immersed surface-link introduced by S. Kamada and K. Kawamura in [5] is used to define a marked graph diagram of an immersed surface-link in [6]. In this paper, with an example obtained from a surface-knot described by a marked graph diagram, it is shown as the main theorem (Theorem 3.6) that *for any positive integer n, there are infinitely many immersed 2-knots with at least n double point singularities every of which is essential double point singularities, that is, infinitely many immersed 2-knots with at least n double point singularities which are not equivalent to the connected sum of any immersed 2-knot and any unknotted immersed sphere.*

This paper is organized as follows: Section 2 is devoted to a review of a marked graph diagram of an immersed surface-link. In particular, an unknotted immersed sphere is defined there. In Section 3, the main theorem is proved.

2. Marked graph representation of immersed surface-links

In this section, we review (oriented) marked graph diagrams representing immersed surface-links described in [6]. A *marked graph* is a 4-valent graph in $\mathbb{R}^3$ each of whose vertices is a vertex with a marker looks like .

Two marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods of markers. As usual, a marked graph in $\mathbb{R}^3$ can be described by a link diagram on $\mathbb{R}^2$ with some 4-valent vertices equipped with markers, called a *marked graph diagram*. An *orientation* of a marked graph G in $\mathbb{R}^3$ is a choice of an orientation for each edge of G. An orientation of a marked graph G is said to be *consistent* if every vertex in G looks like . A marked graph G in $\mathbb{R}^3$ is said to be *orientable* if G admits a consistent orientation. Otherwise, it is said to be *non-orientable*. By an *oriented marked graph* we mean an orientable marked graph in $\mathbb{R}^3$ with a fixed consistent orientation. Two oriented marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhood, marker and consistent orientation. For $t \in \mathbb{R}$, we denote by $\mathbb{R}^3_t$ the hyperplane of $\mathbb{R}^4$ whose fourth coordinate is equal to $t \in \mathbb{R}$, i.e., $\mathbb{R}^3_t = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$. An immersed surface-link $\mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be described in terms of its *cross-sections* $\mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3_t,\ t \in \mathbb{R}$ (cf. [3]). It is shown [5] that any immersed surface-link $\mathcal{L}$, there is an immersed surface-link $\mathcal{L}' \subset \mathbb{R}^3[-2, 2]$ satisfying the following conditions:

(1) The intersections $\mathcal{L}'_1$ and $\mathcal{L}'_{-1}$ are H-trivial links;

(2) All saddle points of $\mathcal{L}'$ are in $\mathbb{R}^3[0]$;

(3) All maximal points of $\mathcal{L}'$ are in $\mathbb{R}^3[2]$;

(4) All minimal points of $\mathcal{L}'$ are in $\mathbb{R}^3[-2]$;

(5) The intersections $\mathcal{L}' \cap (\mathbb{R}^3[1,2])$ and $\mathcal{L}' \cap (\mathbb{R}^3[-2,-1])$ are disjoint unions of a disjoint system of trivial knot cones and a disjoint system of Hopf link cones.

We call $\mathcal{L}'$ a *normal form* of $\mathcal{L}$. Let $\mathcal{L}$ be an immersed surface-link in $\mathbb{R}^4$, and $\mathcal{L}'$ a normal form of $\mathcal{L}$. Then $\mathcal{L}'_0$ is a spatial 4-valent regular graph in $\mathbb{R}^3_0$. We give a marker at each 4-valent vertex (saddle point) that indicates how the saddle point opens up above as illustrated in Fig. 1. We choose an orientation for each edge of $\mathcal{L}'_0$ that coincides with the induced orientation on the boundary of $\mathcal{L}' \cap \mathbb{R}^3 \times (-\infty, 0]$ from the orientation of $\mathcal{L}'$. The resulting oriented marked graph G is called an *oriented marked graph* of $\mathcal{L}$. As usual, G is described by a link diagram D with rigid marked vertices. Such a diagram D is called an *oriented marked graph diagram* or an *oriented ch-diagram* (cf. [13]) of $\mathcal{L}$.

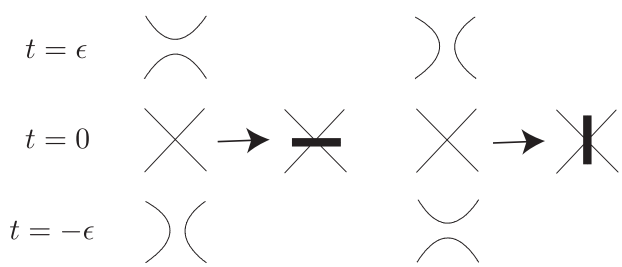
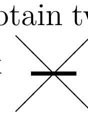
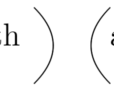
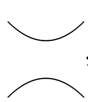


Figure 1: Marking of a vertex

Let D be an oriented marked graph diagram. We obtain two links $L_-(D)$ and $L_+(D)$ from D by replacing each marked vertex  with  and , respectively. Links $L_-(D)$ and $L_+(D)$ are also called the *negative*

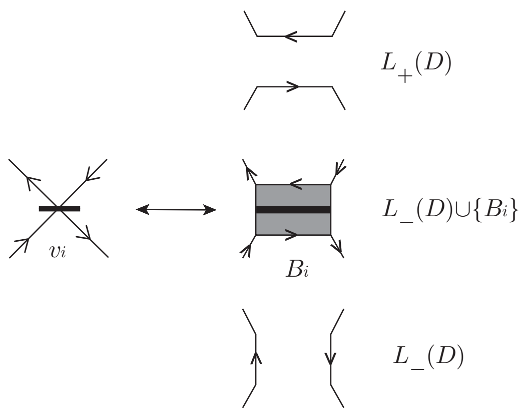


Figure 2: Marked vertex resolutions

resolution and the *positive resolution* of D, respectively. By replacing a neighborhood of each marked vertex v_i $(1 \leq i \leq n)$ with an oriented band B_i as illustrated in Fig. 2. Denote the disjoint union $B_1 \sqcup \cdots \sqcup B_n$ of bands by $\mathcal{B}(D)$. A link L is H-*trivial* if L is a split union of trivial knots and Hopf links. A marked graph diagram D is said to be H-*admissible* if both resolutions $L_-(D)$ and $L_+(D)$ are H-trivial classical link diagrams.

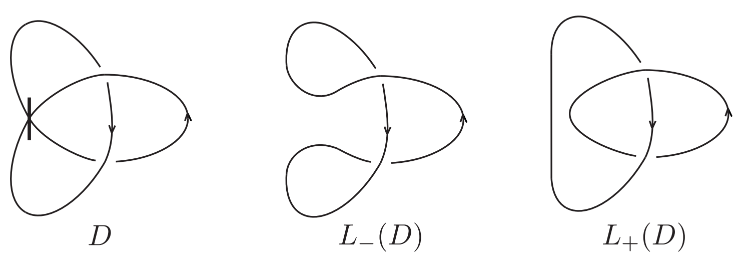


Figure 3: An H-admissible marked graph diagram

From now on, we recall how to construct an immersed surface-link $\mathcal{L}$ in $\mathbb{R}^4$ from a given H-admissible oriented marked graph diagram (cf. [5, 6]). Let D be an H-admissible oriented marked graph diagram. We define a surface-link $\mathcal{F}(D) \subset \mathbb{R}^3 \times [-1, 1]$, called the *proper surface associated with D*, by

$$(\mathbb{R}^3_t, \mathcal{F}(D) \cap \mathbb{R}^3_t) = \begin{cases} (\mathbb{R}^3, L_+(D)) & \text{for } 0 < t \leq 1, \\ (\mathbb{R}^3, L_-(D) \cup \mathcal{B}(D)) & \text{for } t = 0, \\ (\mathbb{R}^3, L_-(D)) & \text{for } -1 \leq t < 0. \end{cases}$$

It is known that a marked graph diagram D is orientable if and only if the proper surface $\mathcal{F}(D)$ associated with D is an orientable surface. Since D has a consistent orientation, the resolutions $L_+(D)$ and $L_-(D)$ have the orientations induced from the orientation of D. We choose an orientation for the proper surface $\mathcal{F}(D)$ so that the induced orientation of the cross-section $L_+(D) = \mathcal{F}(D)_1 = \mathcal{F}(D) \cap \mathbb{R}^3_1$ at $t = 1$ matches the orientation of $L_+(D)$. Let $[a, b]$ be a closed interval with $a < b$. For a link L, let $\hat{L} * [a, b]$ (or $\check{L} * [a, b]$) be a cone with $L[a]$ (or $L[b]$) as the base and a point in $\mathbb{R}^3[b]$ (or $\mathbb{R}^3[a]$), respectively. Let $H = (O_1 \cup \cdots \cup O_m) \cup (P_1 \cup \cdots \cup P_n)$ be an H-trivial link in $\mathbb{R}^3$, where O_i is a trivial knot and P_j is a Hopf link for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

- Let $H_\wedge[a, b]$ be a disjoint union of a disjoint system of trivial knot cones $\hat{O}_i * [a, b] (i = 1, \ldots, m)$ and a disjoint system of Hopf link cones $\hat{P}_j * [a, b] (j = 1, \ldots, n)$ in $\mathbb{R}^3[a, b]$.
- Let $H_\vee[a, b]$ be a disjoint union of a disjoint system of trivial knot cones $\check{O}_i * [a, b] (i = 1, \ldots, m)$ and a disjoint system of Hopf link cones $\check{P}_j * [a, b] (j = 1, \ldots, n)$ in $\mathbb{R}^3[a, b]$.

By capping off $\mathcal{F}(D)$ with $L_+(D)_\wedge[1, 2]$ and $L_-(D)_\vee[-2, -1]$, we obtain an oriented immersed surface-link $\mathcal{S}(D)$ in $\mathbb{R}^4$. We call the oriented immersed surface-link $\mathcal{S}(D)$ the *oriented immersed surface-link associated with* D. It is straightforward from the construction of $\mathcal{S}(D)$ that D is an oriented marked graph diagram of the oriented immersed surface-link $\mathcal{S}(D)$.

Definition 2.1 (cf. [5])**.** A *positive* (or *negative*) *standard singular 2-knot*, denoted by $S(+)$ (or $S(-)$) is the immersed 2-knot of the marked graph diagram D (or D') in Fig. 4, respectively. An *unknotted immersed sphere* is defined to be the connected sum $mS(+)\#nS(-)$ for any non-negative integers m, n with $m + n > 0$.

A double point singularity p of an immersed 2-knot S is *inessential* if S is the connected sum of an immersed 2-knot and an unknotted immersed sphere such that p belongs to the unknotted immersed sphere. Otherwise, p is *essential*.

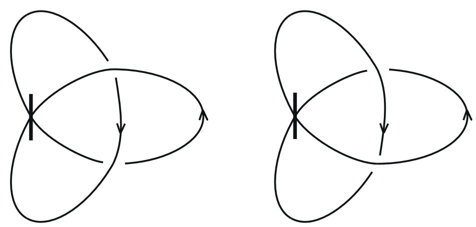


Figure 4: Standard singular 2-knot

3. Confirming immersed 2-knots with essential singularity

In this section, the main theorem will be shown with an example of infinitely many immersed 2-knots with essential singularity. For an immersed 2-knot K, let $E(K) = \mathrm{Cl}(S^4 \setminus \mathrm{N}(K))$. Let $\tilde{E}(K)$ be the infinite cyclic covering of $E(K)$. Then the homology $H(K) = H_1(\tilde{E}(K))$ is a finitely generated Λ-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. This module is called the *first Alexander module* of K (cf. [11]). Let

$$DH(K) = \{x \in H(K) |\ \exists \{\lambda_i\}_{1 \le i \le m} : \text{coprime } (m \ge 2) \text{ with } \lambda_i x = 0,\ \forall i\},$$

called the *annihilator* Λ*-submodule*, which is known to be equal to the integral torsion part of the Alexander module $H(K)$ (cf. [7, Section 3]). Let $\epsilon(K)$ be the first elementary ideal of $DH(K)$. A Λ-ideal is *symmetric* if the ideal is unchanged by replacing t by t^{-1}. Let $DH(K)^* = \hom(DH(K), \mathbb{Q}/\mathbb{Z})$ have the induced Λ-module structure, called the *dual* Λ-module of $DH(K)$. The following lemma is used in our argument.

Lemma 3.1. If K is a 2-knot such that the dual Λ-module $DH(K)^*$ is Λ-isomorphic to $DH(K)$, then the first elementary ideal $\epsilon(K)$ is symmetric.

This lemma is direct from the t-isometric non-singular symmetric pairing

$$\ell : DH(K) \times DH(K) \to \mathbb{Q}/\mathbb{Z},$$

called the Farber-Levine pairing (see [2, 7, 12]), because this pairing induces a t-anti isomorphism $DH(K) \cong DH(K)^*$, so that the assumption on $DH(K)$ implies that there is a t-anti Λ-isomorphism from $DH(K)$ to itself. For example, if the module $DH(K)$ is given by $\Lambda/(2t - 1, m)$ for a non-zero integer m, then $DH(K)^*$ is Λ-isomorphic to $DH(K)$ and by Lemma 3.1, the ideal $\epsilon(K)$ is symmetric. To see that $DH(K)^*$ is Λ-isomorphic to $DH(K)$, take a Λ-exact sequence

$$0 \to \Lambda \xrightarrow{f_2} \Lambda^2 \xrightarrow{f_1} \Lambda \to DH(K) \to 0,$$

where the Λ-homomorphisms $f_i\,(i = 1, 2)$ are given by

$$f_1(e_1) = (2t-1)e,\ f_1(e_2) = me \quad \text{and} \quad f_2(e) = -me_1 + (2t-1)e_2$$

for the standard bases $e \in \Lambda$ and $e_i \in \Lambda^2\,(i = 1, 2)$. Then $DH(K)^*$ is Λ-isomorphic to $Ext^2_\Lambda(DH(K), \Lambda)$ by Levine [12] (cf. [7, Section 3]) and $Ext^2_\Lambda(DH(K), \Lambda)$ is Λ-isomorphic to the cokernel of the Λ-dual homomorphism $f_2^\# : \Lambda^2 \to \Lambda$ of f_2. Thus, it is shown that $DH(K)^*$ is Λ-isomorphic to $\Lambda/(2t-1, m) = DH(K)$.

For any marked graph diagram D of K, the fundamental group $\pi(K)$ of K is generated by the connected components of D, namely, the connected components obtained from D by cutting the under-crossing points and the relations $s_3 = s_2^{-1}s_1s_2$ for all crossings as in (a) or (b) in Fig. 5.

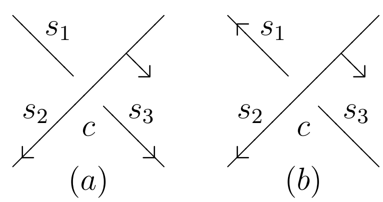


Figure 5: Labels at a crossing

A computation of the Alexander module $H(K)$ and the ideal $\epsilon(K)$ is shown in a concrete example as follows:

Example 3.2. Let T be the ribbon torus-knot of D in Fig. 6. The fundamental group $\pi(T)$ is isomorphic to the group $< x_1, x_2 |\, r_1, r_2 >$, where

$$r_1 : x_2^{-1}x_1x_2 = x_1^{-1}x_2x_1, \quad r_2 : (x_2x_1^{-1})^3x_2(x_2x_1^{-1})^{-3} = x_1.$$

Then the following Λ-semi-exact sequence

$$\Lambda[r_1^*, r_2^*] \xrightarrow{d_2} \Lambda[x_1^*, x_2^*] \xrightarrow{d_1} \Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of the group presentation of $\pi(T)$ is obtained by using the fundamental formula of the Fox differential calculus in [1], where $\Lambda[r_1^*, r_2^*]$ and $\Lambda[x_1^*, x_2^*]$ are free Λ-modules with bases r_i^* $(i = 1, 2)$ and x_j^* $(j = 1, 2)$, respectively, and the Λ-homomorphisms ε, d_1 and d_2 are given as follows:

$$\varepsilon(t) = 1,\ d_1(x_j^*) = t - 1\,(j = 1, 2),\ d_2(r_i^*) = \sum_{j=1}^{u} \frac{\partial r_i}{\partial x_j} x_j^*\,(i = 1, 2)$$

for the Fox differential calculus $\frac{\partial r_i}{\partial x_j}$ regarded as an element of Λ by letting x_j to t. The Alexander module $H(T)$ is identified with the quotient Λ-module $\mathrm{Ker}(d_1)/\mathrm{Im}(d_2)$ (see [8, Theorem 7.1.5]). The Alexander matrix $M_T = (m_{ij})$ defined by $m_{ij} = \frac{\partial r_i}{\partial x_j}$ is a presentation matrix of the Λ-homomorphism d_2 and calculated as follows:

$$M_T = \begin{bmatrix} -2t^{-1} + t^{-2} & 2t^{-1} - t^{-2} \\ 3 - 4t^{-1} & -3 + 4t^{-1} \end{bmatrix}.$$

Hence we have

$$H(T) \cong \Lambda/(2t - 1, 3t - 4),$$

which is equal to $DH(T)$. Thus, the first elementary ideal $\epsilon(T)$ of T is

$$\begin{aligned} \epsilon(T) &=< 2t - 1, 3t - 4 > \\ &=< 2t - 1, 3t - 4, 3(2t - 1) - 2(3t - 4) > \\ &=< 2t - 1, 5 > . \end{aligned}$$

The surface-link T represented by the marked graph diagram D is ambient isotopic to the surface-link T' represented by the motion picture in Fig. 7. Let s' be the circle $l_1 \cup l_2 \cup \{(a, b, c, t) | 1 < t < 2\} \cup \{(d, e, f, t) | 1 < t < 2\}$ in T'. The circle s' bounds a disk d' in $\mathbb{R}^4$ such that the interior $\mathrm{int}d'$ of d' meets T' with 10 crossings and $\mathrm{Int}(\mathrm{int}d', T') = 0$, where Int denotes the intersection number. Since T and T' are ambient isotopic, there is a disk d such that $\partial d \subset T$ and $\mathrm{int}d$ meets T with 10 crossings and $\mathrm{Int}(\mathrm{int}d, T) = 0$. Let $d \times I$ be a thickening of d. Let K be the immersed 2-knot obtained from T by replacing the annulus $T \cap (d \times I)$ by $d \times \partial I$. Then K is the immersed 2-knot with 20 double point singularities. Since the first elementary ideal $\epsilon(K)$ of K is the same as that of T, $\epsilon(K) =< 2t - 1, 5 >$.

The following lemma is useful in a computation for a symmetric ideal.

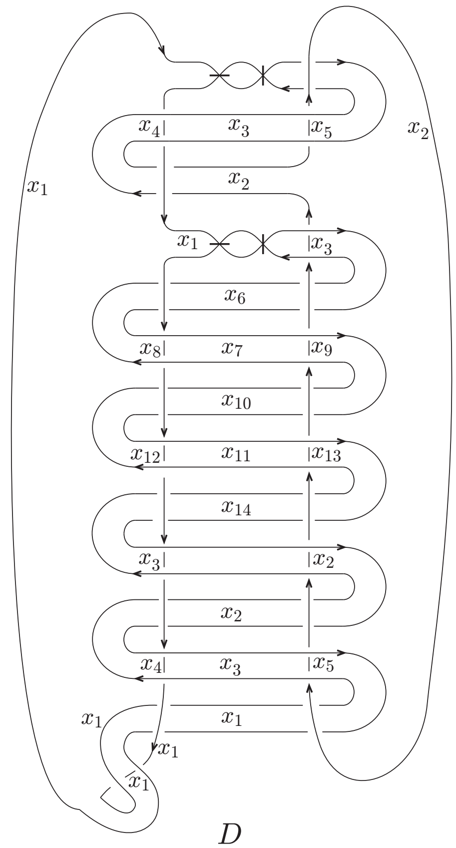


Figure 6: An admissible marked graph diagram D

Lemma 3.3. The following statements are equivalent:

1. The ideal $< 2t-1, m >$ is symmetric.
2. An integer m is $\pm 2^r$ or $\pm 2^r 3$ for any integer $r \geq 0$.

Proof. First, it is easy to show that $< 2t-1, 0 >=< 2t-1 >$ is not symmetric. The ideal $< 2t-1, \pm 3 >=< -t-1, \pm 3 >$ is symmetric. It is observed that

$$< 2t-1, ab >=< t-2, ab > \Rightarrow < 2t-1, a >=< t-2, a > \tag{3.1}$$

for all non-zero integers a, b. Thus, $< 2t-1, \pm 1 >$ is symmetric. Let m be

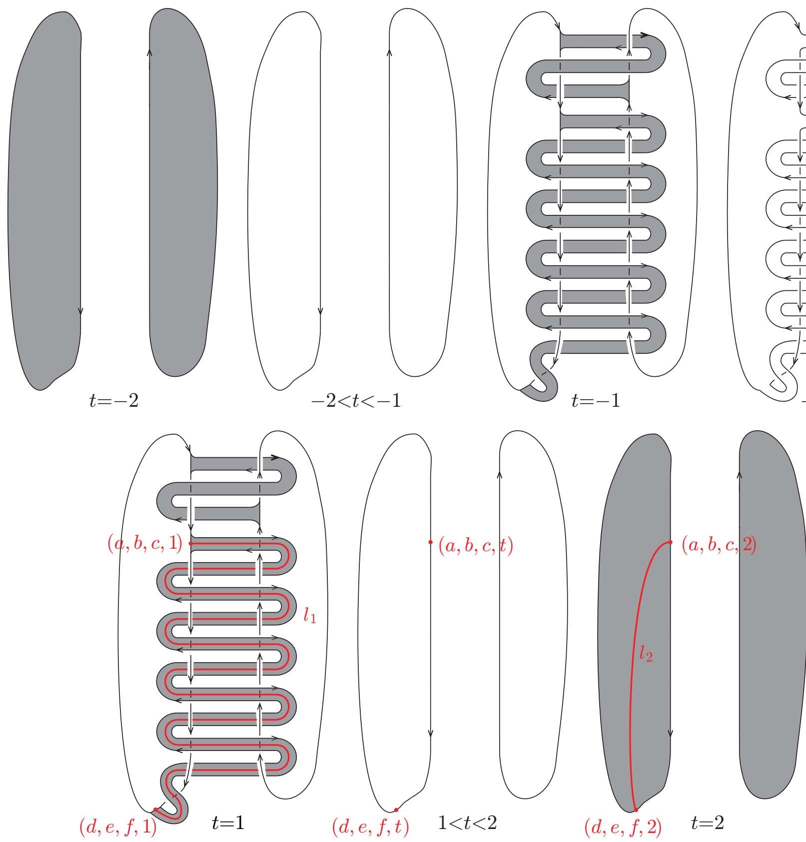


Figure 7: A motion picture

even, that is, $m = 2n$ for some integer n. Then

$$\begin{aligned} < 2t-1, m > &=< 2t-1, 2n > \\ &=< 2t-1, 2n, n(2t-1) - 2nt > \\ &=< 2t-1, n > . \end{aligned}$$

By mathematical induction, if $m = 2^r n$ for $r \geq 0$ and some odd integer n, then

$$< 2t-1, m >=< 2t-1, n > .$$

Let p be a prime with $|p| \geq 5$. Since $\mathbb{Z}_p[t, t^{-1}]$ is a principal ideal domain, $< 2t-1, p >\neq< t-2, p >$. By the contraposition of (3.1), for any non-zero integer m divided by a prime $p \geq 5$, $< 2t-1, m > \neq < t-2, m >$. Suppose that $< 2t-1, 9 >$ is symmetric, i.e., $< 2t-1, 9 >=< t-2, 9 >$. Then

$$\begin{aligned} < t-2, 9 > &=< t-2, 9, 2t-1 > \\ &=< t-2, 9, 2t-1-2(t-2) > \\ &=< t-2, 3 >=< t-5, 3 >, \\ < 2t-1, 9 > &=< t-5, 9 > . \ (\because 2^{-1} \equiv 5 \pmod 9.) \end{aligned}$$

Thus $< t-5, 3 >=< t-5, 9 >$. Then there are $a(t)$, $b(t) \in \mathbb{Z}[t, t^{-1}]$ such that $3 = a(t)(t-5) + b(t)9$. For $b(t)$, there are $b'(t) \in \mathbb{Z}[t, t^{-1}]$ and $c \in \mathbb{Z}$ such that $b(t) = b'(t)(t-5) + c$. Thus

$$3 = a(t)(t-5) + (b'(t)(t-5) + c)9.$$

Then $(a(t) + 9b'(t))(t-5) = 3 - 9c \in \mathbb{Z} \setminus \{0\}$. This is a contradiction. Hence $< 2t-1, 9 >$ is not symmetric. □

Lemma 3.4. There are infinitely many immersed 2-knots with at least one essential double point singularity whose ideals are mutually distinct.

Proof. Let T_n be the ribbon torus-knot of D_n in Fig. 8 ($n \geq 1$). Let K_n be the immersed 2-knot obtained from T_n analogously to the method in Example 3.2. By the same calculation as in Example 3.2, we have $DH(K_n) = H(K_n) \cong \Lambda/(2t-1, n)$. Suppose that the immersed 2-knot K^* is equivalent to the connected sum of a 2-knot K and an unknotted immersed sphere S_0. By Lemma 3.1, the first elementary ideal $\epsilon(K)$ is symmetric for any 2-knot K. Then the identity $\epsilon(K^*) = \epsilon(K)$ is obtained since $\epsilon(S(+)) = \epsilon(S(-)) =< 1 >$, so that the ideal $\epsilon(K^*)$ is symmetric. On the other hand, by Lemma 3.3, $< 2t-1, m >$ is not symmetric except that m is 0, $\pm 2^r$ or $\pm 2^r 3$ ($r \geq 0$). Therefore, the immersed 2-knot K_n obtained from D_n is an immersed 2-knot with at least one essential singularity except that n is 2^{r+2} or $2^r 3$ ($r \geq 0$). Infiniteness of the immersed 2-knots under consideration is seen from infiniteness of the ideals $< 2t-1, m >$ for all m. □

Let J be one of the immersed 2-knots $K_n (n = 1, 2, 3, \dots)$ such that the first elementary ideal $\epsilon(J)$ is asymmetric. Then the following corollary is obtained.

Corollary 3.5. The connected sum $J \# U$ of J and any immersed 2-knot U such that the group orders $|DH(J)|$ and $|DH(U)|$ are coprime is an immersed 2-knot with at least one essential double point singularity.

Proof. Suppose that the immersed 2-knot $J\#U$ is a connected sum of a 2-knot K and an unknotted immersed sphere S_0. Since $DH(K) = DH(J\#U) = DH(J) \otimes DH(U)$ and $|DH(J)|$ and $|DH(U)|$ are coprime, the Farber-Levine pairing $\ell : DH(K) \times DH(K) \to \mathbb{Q}/\mathbb{Z}$ induces the nonsingular t-isometric symmetric pairing on the direct summand $DH(J) = \Lambda/(2t-1, m)$ for some m, so that as in the proof of Lemma 3.4, the ideal $\epsilon(J) =< 2t-1, m >$ must be symmetric, which is a contradiction. □

Finally, the ideal $(2t-1, 5)$ is known to be the first elementary ideal of a ribbon torus-knot in [4].

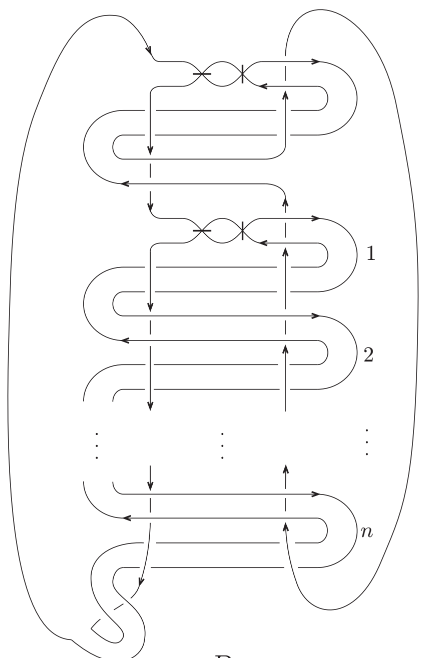


Figure 8: H-admissible marked graph diagrams D_n

By using an immersed 2-knot in Lemma 3.4, the following main theorem is proved.

Theorem 3.6. Let $K = nK_m$ be the connected sum of n copies of an immersed 2-knot K_m with at least one essential double point singularity whose first elementary ideal is $< 2t-1, m >$ for any integer $m \geq 5$ without factors 2 and 3. Then K gives infinitely many immersed 2-knots with at least n double point singularities every of which is essential.

Proof. Assume that there is an immersed 2-knot K' with only $d(< n)$ essential double point singularities such that $K = K'\#S_0$, where S_0 is an unknotted singular 2-knot. We know that $DH_1(S_0) = 0$. Thus

$$\begin{aligned} DH(K') &\cong DH(K') \oplus DH(S_0) \\ &\cong DH(K) \\ &\cong \bigoplus_n (\Lambda/(2t-1, m)). \end{aligned}$$

Therefore

$$e(DH(K)) = e(DH(K')) = n, \tag{3.2}$$

where $e(H)$ is the minimum number of Λ-generators of a finitely generated Λ-module H.

Now, for simplicity, we denote $E(K')$ or $\tilde{E}(K')$ by E or $\tilde{E}$, respectively.

By Wang exact sequence, there is an exact sequence

$$\cdots \to H_d(\tilde{E}) \xrightarrow{t-1} H_d(\tilde{E}) \xrightarrow{p_*} H_d(E) \xrightarrow{\delta_d} H_{d-1}(\tilde{E}) \to \cdots.$$

We have $H_1(E) = H_0(\tilde{E}) = H_0(E) = \mathbb{Z}$, so that we obtain

$$\cdots \to H_1(\tilde{E}) \xrightarrow{t-1} H_1(\tilde{E}) \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to 0.$$

Then $t-1 : H_1(\tilde{E}) \to H_1(\tilde{E})$ is onto. Since $H_1(\tilde{E})$ is a finitely generated Λ-module, the map $t-1$ is an isomorphism by Noetherian property.

Suppose that $H_1(\tilde{E};\mathbb{Q}) = H_1(\tilde{E}) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}}^k \oplus M$, where $\Lambda_{\mathbb{Q}} = \mathbb{Q}[t, t^{-1}]$ which is a principal ideal domain, k is a non-negative integer, and M is the $\Lambda_{\mathbb{Q}}$-torsion part. We have an isomorphism $t-1 : H_1(\tilde{E};\mathbb{Q}) \to H_1(\tilde{E};\mathbb{Q})$. By the $\Lambda_{\mathbb{Q}}$-exact sequence

$$0 \to \Lambda_{\mathbb{Q}} \xrightarrow{t-1} \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}/(t-1) \cong \mathbb{Q} \to 0,$$

the map $t-1 : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ cannot be an epimorphism. Therefore, $k = 0$ and $H_1(\tilde{E};\mathbb{Q})$ is a $\Lambda_{\mathbb{Q}}$-torsion module, which is a Λ-torsion module. The homology $H_2(\tilde{E};\mathbb{Q}/\mathbb{Z})$ is a $\mathbb{Z}$-torsion Λ-module. From the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we obtain a long Λ-exact sequence

$$\cdots \to H_2(\tilde{E};\mathbb{Q}/\mathbb{Z}) \xrightarrow{h_2} H_1(\tilde{E};\mathbb{Z}) \xrightarrow{f_1} H_1(\tilde{E};\mathbb{Q}) \xrightarrow{g_1} H_1(\tilde{E};\mathbb{Q}/\mathbb{Z}) \to \cdots.$$

Let $x \in H_1(\tilde{E}) = H_1(\tilde{E};\mathbb{Z})$. Since $H_1(\tilde{E};\mathbb{Q})$ is a Λ-torsion module, there is a non-zero element $\lambda \in \Lambda$ such that $\lambda f_1(x) = 0$. Then $\lambda x \in \mathrm{Ker}(f_1) = \mathrm{Im}(h_2)$. There is an element $y \in H_2(\tilde{E};\mathbb{Q}/\mathbb{Z})$ such that $\lambda x = h_2(y)$. Since $H_2(\tilde{E};\mathbb{Q}/\mathbb{Z})$ is a $\mathbb{Z}$-torsion Λ-module, there is a non-zero $k \in \mathbb{Z}$ such that $ky = 0$. Then $0 \neq k\lambda \in \Lambda$ and $k\lambda x = kh_2(y) = 0$. Therefore $H_1(\tilde{E})$ is a Λ-torsion module.

For a finitely generated Λ-module H, we define $TH = \{x \in H | \lambda x = 0 \text{ for a non-zero } \lambda \in \Lambda\}$, $BH = H/TH$, and $T_D H = TH/DH$. Since $H_1(\tilde{E}) = TH_1(\tilde{E})$, the Λ-torsion-free part $BH_1(\tilde{E}) = 0$. By the second duality theorem in [7], there are t-anti Λ-epimorphisms

$$\theta : DH_2(\tilde{E}) \to E^1 BH_1(\tilde{E}, \partial\tilde{E}) = E^1 BH_1(\tilde{E}) = 0 \quad \text{and}$$

$$\theta' : DH_0(\tilde{E}, \partial\tilde{E}) = DH_0(\tilde{E}) = 0 \to E^1 BH_3(\tilde{E}),$$

where $E^k H = \mathrm{Ext}^k_\Lambda(H, \Lambda)$ for any non-negative integer k and any Λ-module H, and there is a t-isometric non-singular Λ-pairing

$$\ell : \mathrm{Ker}(\theta) \times \mathrm{Ker}(\theta') = DH_2(\tilde{E}) \times 0 \to \mathbb{Q}/\mathbb{Z}.$$

Thus, $DH_2(\tilde{E}) = 0$. By the first duality theorem in [7], there is a t-Hermitian non-singular pairing

$$L : T_D H_2(\tilde{E}) \times T_D H_1(\tilde{E}, \partial\tilde{E}) \to \mathbb{Q}(\Lambda)/\Lambda.$$

Since there is a Λ-epimorphism from $H_1(\tilde{E}) = H_1(\tilde{E}, \partial\tilde{E})$ to $T_D H_1(\tilde{E}, \partial\tilde{E})$ and $t-1 : H_1(\tilde{E}) \to H_1(\tilde{E})$ is a Λ-isomorphism, the map $t-1 : T_D H_1(\tilde{E}, \partial\tilde{E}) \to T_D H_1(\tilde{E}, \partial\tilde{E})$ is a Λ-epimorphism. The fact that $T_D H_1(\tilde{E}, \partial\tilde{E})$ is a finitely generated Λ-module implies that $t-1 : T_D H_1(\tilde{E}, \partial\tilde{E}) \to T_D H_1(\tilde{E}, \partial\tilde{E})$ is a Λ-isomorphism. Thus we have a Λ-isomorphism $t-1 : T_D H_2(\tilde{E}) \to T_D H_2(\tilde{E})$. Since $DH_2(\tilde{E}) = 0$, the map $t-1 : TH_2(\tilde{E}) \to TH_2(\tilde{E})$ is a Λ-isomorphism. For $x \in TH_2(\tilde{E})$, there is an element $x' \in TH_2(\tilde{E})$ such that $x = (t-1)x'$. Then $p_*(x) = (1-1)p_*(x') = 0$. The module $TH_2(\tilde{E})$ is a submodule of the kernel of $p_* : H_2(\tilde{E}) \to H_2(E)$. So, we obtain the short Λ-exact sequence

$$0 \to BH_2(\tilde{E}) \xrightarrow{t-1} BH_2(\tilde{E}) \xrightarrow{p_*} H_2(E) \cong \mathbb{Z}^d \to 0.$$

We obtain the long exact sequence

$$E^0(\mathbb{Z}^d) \to E^0 BH_2(\tilde{E}) \to E^0 BH_2(\tilde{E}) \to E^1(\mathbb{Z}^d) \to E^1 BH_2(\tilde{E}) \to \cdots .$$

Since E^0H is a Λ-free module for a finitely generated Λ-module H, we have $E^0BH_2(\tilde{E}) \cong \Lambda^k$ for some non-negative integer k. So, the long exact sequence is as follows:

$$0 \to \Lambda^k \xrightarrow{t-1} \Lambda^k \to (\Lambda/(t-1))^d \to G \to \cdots ,$$

where $G = E^1BH_2(\tilde{E})$ is a finite Λ-module. Then we have

$$0 \to (\Lambda/(t-1))^k \to (\Lambda/(t-1))^d \to G \to \cdots .$$

Thus, $E^0BH_2(\tilde{E}) \cong \Lambda^d$.

By the second duality theorem in [7], there are a t-anti Λ-epimorphism $\theta : DH_1(\tilde{E}, \partial\tilde{E}) = DH_1(\tilde{E}) \to E^1BH_2(\tilde{E})$ and a t-isometric symmetric non-singular pairing $\phi : D \times D \to \mathbb{Q}/\mathbb{Z}$, where $D = \mathrm{Ker}(\theta)$. For every prime p and every positive integer i, let $\bar{D}^i_p = \{x \in D |\, p^i x = 0\}$ and $\tilde{D}^i_p = \bar{D}^i_p/(\bar{D}^{i-1}_p + p\bar{D}^{i+1}_p)$. The t-isometric symmetric non-singular pairing ϕ induces a t-isometric symmetric non-singular pairing $\tilde{\phi}^i_p : \tilde{D}^i_p \times \tilde{D}^i_p \to \mathbb{Q}/\mathbb{Z}$ for all prime p and all i (see [10]). Suppose $D \neq 0$. Then there are $p \geq 5$ and i with $\tilde{D}^i_p \neq 0$, so that $\tilde{D}^i_p \cong (\Lambda/(p, 2t-1))^{r_i}$ for some $r_i > 0$. The t-isometric symmetric non-singular pairing $\tilde{\phi}^i_p$ induces a t-anti automorphism of $\tilde{D}^i_p$, so that all the elementary ideals of $\tilde{D}^i_p$ are symmetric. This means that the ideal $(p, 2t-1)$ is symmetric, for it is the $(r_i - 1)$th elementary ideal of $\tilde{D}^i_p$. This contradicts Lemma 3.3. Thus, $D = \mathrm{Ker}(\theta) = 0$. Therefore $DH_1(\tilde{E})$ and $E^1BH_2(\tilde{E})$ are t-anti isomorphic. Then $DH_1(\tilde{E}) \cong E^2DH_1(\tilde{E})$ and $E^2E^1BH_2(\tilde{E})$ are t-anti isomorphic.

By Lemma 3.6 of [7], there is an exact sequence

$$0 \to BH_2(\tilde{E}) \to E^0E^0BH_2(\tilde{E}) \cong \Lambda^d \to E^2E^1BH_2(\tilde{E}) \to 0.$$

This means that $DH_1(\tilde{E}) \cong E^2E^1BH_2(\tilde{E})$ is generated by d elements over Λ. Combining with (3.2), we obtain $n = e(DH_1(\tilde{E})) \leq d$, which is a contradiction. Thus, there is no immersed 2-knot K' such that $K = K' \# S_0$. Infiniteness of the immersed 2-knots under consideration is seen from infiniteness of the ideals $< 2t-1, m >$ for all m. This completes the proof of Theorem 3.6.

□

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[1] R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Ginn and Co., Boston, Mass., 1963.

[2] M. S. Farber, Duality in an infinite cyclic covering and even-dimensional knots, *Math. USSR-Izv.* **11** (1977), 749–781.

[3] R. H. Fox, A quick trip through knot theory, *Toplogy of* 3*-manifolds and Related Topics*, (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962), 120–167.

[4] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, *Osaka J. Math.* **16** (1979), 233–248.

[5] S. Kamada and K. Kawamura, Ribbon-clasp surface-links and normal forms of singular surface-links, *Topology Appl.* **230** (2017), 181–193.

[6] S. Kamada, A. Kawauchi, J. Kim and S. Y. Lee, Presentation of immersed surface-links by marked graph diagrams, *J. Knot Theory Ramifications* **27** (2018), No. 10, 1850052 (10 pages).

[7] A. Kawauchi, Three dualities on the integral homology of infinite cyclic coverings of manifolds, *Osaka J. Math.* **23** (1986), 633–651.

[8] A. Kawauchi, *A survey of knot theory*, Birkhäuser, 1996.

[9] A. Kawauchi, On a cross-section of an immersed sphere-link in 4-space, *Topology Appl.* **230** (2017), 194-217.

[10] A. Kawauchi and S. Kojima, Algebraic classification of linking pairings on 3-manifolds, *Mathematische Annalen* **253** (1980), 29-42.

[11] J. Kim, Y. Joung and S. Y. Lee, On the Alexander biquandles of oriented surface-links via marked graph diagrams, *J. Knot Theory Ramifications* **23**(7) (2014), Article ID:1460007 (26 pages).

[12] J. Levine, Knot modules. I, *Trans. Amer. Math. Soc.* **229** (1977), 1–50.

[13] M. Soma, Surface-links with square-type ch-graphs, Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999), *Topology Appl.* **121** (2002), 231–246.