ENUMERATING HOMOLOGY SPHERES WITH LENGTHS UP TO 10 BY A CANONICAL ORDER

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ABSTRACT

On previous works, we enumerated the prime links and the prime link exteriors with lengths up to 10. In this paper, we make an enumeration of the homology handles and the homology spheres with lengths up to 10 by using the enumeration of the prime link exteriors.

Keywords: Lattice point, Length, Link, Exterior, Homology sphere, Table

1. Introduction

In [3] we suggested a method of enumerating the links, the link exteriors and the closed connected orientable 3-manifolds. The idea is to introduce a well-order on the set of links by embedding it into a well-ordered set of lattice points. This well-order also naturally induces a well-order on the set of prime link exteriors and eventually induces a well-order on the set of (closed connected orientable) 3-manifolds so that the fundamental group of the corresponding prime link exterior gives a complete group invariant for the set of 3-manifolds. By using this method, the first 28, 26 and 26 lattice points of lengths up to 7 corresponding to the prime links, the prime link exteriors and the closed connected orientable 3-manifolds are respectively tabulated in [3]. We enlarged the table of the first 28 lattice points of lengths up to 7 corresponding to the prime links into that of the first 443 lattice points of lengths up to 10 in [5] and enlarged the table of the first 26 lattice points of lengths up to 7 corresponding to the prime link exteriors into that of the first 399 lattice points of lengths up to 10 in [8] and enlarged the table of the first 26 lattice points of lengths up to 7 corresponding to the 3-manifolds into that of the first 133 lattice points of lengths up to 9 in [7]. A tentative goal of this project is to enumerate the lattice points of lengths up to 10 corresponding to the 3-manifolds. In this paper, we enumerate the lattice points of lengths up to 10 corresponding to the homology handles and the homology spheres.

2. Definition of a well-order on the set of links

Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^n$ the product of n copies of $\mathbf{Z}$. We put

$$\mathbf{X} = \coprod_{n=1}^{\infty} \mathbf{Z}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbf{Z},\ n = 1, 2, \ldots\}.$$

We call elements of $\mathbf{X}$ *lattice points*. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put $\ell(\mathbf{x}) = n$ and call it the *length* of $\mathbf{x}$. Let $|\mathbf{x}|$ and $|\mathbf{x}|_N$ be the lattice points determined from $\mathbf{x}$ by the following formulas:

$|\mathbf{x}| = (|x_1|, |x_2|, \ldots, |x_n|)$ and $|\mathbf{x}|_N = (|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$,
where $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$ and $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

We define a well-order (called a *canonical order* [3]) on $\mathbf{X}$ as follows:

Definition 2.1. We define a well-order on $\mathbf{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 \cdots$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we define $\mathbf{x} < \mathbf{y}$ if we have one of the following conditions (1)-(4):

(1) $\ell(\mathbf{x}) < \ell(\mathbf{y})$.
(2) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$ by the lexicographic order on the natural number order.
(3) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$ by the lexicographic order on the natural number order.
(4) $|\mathbf{x}| = |\mathbf{y}|$ and $\mathbf{x} < \mathbf{y}$ by the lexicographic order on the well-order of $\mathbf{Z}$ defined above.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put

$$\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|.$$

Let $\beta(\mathbf{x})$ be the $(\max|\mathbf{x}| + 1)$-string braid determined from $\mathbf{x}$ by the identity

$$\beta(\mathbf{x}) = \sigma_{|x_1|}^{\mathrm{sign}(x_1)} \sigma_{|x_2|}^{\mathrm{sign}(x_2)} \cdots \sigma_{|x_n|}^{\mathrm{sign}(x_n)},$$

where we define $\sigma_{|0|}^{\mathrm{sign}(0)} = 1$. We note that $\max|\mathbf{x}| + 1$ is the minimum string number of the braid indicated by the right-hand side of the identity. Let $\mathrm{cl}\beta(\mathbf{x})$ be the closure of the braid $\beta(\mathbf{x})$. Let $\mathbf{L}$ be the set of all links modulo equivalence, where two links are *equivalent* if there is a (possibly orientation-reversing) homeomorphism sending one to the other. Then we have a map

$$\mathrm{cl}\beta : \mathbf{X} \to \mathbf{L}$$

sending $\mathbf{x}$ to $\mathrm{cl}\beta(\mathbf{x})$. By Alexander's braiding theorem, the map $\mathrm{cl}\beta$ is surjective. For $L \in \mathbf{L}$, we define a map

$$\sigma : \mathbf{L} \to \mathbf{X}$$

by $\sigma(L) = \min\{\mathbf{x} \in \mathbf{X} \,|\, \mathrm{cl}\beta(\mathbf{x}) = L\}$. Then σ is a right inverse of $\mathrm{cl}\beta$ and hence is injective. Now we have a well-order on $\mathbf{L}$ by the following definition:

Definition 2.2. For $L, L' \in \mathbf{L}$, we define $L < L'$ if $\sigma(L) < \sigma(L')$.

For a link $L \in \mathbf{L}$, we call $\ell(\sigma(L))$ the *length* of L.

3. A method of a tabulation of prime links and prime link exteriors

Let $\mathbf{L}^{\mathrm{p}}$ be the subset of $\mathbf{L}$ consisting of the prime links, where we consider that the 2−component trivial link is not prime. We use the injection σ for our method of a tabulation of $\mathbf{L}^{\mathrm{p}}$. For $k \in \mathbf{Z}$, let k^n and $-k^n$ be the lattice points determined by

$$k^n = (\underbrace{k, k, \ldots, k}_{n}) \quad \text{and} \quad -k^n = (-k)^n,$$

respectively.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbf{X}$, let $\mathbf{x}^T$, $-\mathbf{x}$, $(\mathbf{x},\ \mathbf{y})$ and $\delta(\mathbf{x})$ be the lattice points determined by the following formulas:

$$\begin{aligned}
\mathbf{x}^T &= (x_n, \ldots, x_2, x_1), \\
-\mathbf{x} &= (-x_1, -x_2, \ldots, -x_n), \\
(\mathbf{x},\ \mathbf{y}) &= (x_1, \ldots, x_n, y_1, \ldots, y_m), \\
\delta(\mathbf{x}) &= (x'_1, x'_2, \ldots, x'_n), \\
\text{where} \quad x'_i &= \begin{cases} \mathrm{sign}(x_i)(\max|\mathbf{x}| + 1 - |x_i|) & (x_i \neq 0) \\ 0 & (x_i = 0). \end{cases}
\end{aligned}$$

A point of our argument on a tabulation of prime links is to define some transformations between lattice points. We make this definition as follows:

Definition 3.1. Let $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w} \in \mathbf{X},\ k,\ l,\ n \in \mathbf{Z}$ with $n > 0$ and $\varepsilon = \pm 1$. An *elementary transformation* on lattice points is one of the following operations (1)-(12) and their inverses $(1)^-$-$(12)^-$.

(1) $(\mathbf{x},\ k,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$
(2) $(\mathbf{x},\ k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$, where $|k| > \max|\mathbf{x}|,\ \max|\mathbf{y}|$.
(3) $(\mathbf{x},\ k,\ l,\ \mathbf{y}) \to (\mathbf{x},\ l,\ k,\ \mathbf{y})$, where $|k| > |l| + 1$ or $|l| > |k| + 1$.
(4) $(\mathbf{x},\ \varepsilon k^n,\ k+1,\ k,\ \mathbf{y}) \to (\mathbf{x},\ k+1,\ k,\ \varepsilon(k+1)^n,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(5) $(\mathbf{x},\ k,\ \varepsilon(k+1)^n,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ -(k+1),\ \varepsilon k^n,\ k+1,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(6) $(\mathbf{x},\ \mathbf{y}) \to (\mathbf{y},\ \mathbf{x})$
(7) $\mathbf{x} \to \mathbf{x}^T$

(8) $\mathbf{x} \to -\mathbf{x}$
(9) $\mathbf{x} \to \delta(\mathbf{x})$
(10) $(1^n,\ \mathbf{x},\ \varepsilon,\ \mathbf{y}) \to (1^n,\ \mathbf{y},\ \varepsilon,\ \mathbf{x})$, where $\min|\mathbf{x}| \geqq 2$ and $\min|\mathbf{y}| \geqq 2$.
(11) $(k^2,\ \mathbf{x},\ \mathbf{y},\ -k^2,\ \mathbf{z},\ \mathbf{w}) \to (-k^2,\ \mathbf{x},\ \mathbf{w}^T,\ k^2,\ \mathbf{z},\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$, $\max|\mathbf{z}| < k < \min|\mathbf{w}|$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ or $\mathbf{w}$ may be empty.
(12) $(\mathbf{x},\ k,\ (k{+}1)^2,\ k,\ \mathbf{y}) \to (\mathbf{x},\ -k,\ -(k{+}1)^2,\ -k,\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$ and $\mathbf{x}$ or $\mathbf{y}$ may be empty.

A meaning of the transformations of Definition 3.1 is given by the following lemma (See [3, 5]):

Lemma 3.2. If a lattice point $\mathbf{x}$ is transformed into a lattice point $\mathbf{y}$ by an elementary transformation, then we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ (modulo a split union of a trivial link for (1), (2), (9)).

The outline of a tabulation of prime links is the following (See [3, 5] for the details): Let Δ be the subset of $\mathbf{X}$ consisting of 0, 1^m for $m \geqq 2$ and $(x_1, x_2, \ldots, x_n)$, where $n \geqq 4$, $x_1 = 1$, $1 \leqq |x_i| \leqq \frac{n}{2}$, $|x_n| \geqq 2$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, \max|\mathbf{x}|\}$. Then we have $\sharp\{\mathbf{y} \in \Delta | \mathbf{y} < \mathbf{x}\} < \infty$ for every $\mathbf{x} \in \Delta$ and have $\sigma(\mathbf{L}^{\mathrm{p}}) \subset \Delta$. First, we enumerate the lattice points of Δ under the canonical order and then we omit $\mathbf{x} \in \Delta$ from the sequence if $\mathrm{cl}\beta(\mathbf{x})$ is a non-prime link or a link which has already appeared in the table of prime links. By using Lemma 3.2, we see that if $\mathbf{x}$ is transformed into a smaller one, then $\mathbf{x}$ must be removed from the sequence. We can find most of the omittable lattice points in this way. We show a table of prime links with lengths up to 10 below. Note that there are 7 omissions and 1 overlap in Conway's prime link table of 10 crossings and the seven links are indicated by the formulas: $\sigma(10^2_A) = (1, -2, 1, -2, 1, -3-2^3, -3)$, $\sigma(10^3_A) = (1^3, 2, -1^3, 2, -1, 2)$, $\sigma(10^3_B) = (1^4, 2^2, -1^2, 2^2)$, $\sigma(10^4_A) = (1^2, 2, 1^2, 2, 3, 2^2, 3)$, $\sigma(10^4_B) = (1^2, 2, 1^2, 2, 3, -2^2, 3)$, $\sigma(10^4_C) = (1^2, 2, 1^2, 2, -3, 2^2, -3)$, $\sigma(10^4_D) = (1^2, 2, -1^2, 2, -3, 2^2, -3)$.

$O < 2^2_1 < 3_1 < 4^2_1 < 4_1 < 5_1 < 5^2_1 < 6^2_1 < 5_2 < 6_2 < 6^3_3 < 6^3_1 < 6_3 < 6^3_2 < 6^2_3 < 7_1 < 6^2_2 < 7^2_1 < 7^2_7 < 7^2_8 < 7^2_4 < 7^2_2 < 7^2_5 < 7^2_6 < 6_1 < 7_6 < 7_7 < 7^3_1 < 8^2_1 < 7_3 < 8_2 < 8^3_7 < 8^3_8 < 8^3_1 < 8_{19} < 8_{20} < 8_5 < 7_5 < 8_7 < 8_{21} < 8_{10} < 8^3_9 < 8^3_5 < 8_{16} < 8_9 < 8^3_2 < 8_{17} < 8^3_6 < 8^3_{10} < 8^3_4 < 8_{18} < 7^2_3 < 8^2_5 < 8^2_{16} < 8^2_{15} < 8^2_9 < 8^2_8 < 8^2_{12} < 8^2_{13} < 8^2_7 < 8^2_{10} < 8^2_{11} < 8^4_3 < 8^4_2 < 8^4_1 < 8^2_{14} < 8_{12} < 9_1 < 8^2_2 < 9^2_1 < 9^2_{43} < 9^2_{44} < 9^2_{13} < 9^2_{49} < 9^2_{51} < 9^2_{19} < 9^2_{50} < 8^2_3 < 9^2_2 < 9^2_{52} < 9^2_{20} < 9^2_{55} < 9^2_{31} < 9^2_{53} < 9^2_{54} < 8^2_4 < 9^2_{23} < 9^2_{57} < 9^2_{35} < 9^2_{40} < 9^2_5 < 9^2_{14} < 9^2_{21} < 9^2_{34} < 9^2_{37} < 9^2_{59} < 9^2_{29} < 9^2_{39} < 9^2_{61} < 9^2_{41} < 9^2_{42} < 8_6 < 9_{11} < 9_{43} < 9_{44} < 9_{36} < 9_{42} < 7_2 < 8_{14} < 9_{26} < 8_4 < 8^3_3 < 9^3_6 < 9^3_{13} < 9^3_{14} < 9^3_2 < 9^3_{19} < 9^3_{18} < 9^3_8 < 9_{45} < 9_{32} < 9^3_{11} < 8_8 < 9_{20} < 9^3_1 < 7_4 < 8_{11} < 9_{27} < 8_{13} < 8_{15} < 9_{24} < 9_{30} < 9^3_{17} < 9^3_{16} < 9^3_{15} < 9^3_4 < 9^3_{10} < 9^3_{20} < 9^3_{12} < 9^3_{21} < 9_{33} < 9_{46} < 9_{34} < 9_{47} < 9_{31} < 9_{28} < 9_{40} < 9^2_{11} < 9_{17} < 9_{22} < 9^3_5 < 9^3_9 < 9_{29} < 9^2_{12} < 8^2_6 < 9^2_{25} < 10^2_1 < 9_3 < 10_2 < 10^3_{44} < 10^3_{45} < 10^3_1 < 10_{124} < 10_{126} < 10_{46} < 10_{125} < 10^3_{50} < 10^3_{51} < 10^3_7 < 9_6 < 10_5 < 10_{127} < 10_{47} < 10^3_{56} < 10^3_{27} < 10_{139} < 10_{143} < 9_9 < 10_{62} < 10_{141} < 10_{148} < 10_{85} < 10^3_A < 10^3_{52} < 10^3_{31} < 10_{155} < 10_{100} < 10_9 < 10^3_2 < 10^3_8 < 10_{149} < 10_{82} < 10^3_{58} < 10^3_{35} < 10^3_{62} < 10^3_B < 10^3_{19} < 9_{16} < 10_{64} < 10^3_{60} < 10^3_{38} < 10_{94} < 10_{161} < 10_{159} < 10_{106} < 10_{112} < 10^3_{64} < 10^3_{21} < 10^3_{73} < 10^3_{41} < 10_{116} < 10^3_{43} < 9^2_3 < 10^2_6 < 10^2_{133} <$

$10^2_{134} < 10^2_{38} < 10^2_{132} < 10^2_{141} < 10^2_{140} < 10^2_{46} < 10_{17} < 10_{48} < 10^3_{30} < 10_{91} < 10^3_{33} <$
$10_{152} < 10_{79} < 10_{157} < 10_{104} < 10_{99} < 10^3_{42} < 10_{118} < 10_{109} < 10_{123} < 9^2_7 < 10^2_{11} <$
$9^2_4 < 9^2_{27} < 10^2_{56} < 10^2_{136} < 10^2_{139} < 10^2_{44} < 10^2_{138} < 10^2_{169} < 10^2_{163} < 10^2_{76} < 10^2_{162} <$
$10^2_{135} < 10^2_{155} < 10^2_{88} < 9^2_{45} < 10^2_{128} < 9^2_{56} < 9^2_{47} < 10^2_{160} < 10^2_{154} < 10^2_{94} < 10^2_{124} <$
$10^2_{137} < 10^2_{98} < 10^2_{176} < 9^2_{58} < 10^2_{177} < 10^2_{110} < 9^2_6 < 10^2_{10} < 10^2_{42} < 10^2_{18} < 9^2_9 < 9^2_{48} <$
$9^2_{18} < 10^2_{125} < 10^2_{31} < 10^2_{63} < 9^2_{26} < 10^2_{41} < 9^2_8 < 10^4_{12} < 10^4_{11} < 10^4_{10} < 9^4_1 < 10^4_2 <$
$9^2_{22} < 9^2_{38} < 10^2_{90} < 9^2_{46} < 10^2_{129} < 10^2_{35} < 9^2_{16} < 10^4_A < 10^4_B < 10^4_C < 10^4_D < 10^4_6 <$
$10^2_{167} < 9^2_{60} < 9^2_{30} < 10^2_{74} < 10^2_{85} < 10^2_{170} < 10^2_{172} < 10^2_{107} < 10^2_{174} < 10^2_{156} < 10^2_{101} <$
$10^2_{179} < 10^2_{175} < 10^2_{118} < 10^2_{183} < 10^2_{93} < 9^2_{36} < 10^2_{142} < 10^2_{108} < 10^2_{117} < 10^2_{97} < 10^4_{15} <$
$10^4_{13} < 10^4_{19} < 10^4_8 < 10^4_{20} < 10^2_{165} < 10^2_{72} < 9^2_{17} < 10^4_{16} < 10^4_{14} < 10^4_{17} < 10^4_4 < 10^2_{114} <$
$10^2_{180} < 10^2_{181} < 10^2_{178} = 10^2_{182} < 10^2_{120} < 10^2_{184} < 10^2_{19} < 9^2_{28} < 10^2_{64} < 9^2_{33} < 10^2_{81} <$
$10^2_{130} < 10^2_{131} < 10^2_{37} < 10^2_{168} < 10^2_{161} < 10^2_{75} < 10^4_{18} < 10^4_7 < 10^2_{86} < 10^2_{119} < 10^2_{91} <$
$10^2_{17} < 10^2_{30} < 10^4_5 < 9_{15} < 10_{29} < 10^3_{49} < 10^3_{48} < 10^3_5 < 10^2_{26} < 9^2_{15} < 10^2_{36} < 10^2_{62} <$
$10^2_{43} < 10^2_{79} < 10^2_{149} < 10^2_{144} < 10^2_{143} < 10^2_{150} < 10^2_{145} < 10^2_{50} < 10^2_{48} < 10^2_{83} < 9^2_{32} <$
$10^2_{104} < 10^2_A < 10^2_{171} < 10^2_{106} < 10^2_{173} < 10^2_{87} < 10^2_{158} < 10^2_{115} < 10^2_{116} < 10^2_{109} < 10^2_{27} <$
$10^2_{45} < 10^2_{84} < 9^2_{24} < 10^2_{66} < 10^2_{151} < 10^2_{148} < 10^2_{147} < 10^2_{152} < 10^2_{54} < 10^2_{105} < 10^2_{96} <$
$10^4_9 < 10^4_{21} < 10^2_{121} < 8_1 < 9_{21} < 10_{42} < 9_8 < 9_{25} < 10_{71} < 10^3_{29} < 10^3_{61} < 9_{14} < 8_3 <$
$9_{12} < 10_{44} < 10^3_{12} < 10_{43} < 10^2_{25} < 10^2_{34} < 10^2_{39} < 10^2_{55} < 10^2_{78} < 10^4_1 < 10^2_{146} < 10^2_{52} <$
$10^2_{100} < 10^2_{103} < 10^2_{112} < 10_{41} < 9_{19} < 10_{137} < 10_{59} < 10_{136} < 10_{138} < 10_{70} < 10^3_6 <$
$9^3_7 < 10^3_{53} < 10^3_{10} < 10^3_{54} < 10^3_{37} < 10^3_{59} < 10_{45} < 10^3_{13} < 10^3_{24} < 10_{88} < 10^3_{34} < 10^2_{23}$

Table 1

Next we enumerate the prime link exteriors with lengths up to 10. Since a knot is determined by its exterior by the Gordon-Luecke Theorem [2], we classify the exteriors of two or more component links.

We obtain a table of prime link exteriors, by omitting the links in Table 2 from the links in Table 1 and replacing the rest of the links with their exteriors because the exteriors of the following 44 links have already appeared (See [8]).

7^2_7, 7^2_8, 8^3_7, 8^3_8, 8^2_{16}, 8^2_{15}, 9^2_{43}, 9^2_{44}, 9^2_{49}, 9^3_{13}, 9^3_{14}, 9^3_{19}, 9^3_{18}, 9^3_{17}, 10^3_{44}, 10^3_{45}, 10^2_{132}, 9^2_{45}, 10^2_{128}, 9^2_{56}, 9^2_{47}, 10^2_{160}, 10^2_{124}, 9^2_{48}, 10^2_{125}, 10^4_{12}, 9^2_{46}, 10^2_{129}, 10^4_A, 10^4_B, 10^4_C, 10^4_D, 10^2_{167}, 10^4_{13}, 10^4_{17}, 10^2_{130}, 10^2_{131}, 10^2_{168}, 10^2_{161}, 10^4_{18}, 10^3_{49}, 10^3_{48}, 10^3_{61}, 10^3_{59}.

Table 2

4. A method of a tabulation of 3-manifolds

The prime link exteriors in the exterior table are all simple manifolds and hence are hyperbolic or special Seifert manifolds. Thus, the fundamental groups of these prime link exteriors are mutually non-isomorphic (cf. Proposition (4.6) of [3]), so that the prime links in the exterior table are π-minimal links, where a π-minimal link means a prime link whose fundamental group is the first appearing group up to isomorphisms in the canonical ordered set of prime links. We make a list of closed connected orientable 3-manifolds by constructing a sequence of 3-manifolds obtained by the 0-surgery of the links in the exterior table and removing the manifolds which have already appeared (See [3]). Note that every 3-manifold can be represented as the 0-surgery of a link by considering Spin structure and especially of a prime link with unique exterior property by using the imitation theory, which implies that every manifold appears in our list. Let $\chi(L,0)$ denote the manifold obtained by the 0-surgery of a link L. We classify $\chi(L,0)$ for the links L in the exterior table in terms of the first homology group $H_1(\chi(L,0))$. There are 16 types of groups $\mathbf{Z}$, 0, $\mathbf{Z}_2\oplus\mathbf{Z}_2$, $\mathbf{Z}\oplus\mathbf{Z}$, $\mathbf{Z}_3\oplus\mathbf{Z}_3$, $\mathbf{Z}_2$, $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$, $\mathbf{Z}_4\oplus\mathbf{Z}_4$, $\mathbf{Z}_4$, $\mathbf{Z}\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2$, $\mathbf{Z}_5\oplus\mathbf{Z}_5$, $\mathbf{Z}_6$, $\mathbf{Z}_8$, $\mathbf{Z}\oplus\mathbf{Z}_3\oplus\mathbf{Z}_3$, $\mathbf{Z}\oplus\mathbf{Z}_2$, $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ and we have respectively 140, 50, 60, 69, 21, 12, 6, 9, 7, 6, 1, 5, 7, 1, 4, 1 links with these types of groups. We enumerate the 3-manifolds $\chi(L,0)$ with $H_1(\chi(L,0))\cong\mathbf{Z}$ or 0.

Case 1. $H_1(\chi(L,0))\cong\mathbf{Z}$.

We enumerate the manifolds $\chi(L,0)$ with $H_1(\chi(L,0))\cong\mathbf{Z}$. The links with this condition are the following:

$O<3_1<4_1<5_1<5_2<6_2<6_3<7_1<6_1<7_6<7_7<7_3<8_2<8_{19}<8_{20}<8_5<7_5<8_7<8_{21}<8_{10}<8^3_5<8_{16}<8_9<8_{17}<8^3_6<8_{18}<8_{12}<9_1<8_6<9_{11}<9_{43}<9_{44}<9_{36}<9_{42}<7_2<8_{14}<9_{26}<8_4<9^3_2<9_{45}<9_{32}<8_8<9_{20}<9^3_1<7_4<8_{11}<9_{27}<8_{13}<8_{15}<9_{24}<9_{30}<9^3_{10}<9_{33}<9_{46}<9_{34}<9_{47}<9_{31}<9_{28}<9_{40}<9_{17}<9_{22}<9_{29}<9_3<10_2<10_{124}<10_{126}<10_{46}<10_{125}<9_6<10_5<10_{127}<10_{47}<10_{139}<10_{143}<9_9<10_{62}<10_{141}<10_{148}<10_{85}<10^3_A<10^3_{31}<10_{155}<10_{100}<10_9<10_{149}<10_{82}<10^3_{62}<10^3_B<10^3_{19}<9_{16}<10_{64}<10^3_{60}<10^3_{38}<10_{94}<10_{161}<10_{159}<10_{106}<10_{112}<10_{116}<10^3_{43}<10_{17}<10_{48}<10^3_{30}<10_{91}<10_{152}<10_{79}<10_{157}<10_{104}<10_{99}<10^3_{42}<10_{118}<10_{109}<10_{123}<9_{15}<10_{29}<8_1<9_{21}<10_{42}<9_8<9_{25}<10_{71}<10^3_{29}<9_{14}<8_3<9_{12}<10_{44}<10^3_{12}<10_{43}<10_{41}<9_{19}<10_{137}<10_{59}<10_{136}<10_{138}<10_{70}<10^3_{37}<10_{45}<10^3_{13}<10_{88}<10^3_{34}$.

The 3-manifolds obtained by the 0-surgery of the above three component links are homeomorphic to those of some knots and we have:

$\chi(8^3_5,0)\cong\chi(9_{46},0)$,	$\chi(8^3_6,0)\cong\chi(3_1\sharp 3^*_1,0)$,	$\chi(9^3_2,0)\cong\chi(6_3,0)$,
$\chi(9^3_1,0)\cong\chi(6_2,0)$,	$\chi(9^3_{10},0)\cong\chi(K_1,0)$,	$\chi(10^3_A,0)\cong\chi(8_{20},0)$,
$\chi(10^3_{31},0)\cong\chi(K_2,0)$,	$\chi(10^3_{62},0)\cong\chi(O,0)$,	$\chi(10^3_B,0)\cong\chi(10_{140},0)$,
$\chi(10^3_{19},0)\cong\chi(K_3,0)$	$\chi(10^3_{60},0)\cong\chi(6_1,0)$,	$\chi(10^3_{38},0)\cong\chi(10_{153},0)$,
$\chi(10^3_{43},0)\cong\chi(K_4,0)$,	$\chi(10^3_{30},0)\cong\chi(K_5,0)$,	$\chi(10^3_{42},0)\cong\chi(K_6,0)$,
$\chi(10^3_{29},0)\cong\chi(K_7,0)$,	$\chi(10^3_{12},0)\cong\chi(7_6,0)$,	$\chi(10^3_{37},0)\cong\chi(3^*_1\sharp 5_2,0)$,
$\chi(10^3_{13},0)\cong\chi(7_7,0)$,	$\chi(10^3_{34},0)\cong\chi(4_1\sharp 4_1,0)$.	

So we omit 9_{46}, 9^3_2, 9^3_1, 10^3_A, 10^3_{62}, 10^3_{60}, 10^3_{12}, 10^3_{13} from the sequence and we show the pictures of $K_1,\ldots,K_7$ in Appendix. For the rest of the links, we can see, by calculating the Alexander polynomials, that the manifolds are different from each other except the cases shown in the following box.

$\Delta(\chi(L,0))$	L	$\Delta(\chi(L,0))$	L
1	$O < 9^3_{10} < 10^3_{19}$	$7-6+3-1$	$8_{10} < 10_{143}$
$3+0-1$	$10^3_{31} < 10^3_{30}$	$11-6+1$	$10_{137} < 10^3_{34}$
$3-2+1$	$8_{20} < 8^3_6 < 10^3_B < 10^3_{43}$	$11-8+2$	$8_{14} < 9_8$
$5-2$	$6_1 < 8^3_5 < 10^3_{42} < 10^3_{29}$	$11-9+5-1$	$9_{20} < 10_{149}$
$5-4+1$	$8_{21} < 10_{136}$	$13-10+5-1$	$8_{18} < 9_{24}$
$5-4+3-1$	$8_5 < 10_{141}$	$15-12+5-1$	$9_{28} < 9_{29}$
$7-5+3-1$	$8_9 < 10_{155}$	$23-18+7-1$	$9_{40} < 10_{59}$

For the above links, we replace the 0-framed three component links with the 0-framed knots, as we described before, and we substitute the fifth root of unity $q = e^{\frac{2\pi i}{5}}$ for the Jones polynomials of the knots, and we have the following box.

$L \to K$	$V_K(q)$	$L \to K$	$V_K(q)$
O	1	8_9	$8+4q^2+4q^3$
$9^3_{10} \to K_1$	$3q-3q^2+q^3$	10_{155}	$5+4q+7q^3$
$10^3_{19} \to K_3$	$2+2q-2q^2+4q^3$	8_{10}	$-5-6q-8q^3$
$10^3_{31} \to K_2$	$-3-q-q^2-4q^3$	10_{143}	$3-5q+4q^2-q^3$
$10^3_{30} \to K_5$	$5+3q^2+3q^3$	10_{137}	$5+4q+7q^3$
8_{20}	$2+q+3q^3$	$10^3_{34} \to 4_1\sharp 4_1$	$8+4q^2+4q^3$
$8^3_6 \to 3_1\sharp 3^*_1$	$4+q^2+q^3$	8_{14}	$1-5q+5q^2-5q^3$
$10^3_B \to 10_{140}$	$2q-q^2$	9_8	$-6-4q-q^2-8q^3$
$10^3_{43} \to K_4$	$1+4q-3q^2+4q^3$	9_{20}	$-10+2q-6q^2-5q^3$
6_1	$4+2q+2q^2+3q^3$	10_{149}	$-13-4q-5q^2-12q^3$
$8^3_5 \to 9_{46}$	$1+q-2q^2+q^3$	8_{18}	$15+8q^2+8q^3$
$10^3_{42} \to K_6$	$8+4q^2+4q^3$	9_{24}	$13+q+7q^2+10q^3$
$10^3_{29} \to K_7$	$-3+3q-4q^2$	9_{28}	$-5-10q+3q^2-12q^3$
8_{21}	$2-3q+3q^2-q^3$	9_{29}	$-3+9q-6q^2+6q^3$
10_{136}	$-5-2q-3q^2-4q^3$	9_{40}	$-7-13q+5q^2-19q^3$
8_5	$-5+q-4q^2-q^3$	10_{59}	$1+13q-6q^2+13q^3$
10_{141}	$3+6q^2+2q^3$		

If $\chi(K,0) \cong \chi(K',0)$, then we have $V_K(q) = V_{K'}(q)$ or $\overline{V_{K'}(q)}$ by Kirby and Melvin's theorem [9],p.530. So the above calculations show that all the manifolds are different from each other.

Case 2. $H_1(\chi(L,0)) \cong 0$.

We enumerate the manifolds $\chi(L,0)$ with $H_1(\chi(L,0)) \cong 0$. The links with this condition are the following:

$2^2_1 < 7^2_1 < 7^2_2 < 8^2_8 < 8^2_7 < 9^2_{19} < 9^2_{50} < 9^2_{52} < 9^2_{54} < 9^2_{35} < 9^2_{21} < 9^2_{34} < 9^2_{39} < 9^2_{42} < 9^2_{11} < 9^2_{12} < 9^2_3 < 10^2_{134} < 10^2_{140} < 10^2_{136} < 10^2_{139} < 10^2_{44} < 10^2_{138} < 10^2_{88} < 10^2_{137} <$

$10^2_{98} < 10^2_{42} < 10^2_{41} < 9^2_8 < 10^4_2 < 10^2_{85} < 10^2_{101} < 10^2_{118} < 10^2_{183} < 10^2_{117} < 10^2_{114} < 10^2_{180} < 10^2_{64} < 10^2_{86} < 10^2_{119} < 10^2_{62} < 10^2_{43} < 10^2_{87} < 10^2_{115} < 10^2_{109} < 10^2_{45} < 10^2_{121} < 10^2_{39} < 10^4_1 < 10^2_{112}$.

We see that $\chi(10^4_2, 0) \cong \chi(7^2_2, 0)$, $\chi(10^4_1, 0) \cong \chi(7^2_1, 0)$, $\chi(10^2_{140}, 0) \cong \chi(9^2_{50}, 0)$, $\chi(10^2_{134}, 0) \cong \chi(9^2_{54}, 0)$ and remove 10^4_2, 10^4_1, 10^2_{140}, 10^2_{134} from our sequence. The last two homeomorphisms can be shown as follows: they are Mazur manifolds of types $\chi(10^2_{140}, 0) \cong W^+(0,1)$, $\chi(9^2_{50}, 0) \cong W^+(1,0)$, $\chi(10^2_{134}, 0) \cong W^+(-1,0)$, $\chi(9^2_{54}, 0) \cong W^+(0,-1)$ and we have $W^+(0,1) \cong W^+(1,0)$ and $W^+(-1,0) \cong W^+(0,-1)$ by Proposition 1 in [1]. For the rest of the links L, we calculate $\tau_5(\chi(L,0))$, the 3-manifold invariant of Witten and Reshetikhin-Turaev in Kirby and Melvin's paper [9], which is described as

$$\tau_5(\chi(L,0)) = C(1 + (s^{-2} + 2 + s^2) \cdot f_L),$$

where $C = \frac{4}{5}\sin^2\frac{\pi}{5}$, $s = e^{\frac{\pi i}{5}}$, and f_L is a complex number determined by L. We show the values of f_L for the rest of the links L in the following box.

L	f_L	L	f_L
2^2_1	1	10^2_{98}	$-3+13s-12s^2+6s^3$
7^2_1	$2+3s-2s^2+s^3$	10^2_{42}	$6-5s+5s^3$
7^2_2	$2s+2s^2-s^3$	10^2_{41}	$-4+5s^2-5s^3$
8^2_8	$1+5s^2-5s^3$	9^2_8	$-1+4s-s^2-2s^3$
8^2_7	$1+5s-5s^2$	10^2_{85}	$-4+10s^2-10s^3$
9^2_{19}	$3-4s+s^2+2s^3$	10^2_{101}	$-5+2s+7s^2-6s^3$
9^2_{50}	$2s-3s^2-s^3$	10^2_{118}	$-9+15s-10s^2$
9^2_{52}	$-3+3s-2s^2+s^3$	10^2_{183}	$1-5s+5s^2-5s^3$
9^2_{54}	$-3s+2s^2-s^3$	10^2_{117}	$-4+10s^2-10s^3$
9^2_{35}	$7s-8s^2+4s^3$	10^2_{114}	$-5+2s+12s^2-11s^3$
9^2_{21}	$-2+6s-4s^2+2s^3$	10^2_{180}	$1+5s-5s^2+5s^3$
9^2_{34}	$-3+3s+3s^2-4s^3$	10^2_{64}	$-4+10s-10s^2+5s^3$
9^2_{39}	$-3+8s-7s^2+s^3$	10^2_{86}	$-4+15s-15s^2+5s^3$
9^2_{42}	$-4+5s+5s^2-5s^3$	10^2_{119}	$-3+18s-17s^2+6s^3$
9^2_{11}	$-3+8s-7s^2+s^3$	10^2_{62}	$1-5s+10s^2-5s^3$
9^2_{12}	$-3s+7s^2-6s^3$	10^2_{43}	$-8s+12s^2-6s^3$
9^2_3	$-2+6s-4s^2+2s^3$	10^2_{87}	$2-7s+13s^2-9s^3$
10^2_{136}	$2-2s+3s^2+s^3$	10^2_{115}	$1-10s+15s^2-10s^3$
10^2_{139}	$-4+5s$	10^2_{109}	$1-10s+15s^2-10s^3$
10^2_{44}	$-4+10s-10s^2+5s^3$	10^2_{45}	$-8+13s-7s^2+s^3$
10^2_{138}	$2s-3s^2+4s^3$	10^2_{121}	$-4+15s^2-15s^3$
10^2_{88}	$-3+13s-12s^2+6s^3$	10^2_{39}	$6-5s+5s^3$
10^2_{137}	$1-5s+5s^2$	10^2_{112}	$7-7s-2s^2+6s^3$

We conclude that the manifolds are different from each other except the following 11 cases:

$\tau_5(\chi(9^2_{50},0)) \equiv \tau_5(\chi(10^2_{136},0),\ \tau_5(\chi(9^2_{52},0)) \equiv \tau_5(\chi(9^2_{54},0)),$
$\tau_5(\chi(9^2_{21},0)) \equiv \tau_5(\chi(9^2_{3},0)),\ \tau_5(\chi(9^2_{34},0)) \equiv \tau_5(\chi(9^2_{12},0)),\ \tau_5(\chi(9^2_{39},0)) \equiv \tau_5(\chi(9^2_{11},0)),$
$\tau_5(\chi(10^2_{139},0)) \equiv \tau_5(\chi(10^2_{183},0)),\ \tau_5(\chi(10^2_{44},0)) \equiv \tau_5(\chi(10^2_{64},0)),$
$\tau_5(\chi(10^2_{88},0)) \equiv \tau_5(\chi(10^2_{98},0)), \tau_5(\chi(10^2_{42},0)) \equiv \tau_5(\chi(10^2_{39},0)),$
$\tau_5(\chi(10^2_{85},0)) \equiv \tau_5(\chi(10^2_{117},0)),\ \tau_5(\chi(10^2_{115},0)) \equiv \tau_5(\chi(10^2_{109},0)),$

where $\alpha \equiv \beta$ means $\alpha = \beta$ or $\alpha = \bar{\beta}$ for complex values α, β and we say α is equivalent to β. For these manifolds, we compute $\tau_7(\chi(L,0))$ in [9], which is described as

$$\tau_7(\chi(L,0)) = C(1 - 2[3] + [3]^2 + [2] \cdot f_L),$$

where $C = \frac{4}{7}\sin^2\frac{\pi}{7}$, $[2] = s^{-1} + s$, $[3] = s^{-2} + 1 + s^2$, $s = e^{\frac{\pi i}{7}}$, and f_L is a complex number determined by L. We show the values of f_L for these manifolds in the following box.

L	f_L	L	f_L
9^2_{50}	$-2 - s + s^2 + 4s^3 + s^5$	10^2_{183}	$1 - 7s + 10s^2 - 8s^3 + s^4 - 3s^5$
10^2_{136}	$4 + s + 5s^2 + s^3 + 2s^4 - 7s^5$	10^2_{44}	$8 - 4s^2 + 13s^3 - 13s^4 + 11s^5$
9^2_{52}	$-3 + s - 2s^2 + s^3 + 2s^4$	10^2_{64}	$1 - 11s^2 + 13s^3 - 20s^4 + 11s^5$
9^2_{54}	$5 - s + 8s^2 - 3s^3 - 6s^5$	10^2_{88}	$11 - 6s - 9s^2 + 22s^3 - 26s^4 + 14s^5$
9^2_{21}	$2s - 7s^2 + 3s^3 - 4s^4 + 3s^5$	10^2_{98}	$18 - 6s - 2s^2 + 15s^3 - 26s^4 + 14s^5$
9^2_{3}	$7 + 2s + 3s^3 + 3s^4 - 4s^5$	10^2_{42}	$-6 + 10s^2 - 15s^3 + 15s^4 - 17s^5$
9^2_{34}	$-3 + 8s + 5s^2 - 6s^3 + 9s^4 - 14s^5$	10^2_{39}	$1 + 14s - 11s^2 + 13s^3 + s^4 - 10s^5$
9^2_{12}	$-2 - 8s + 15s^2 - 17s^3 + 7s^4 - 6s^5$	10^2_{85}	$-13 + 24s^2 - 43s^3 + 43s^4 - 31s^5$
9^2_{39}	$-3 + 8s - 16s^2 + 15s^3 - 12s^4 + 7s^5$	10^2_{117}	$1 + 38s^2 - 50s^3 + 50s^4 - 38s^5$
9^2_{11}	$4 + 8s - 9s^2 + 15s^3 - 5s^4$	10^2_{115}	$22 - 42s + 59s^2 - 43s^3 + 15s^4 - 3s^5$
10^2_{139}	$1 + 3s^2 - 8s^3 + 8s^4 - 10s^5$	10^2_{109}	$15 - 49s + 52s^2 - 50s^3 + 15s^4 - 3s^5$

Since the values for the above 11 pairs are not equivalent to each other, we have the enumeration of 3-manifolds in Case 2.

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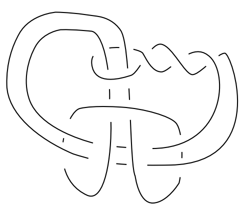
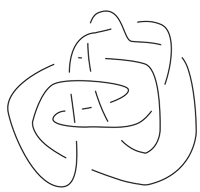
Appendix

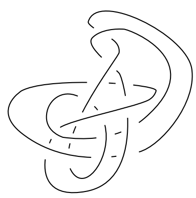
We show the pictures of $K_1, \ldots, K_7$ for Case 1 in Section 4.



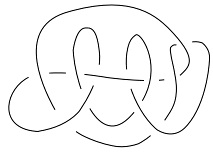
K_1



K_2

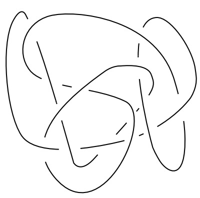


K_3

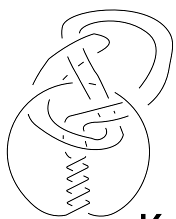


K_4

K_5



K_6



K_7