The Alexander polynomials of immersed concordant links *

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ABSTRACT

For two links bounded by an immersed concordance of annuli, we relate the number of the double points of the immersed annuli to the beta-distance and the torsion multi-variable Alexander polynomials of the links. This result unifies T. Kanenobu's announced result on the Alexander polynomial of a link with unlinking number one and the author's result on the Alexander polynomials of concordant links.

Keywords: Link, Immersed concordance, Concordance, 4-dimensional clasp distance, Gordian distance, beta-distance, Torsion Alexander polynomial

1. Introduction

As a link an oriented link L in the 3-sphere S^3 with components $K_i\,(i=1,2,\ldots,r)$ for $r \geq 2$ is considered. A link L is *immersed concordant* to a link L' with components $K'_i\,(i=1,2,\ldots,r)$ if there is a smooth proper immersion

$$\alpha : rS^1 \times I \to S^3 \times I, \quad I=[0,1]$$

such that $\alpha(rS^1\times 0) = L\times 0$, $\alpha(rS^1 \times 1) = L' \times 1$, and the image $\alpha(S_\alpha)$ of the singular set S_α of α consists of finitely many transversely intersected double points in $S^3 \times (0,1)$, where rS^1 denotes the disjoint union of r copies of S^1. It is understood that the pairs $(S^3 \times 0, L \times 0)$ and $(S^3 \times 1, L' \times 1)$ are identified with the orientation reversing pair $(-S^3, -L)$ and the oriented pair (S^3, L'), respectively. The image $\mathcal{A}$ of α consisting of r immersed annuli is called an *immersed concordance* from L to L'. The links L and L' are said to be *concordant* if α is an embedding.[1] It can be seen

*Dedicated to Professor Francisco González-Acuña on his 70th birthday.

[1]In an earlier work, "concordant"is called "cobordant".

that for any links L and L' with the same number r of components, there is always an immersed concordance from L to L'. Let $\mathcal{A} = \cup_{i=1}^{r} \mathcal{A}_i$ where $\mathcal{A}_i$ is the immersed annulus in $\mathcal{A}$ connecting K_i to K'_i by re-indexing the components K'_i $(i = 1, 2, \ldots, r)$ of L'. The *double point number* $c(\mathcal{A})$ of $\mathcal{A}$ is defined to be the cardinality of $\alpha(S_\alpha)$. The 4-*dimensional clasp distance* $c^4(L, L')$ between L and L' are defined by

$$c^4(L, L') = \min\{c(\mathcal{A}) |\ \mathcal{A} \,\text{is an immersed concordance from}\, L \,\text{to}\, L'.\}$$

In particular, the 4-*dimensional clasp number* $c^4(L)$ of L is the 4-dimensional clasp distance $c^4(L, L')$ with L' a trivial link. By definition of the 4-dimensional clasp number, the inequalities

$$|c^4(L) - c^4(L')| \leq c^4(L, L') \leq c^4(L) + c^4(L')$$

are established for all links L, L' with the same number of components. It is obvious that L and L' are concordant if and only if $c^4(L, L') = 0$. The *Gordian distance* $u(L, L')$ between L and L' is the minimal number of crossing changes needed to obtain L' from L. In particular, the *unlinking number* $u(L)$ of L is the Gordian distance $u(L, L')$ with L' a trivial link. By definition of the unlinking number, the inequalities

$$|u(L) - u(L')| \leq u(L, L') \leq u(L) + u(L')$$

are established for all links L, L' with the same number of components. If a link L' is obtained from a link L by n crossing changes, then we have an immersed concordance $\mathcal{A}$ from L to L' with $c(\mathcal{A}) = n$ by considering the trace of the crossing changes from L to L' in $S^3 \times I$. Thus, we have

$$u(L, L') \geq c^4(L, L').$$

Let $E = E(L) = \mathrm{cl}(S^3 \backslash N(L))$ be the compact exterior of L for a tubular neighborhood $N(L)$ of L in S^3. The first homology $H_1(E)$ is a free abelian group of rank r and has a basis given by choices of oriented meridians of L. We explain here the graded multi-variable Alexander polynomials of L ([6, Chapter 7]). Let $\Lambda = Z[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$ be the integral group ring $Z[H_1(E)]$ where t_i represents a meridian of K_i. The *link module* of L is the Λ-module $H_1(\tilde{E})$ for the universal free abelian covering $\tilde{E} \to E$, namely the regular covering associated with the Hurewicz epimorphism $\pi_1(E) \to H_1(E)$. By the Noetharian property of Λ, the link module $H_L = H_1(\tilde{E})$ is finitely generated over Λ. For a finitely generated Λ-module H, let $\beta(H)$ denote the Λ-rank of H, namely the $Q(\Lambda)$-dimension of the $Q(\Lambda)$-vector space $H \bigotimes_\Lambda Q(\Lambda)$ for the quotient field $Q(\Lambda)$ of Λ. Let TH be the Λ-torsion part of H, which is also finitely generated over Λ by the Noetharian property of Λ. Let DH be the Λ-submodule of TH consisting of all elements x such that $f_i x = 0$ for coprime elements $f_i \in \Lambda$ $(i = 1, 2, \ldots, s)$ for some $s \geq 2$. Let $BH = H/TH$ be the Λ-torsion-free part of H. For any non-negative integer d, the *d-th characteristic polynomial* $\Delta^{(d)}(H)$ is defined in [6] by using a Λ-presentation matrix of H. The zeroth polynomial $\Delta^{(0)}(H)$ is simply denoted by $\Delta(H)$. It is standard to use the notation $f \doteq f'$ for elements $f, f' \in \Lambda$ which are equal up to the units $\pm t_1^{n_1} t_2^{n_2} \dots t_r^{n_r}$ $(n_1, n_2, \dots, n_r \in Z)$ of Λ. Let $\Delta^T(H) = \Delta(TH)$. For convenience, we list some known facts on properties of the graded characteristic polynomials which are often used in this paper.

Facts on properties of the graded characteristic polynomials.

(1) For every short Λ-exact sequence $0 \to T' \to T \to T'' \to 0$ of finitely generated torsion Λ-modules T', T, T'', we have $\Delta(T) \doteq \Delta(T')\Delta(T'')$.

(2) For every finitely generated Λ-module H, we have $\Delta^{(d)}(H) = 0$ for all $d < \beta(H)$ and $\Delta^{(d)}(TH) \doteq \Delta^{(d+\beta(H))}(H)$ for all $d \geq 0$.

(3) For every finitely generated Λ-module H with $H = DH$, we have $\Delta(H) \doteq \pm 1$.

(4) Let $(\tilde{X}, \tilde{X}')$ be a regular Z^r-covering of a compact polyhedral pair (X, X'). If $H_d(X, X') = 0$, then the Λ-module $T_d = H_d(\tilde{X}, \tilde{X}')$ for $\Lambda = Z[Z^r]$ is a finitely generated torsion Λ-module with $\Delta(T_d)(1, 1, \dots, 1) = \pm 1$.

Explanation on how Facts (1)-(4) are obtained. For the proof of (1), see [5, Lemma 2.4]. For the proof of (2), apply [6, 7.2.7 (3)] to the natural short exact sequence $0 \to TH \to H \to BH \to 0$. The proof of (3) is given by the induction on the minimal number $m(H)$ of Λ-generators of H. In fact, if $m(H) = 1$, then there are coprime elements $f_i \in \Lambda$ $(i = 1, 2, \dots, s)$ with $s \geq 2$ and a Λ-generator x of H such that $f_i x = 0$ for all i, so that there is a Λ-epimorphism $\Lambda/f_i\Lambda \to H$. By Fact (1), $\Delta(H)$ is a factor of the element f_i for every i. Using that the elements f_i $(i = 1, 2, \dots, s)$ are coprime, we see that $\Delta(H) \doteq \pm 1$. If $m(H) \geq 2$, then we choose a Λ-submodule $H' \subset H$ with $e(H') = 1$ and $m(H/H') = m(H) - 1$. By induction on $m(H)$, we have $\Delta(H') \doteq \Delta(H/H') \doteq \pm 1$, so that $\Delta(H) \doteq \Delta(H')\Delta(H/H') \doteq \pm 1$ by Fact (1) as desired. The proof of (4) is given in [5, Lemma 2.1] (see also [2] for a similar argument). This concludes the explanation.

The identity $\Delta^T(H) \doteq \Delta^{(\beta(H))}(H)$ given by Fact (3) is always a non-zero element of Λ and useful in our argument. Let $\beta(L) = \beta(H_L)$ which is called the *β-rank* of L. Then $0 \leq \beta(L) \leq r - 1$. The d-th *Alexander polynomial* $\Delta_L^{(d)} = \Delta_L^{(d)}(t_1, t_2, \dots, t_r)$ of L is just the d-th characteristic polynomial $\Delta^{(d)}(H_L)$. Then $\Delta_L^{(d)} = 0$ for all d with $0 \leq d \leq \beta(L)$ and the $\beta(L)$-th Alexander polynomial $\Delta_L^{(\beta(L))}$ is equal to the zeroth Alexander polynomial $\Delta^T(H_L)$, which we call the *torsion Alexander polynomial* of L and denoted by Δ_L^T. The zeroth Alexander polynomial Δ_L is usually called the *Alexander polynomial* of L. By definition, $\Delta_L \neq 0$ if and only if $\beta(L) = 0$. The *boundary polynomial* $b(L) \in \Lambda$ of L is the zeroth characteristic polynomial of the torsion Λ-module $H_1(\partial\tilde{E})$. The *peripheral polynomial* $p(L) \in \Lambda$ of L is the zeroth characteristic polynomial of the image $\mathrm{im}H_1(\partial\tilde{E})$ of the natural Λ-homomorphism $H_1(\partial\tilde{E}) \to H_1(\tilde{E})$ which is a torsion Λ-module. The peripheral polynomial $p(L)$ of L

is a factor of the boundary polynomial $b(L)$ as well as a factor of the torsion Alexander polynomial Δ_L^T, and will be shown in Lemma 2.1 (4) to be a concordance invariant. Let $\mathcal{A}$ be an immersed concordance from L to L'. For every i ($i = 1, 2, \ldots, r$), let

$$c_i = c_i(\mathcal{A}) = \sum_{1 \le j \le r, j \ne i} c(\mathcal{A}_i, \mathcal{A}_j),$$

where $c(\mathcal{A}_i, \mathcal{A}_j)$ denotes the number of the intersection double points of the immersed annuli $\mathcal{A}_i$ and $\mathcal{A}_j$. We note that

$$c_i - \sum_{1 \le j \le r, j \ne i} |\mathrm{Link}(K_i, K_j) - \mathrm{Link}(K'_i, K'_j)|$$

is a non-negative even integer by the well-known properties on the intersection number and the linking number. For an i, the pair $(K_i, L \backslash K_i)$ is *algebraically split* if the linking number $\mathrm{Link}(K_i, K_j) = 0$ for all $j \ne i$, and the link L is *algebraically split* if $(K_i, L \backslash K_i)$ is algebraically split for every i. For example, if $(K_i, L \backslash K_i)$ is algebraically non-split and $(K'_i, L' \backslash K'_i)$ is algebraically split, then we have $c_i \geq 1$. Let

$$\delta_i = \begin{cases} c_i & \text{if neither } (K_i, L\backslash K_i) \text{ nor } (K'_i, L'\backslash K'_i) \\ & \qquad \text{or is algebraically split,} \\ c_i - 1 & \text{if either, but not both, of } (K_i, L\backslash K_i) \\ & \qquad \text{or } (K'_i, L'\backslash K'_i) \text{ is algebraically split,} \\ \max\{0, c_i - 2\} & \text{if both } (K_i, L\backslash K_i) \text{ and } (K'_i, L'\backslash K'_i) \\ & \qquad \text{are algebraically split.} \end{cases}$$

The *peripheral polynomial* of the immersed concordance $\mathcal{A}$ is the element

$$p(\mathcal{A}) = \prod_{i=1}^{r} (t_i - 1)^{\delta_i} \in \Lambda.$$

For any non-zero element $f \in \Lambda$ we can write it as

$$f = f(t_1, t_2, \ldots, t_r) = \prod_{i=1}^{r} (t_i - 1)^{n_i} g$$

for some integers $n_i \geq 0$ ($i = 1, 2, \ldots, r$) and an element $g \in \Lambda$ which does not have any element $t_i - 1$ ($i = 1, 2, \ldots, r$) as a factor. Then the *elementary factor* $e(f)$ of f is defined by $e(f) = \prod_{i=1}^{r} (t_i - 1)^{\varepsilon_i}$ where $\varepsilon_i \in \{0, 1\}$ is the modulo 2 reduction of n_i. If $f = 0$, then we take $e(f) = 0$. The *peripheral polynomial* $p(L, L')$ of the link pair (L, L') is defined to be the elementary factor $p(L, L') = e(p(\mathcal{A}))$ of the peripheral polynomial $p(\mathcal{A})$ of the immersed concordance $\mathcal{A}$, which will be shown later to be independent of a choice of $\mathcal{A}$ and determined only by the link pair (L, L'). For a Laurent polynomial $f = f(t_1, t_2, \ldots, t_r) \in \Lambda$, we denote

$$f^* = f(t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1}) \in \Lambda.$$

It is a classical result due to Blanchfield [1] that $\Delta_L^{(d)} \doteq \Delta_L^{(d)*}$ for all d, so that in particular, $\Delta_L \doteq \Delta_L^*$ and $\Delta_L^T \doteq (\Delta_L^T)^*$. In [5], the author showed the following result in a slightly different form, whose difference is explained soon after Lemma 2.1. (see also Theorem 5.4.2 in [13].)

Concordance Theorem. If L and L' are concordant, then we have

$$\Delta_L^T \Delta_{L'}^T \doteq f f^*$$

for an element $f \in \Lambda$.

T. Kanenobu announced the following result in [4].

Unlinking Number One Theorem. For an r-component link L with $u(L) = 1$, we have $\Delta_L = 0$ for $r > 2$, and for $r = 2$ there is an element $f \in \Lambda$ such that

$$\Delta_L \doteq \begin{cases} f f^* & \text{if } \mathrm{Link}(L) = \pm 1 \\ (t_1 - 1)(t_2 - 1) f f^* & \text{if } \mathrm{Link}(L) = 0. \end{cases}$$

The *gap factor* of a link pair (L, L') is the elementary factor $g(L, L') = e(\Delta_L^T \Delta_{L'}^T)$ of the product $\Delta_L^T \Delta_{L'}^T$. The *β-distance* of a link pair (L, L') is the number

$$\beta(L, L') = |\beta(L) - \beta(L')| \leq r - 1.$$

The following theorem is our main theorem.

Theorem 1.1 We have

$$u(L, L') \geq c^4(L, L') \geq \beta(L, L') \geq 0.$$

Further, if $c^4(L, L') = \beta(L, L')$, then we have

$$\Delta_L^T \Delta_{L'}^T \doteq g(L, L') f f^* \quad \text{and} \quad g(L, L') = e(p(L) p(L, L') p(L'))$$

for an element $f \in \Lambda$.

It is noted that another estimate of the Gordian distance $u(L, L')$ from below is given by different invariants of a link in [7]. As a corollary to Theorem 1.1, we obtain the following result.

4-dimensional Clasp Number One Theorem For an r-component link L with $c^4(L) = 1$, we have $\Delta_L = 0$ for $r > 2$, and for $r = 2$ there is an element $f \in \Lambda$ such that

$$\Delta_L \doteq \begin{cases} ff^* & \text{if } \mathrm{Link}(L) = \pm 1 \\ (t_1 - 1)(t_2 - 1)ff^* & \text{if } \mathrm{Link}(L) = 0. \end{cases}$$

We note that if $r = 2$, $c^4(L) = 1$ (or $u(L) = 1$) and $\mathrm{Link}(L) \neq 0$, then we have $\mathrm{Link}(L) = \pm 1$ because of the following inequalities

$$u(L) \geq c^4(L) \geq |\mathrm{Link}(L)|.$$

Then Unlinking Number One Theorem follows directly from 4-dimensional Clasp Number One Theorem because if $\Delta_L \neq 0$ and $u(L) = 1$, then $r = 2$ and $u(L) = c^4(L) = \beta(L, O) = \beta(O) = 1$.

In § 2, some computations of the boundary polynomial and the peripheral polynomial are done and it is explained how Concordance Theorem and 4-dimensional Clasp Number One Theorem are derived from Theorem 1.1. Throughout § 3, the proof of Theorem 1.1 is done. In § 4, we show three corollaries to Theorem 1.1 and examples on the unlinking number, the 4-dimensional clasp number and the peripheral polynomials.

2. Computing the boundary polynomial and the peripheral polynomials

The following lemma is used to compute the boundary polynomial and the peripheral polynomial of a link.

Lemma 2.1 We have the following (1)-(3).

(1) The boundary polynomial $b(L)$ of a link L is given by $b(L) = \prod_{i=1}^{r}(t_i - 1)^{\varepsilon_i}$ where $\varepsilon_i \in \{0, 1\}$ $(i = 1, 2, \ldots, r)$, and $\varepsilon_i = 1$ if and only if the pair $(K_i, L \backslash K_i)$ is algebraically split.

(2) If $\beta(L) = 0$, then $p(L) = b(L)$.

(3) If $\beta(L) = r - 1$, then L is algebraically split, $b(L) = \prod_{i=1}^{r}(t_i - 1)$ and $p(L) = 1$.

(4) Let $f_L \in \Lambda$ be the factor of Δ_L^T obtained from Δ_L^T by removing all the non-unit prime factors $g_1 \in \Lambda$ of Δ_L^T with $g_1(1, 1, \ldots, 1) = 1$. Then $p(L)$ is a factor of f_L. If L is concordant to a link L', then we have $f_L \doteq f_{L'}$ and $p(L) \doteq p(L')$.

In (4), we take an element $\tilde{f}_L \in \Lambda$ with $\Delta_L^T = f_L \tilde{f}_L$. Then $\tilde{f}_L(1, 1, \ldots, 1) = \pm 1$. Counting the identity $f_L \doteq f_{L'}$ given by (4) and also known by [5], we obtain the identities $\tilde{f}_L \doteq \tilde{f}_{L'} hh^*$ and $\Delta_L^T \doteq \Delta_{L'}^T hh^*$ for an element $h \in \Lambda$ with $h(1, 1, \ldots, 1) =$

± 1 from Concordance Theorem. This last identity is an exact form of Concordance Theorem given in [5].

Proof To see (1), we note that the lift of the torus component T_i of ∂E around K_i to $\partial\tilde{E}$ consists of components homeomorphic to $S^1 \times R$ or R^2 according to whether $(K_i, L\backslash K_i)$ is algebraically split or not. Hence we have $H_1(\partial\tilde{E}) \cong \bigoplus_{k=1}^{s} \Lambda/(t_{i_k}-1)$ for the members i_k $(k = 1, 2, \ldots, s)$ in $\{1, 2, \ldots, r\}$ with $\varepsilon_{i_k} = 1$ and $b(L) = \prod_{k=1}^{s}(t_{i_k}-1)$. To see (2), since

$$\beta(L) = 0,\ H_0(\tilde{E}) = DH_0(\tilde{E}) \cong \Lambda/(t_1 - 1, t_2 - 1, \ldots, t_r - 1),\ DH_1(\partial\tilde{E}) = 0,$$

Blanchfield duality [1] implies that there is a $(t_1, t_2, \ldots, t_r)$-anti monomorphism

$$BH_2(\tilde{E}, \partial\tilde{E}) \to \mathrm{hom}(BH_1(\tilde{E}), \Lambda) = 0$$

and there is a $(t_1, t_2, \ldots, t_r)$-anti isomorphism

$$TH_2(\tilde{E}, \partial\tilde{E})/DH_2(\tilde{E}, \partial\tilde{E}) \cong \mathrm{hom}(TH_0(\tilde{E})/DH_0(\tilde{E}), Q(\Lambda)/\Lambda) = 0.$$

Thus, we see that $H_2(\tilde{E}, \partial\tilde{E}) = DH_2(\tilde{E}, \partial\tilde{E})$. Then, because $DH_1(\partial\tilde{E}) = 0$, the boundary map $\partial_* : H_2(\tilde{E}, \partial\tilde{E}) \to H_1(\partial\tilde{E})$ is the zero-map. Hence the natural homomorphism $H_1(\partial\tilde{E}) \to H_1(\tilde{E})$ is injective, so that we have $p(L) = b(L)$. To see (3), let $\Lambda_{(1)} = Z[t, t^{-1}]$ be the t-variable Laurent polynomial ring, and $\tilde{E}_{(1)} \to E$ the infinite cyclic covering associated with the epimorphism $H_1(E) \to Z$ sending every oriented meridian of L in ∂E to $1 \in Z$. Then the $\Lambda_{(1)}$-rank $\beta_{(1)}(L)$ of the $\Lambda_{(1)}$-module $H_1(\tilde{E}_{(1)})$ is $r - 1$ (cf. [6, 7.3.12]), which implies that the torsion Alexander polynomial $\Delta_{(1)}$ of the $\Lambda_{(1)}$-module $H_1(\tilde{E}_{(1)})$ has $\Delta_{(1)}(1) = \pm 1$. Thus, the natural homomorphism

$$H_1(\partial\tilde{E}_{(1)}) = (\Lambda_{(1)}/(t-1))^r \to H_1(\tilde{E}_{(1)})$$

must be trivial, meaning that the longitude of K_i in ∂E is the boundary of a 2-chain in E and hence $\mathrm{Link}(K_i, L\backslash K_i) = 0$, for every i. Thus, L is algebraically split. By (1), we have $b(L) = \prod_{i=1}^{r}(t_i - 1)$. To see (4), we note that the peripheral polynomial $p(L)$ is a factor of Δ_L^T which does not have any non-unit prime element $g_1 \in \Lambda$ with $g_1(1, 1, \ldots, 1) = 1$. Hence $p(L)$ is a factor of f_L. To see that $p(L)$ is a concordance invariant, let $E(\mathcal{A}) = \mathrm{cl}(S^3 \times I\backslash N(\mathcal{A}))$ be the exterior of a concordance $\mathcal{A}$ from L to L' where $N(\mathcal{A})$ denotes a tubular neighborhood of $\mathcal{A}$ in $S^3 \times I$. Let $R_{\mathcal{A}} = \mathrm{cl}(\partial E(\mathcal{A})\backslash(E(L) \times 0 \cup E(L') \times 1))$ which is homeomorphic to both $(\partial E(L)) \times I$ and $(\partial E(L')) \times I$. The universal free abelian coverings $\tilde{E}(L) \to E(L)$ and $\tilde{E}(L') \to E(L')$ extend to the universal free abelian covering $\tilde{E}(\mathcal{A}) \to E(\mathcal{A})$. Since the natural homomorphism $H_1(\partial\tilde{E}(L)) \to H_1(\tilde{R}_{\mathcal{A}})$ is a Λ-isomorphism, the natural homomorphism $H_1(\tilde{E}(L)) \to H_1(\tilde{E}(\mathcal{A}))$ sends the image $\mathrm{im}H_1(\partial\tilde{E}(L))$ of the natural homomorphism $H_1(\partial\tilde{E}(L)) \to H_1(\tilde{E}(L))$ onto the image $\mathrm{im}H_1(\tilde{R}_{\mathcal{A}})$ of the natural homomorphism $H_1(\tilde{R}_{\mathcal{A}}) \to H_1(\tilde{E}(\mathcal{A}))$. Let $\Delta_{\mathcal{A}}^T$ be the zeroth polynomial of

$TH_1(\tilde{E}(\mathcal{A}))$, and $p(\mathcal{A})$ the zeroth polynomial of $\mathrm{im}H_1(\tilde{R}_{\mathcal{A}})$. Let $f_{\mathcal{A}} \in \Lambda$ be the factor of $\Delta^T_{\mathcal{A}}$ obtained from $\Delta^T_{\mathcal{A}}$ by removing all the non-unit prime factors $g_1 \in \Lambda$ of $\Delta^T_{\mathcal{A}}$ with $g_1(1,1,\dots,1) = 1$. We have

$$H_*(E(\mathcal{A}), E(L) \times 0) = H_*(E(\mathcal{A}), E(L') \times 1) = 0.$$

By Fact (4), the Λ-module $T_* = H_*(\tilde{E}(\mathcal{A}), \tilde{E}(L) \times 0)$ is a torsion Λ-module with $\Delta(T_*)(1,1,\dots,1) = \pm 1$. In the homology exact sequence of the pair $(\tilde{E}(\mathcal{A}), \tilde{E}(L) \times 0)$, let T'_2 be the image of the connecting homomorphism

$$\partial_* : T_2 = H_2(\tilde{E}(\mathcal{A}), \tilde{E}(L) \times 0) \to TH_1(\tilde{E}(L) \times 0),$$

and T'_1 the image of

$$j_* : TH_1(\tilde{E}(\mathcal{A}) \times 0) \to H_1(\tilde{E}(\mathcal{A}), \tilde{E}(L) \times 0) = T_1.$$

By Fact (1), $\Delta(T'_d)(1,1,\dots,1) = \pm 1$ $(d = 1,2)$. From the short Λ-exact sequence

$$0 \to TH_1(\tilde{E}(L) \times 0)/T'_2 \to TH_1(E(\mathcal{A})) \to T'_1 \to 0$$

and Fact (1), the identity $f_L \doteq f_{\mathcal{A}}$ is obtained. The zeroth polynomial of the Λ-module $T''_2 = T'_2 \cap \mathrm{im}H_1(\partial\tilde{E}(L))$ is a unit of Λ, because by Fact (1) the zeroth polynomial $\Delta(T''_2)$ is a common factor of the zeroth polynomials $\Delta(T'_2)$ and $p(L)$ which is coprime. Since the Λ-epimorphism $\mathrm{im}H_1(\partial\tilde{E}(L)) \to \mathrm{im}H_1(\tilde{R}_{\mathcal{A}})$ induces a Λ-isomorphism

$$\mathrm{im}H_1(\partial\tilde{E}(L))/T''_2 \cong \mathrm{im}H_1(\tilde{R}_{\mathcal{A}}),$$

we have $p(L) \doteq p(\mathcal{A})$ by Fact (1). Similarly, we have $f_{L'} \doteq f_{\mathcal{A}}$ and $p(L') \doteq p(\mathcal{A})$. Thus, we have $f_L \doteq f_{L'}$ and $p(L) \doteq p(L')$. This completes the proof.

The following lemma shows that the peripheral polynomial of a link pair is calculable by the linking numbers modulo 2 of the immersed concordant links and hence is independent of a choice of an immersed concordance.

Lemma 2.2 The peripheral polynomial $p(L, L')$ of a pair (L, L') has the form

$$p(L, L') = \prod_{i=1}^{r} (t_i - 1)^{\varepsilon_i},$$

where $\varepsilon_i \in \{0,1\}$ is determined by

$$\varepsilon_i = \begin{cases} \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K'_i, L'\backslash K'_i) \pmod 2 & \text{if neither } (K_i, L\backslash K_i) \\ & \text{nor } (K'_i, L'\backslash K'_i) \text{ is algebraically split,} \\ \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K'_i, L'\backslash K'_i) - 1 \pmod 2 & \text{if either, but not} \\ & \text{both, of } (K_i, L\backslash K_i) \text{ or } (K'_i, L'\backslash K'_i) \text{ is algebraically split,} \\ 0 & \text{if both } (K_i, L\backslash K_i) \text{ and } (K'_i, L'\backslash K'_i) \text{ are algebraically split.} \end{cases}$$

Proof As observed in the introduction, we have

$$c_i \equiv \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K_i', L'\backslash K_i') \pmod 2.$$

In particular, if both $(K_i, L\backslash K_i)$ and $(K_i', L'\backslash K_i')$ are algebraically split, then c_i is even and $\max\{0, c_i - 2\} \equiv 0 \pmod 2$. This completes the proof.

The following corollary is useful in deriving 4-dimensional Clasp Number One Theorem from Theorem 1.1.

Corollary 2.3 Assume that $c^4(L, L') = \beta(L, L') = r - 1$ with $\beta(L) = 0$ and $\beta(L') = r - 1$. Then we have the following (1) and (2).
(1) If L is algebraically split, then we have $g(L, L') = \prod_{i=1}^{r}(t_i - 1)$.
(2) If $\mathrm{Link}(K_i, L\backslash K_i)$ is odd for every i, then we have $g(L, L') = 1$.

Proof By the assumption of (1), we have $p(L) = \prod_{i=1}^{r}(t_i - 1)$ and $p(L') = 1$ by (1), (2) and (3) of Lemma 2.1, and $p(L, L') = 1$ by Lemma 2.2, obtaining (1) by Theorem 1.1. For the assumption of (2), we have $p(L) = p(L') = 1$ by (1), (2) and (3) of Lemma 2.1, and $p(L, L') = 1$ by Lemma 2.2, obtaining (2) by Theorem 1.1.

We are in a position to explain how Concordance Theorem and 4-dimensional Clasp Number One Theorem are derived from Theorem 1.1.

Deriving Concordance Theorem and 4-dimensional Clasp Number One Theorem from Theorem 1.1. To derive Concordance Theorem from Theorem 1.1, assume that L and L' are concordant. Then we have $c^4(L, L') = \beta(L, L') = 0$ and $p(L, L') = 1$. Since $p(L)$ is a concordance invariant, we have $p(L) \doteq p(L')$, so that $g(L, L') = 1$. Thus, Concordance Theorem is obtained from Theorem 1.1. To derive 4-dimensional Clasp Number One Theorem from Theorem 1.1, let L be an r-component link with $c^4(L) = 1$. By Theorem 1.1, we have

$$1 \geq \beta(L, O) \geq \beta(O) - \beta(L) = r - 1 - \beta(L)$$

for an r-component trivial link O. Hence if $r > 2$, then $\beta(L) \geq r - 2 > 0$ and $\Delta_L = 0$. For $r = 2$, assume $\Delta_L \neq 0$. Then $\beta(L) = 0$ and $c^4(L) = \beta(L, O) = 1$. Since $\Delta^T(O) = 1$ and $|\mathrm{Link}(L)| \leq c^4(L) = 1$, the desired result follows directly from Theorem 1.1 and Corollary 2.3.

3. Proof of Theorem 1.1

Throughout this section, the proof of Theorem 1.1 will be done. Let $\mathcal{A}$ be an immersed concordance from L to L' with $n = c(\mathcal{A}) = c^4(L, L')$. We assume that

$$\beta(L') = \beta(L) + b$$

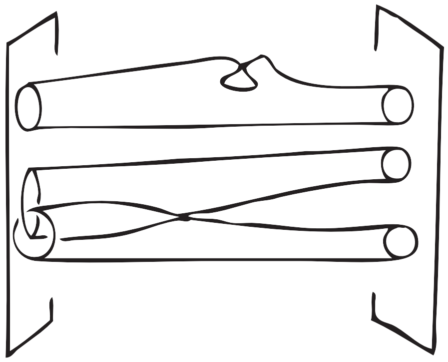

Figure 1: An immersed concordance $\mathcal{A}$

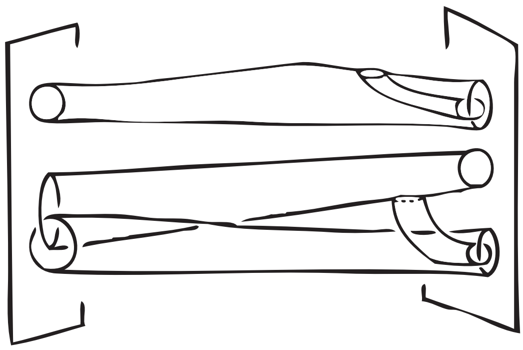

Figure 2: The surface F

for $b = \beta(L, L') = |\beta(L) - \beta(L')|$. Since every transversely intersected double point in $\mathcal{A}$ is topologically represented by the cone vertex of a Hopf link, we slide the double points in $\mathcal{A}$ into $L' \times 1$. Then we obtain from $\mathcal{A}_i$ a connected oriented proper planar surface F_i in $S^3 \times I$ for every i such that $F_i \cap F_j = \emptyset$ for every $i \neq j$ and the boundary ∂F of the surface $F = \cup_{i=1}^{r} F_i$ is given by $\partial F = (-L) \times 0 \cup L'' \times 1$, where L'' is the link which is regarded as a connected sum $L'' = L' \#_{j=1}^{s} S_j^H \#_{k=1}^{u} U_k^H$ for thelink L' and the Hopf links S_j^H $(j = 1, 2, \dots, s)$ and U_k^H $(k = 1, 2, \dots, u)$ with the following conditions (1)-(3) (see Fig. 1 for an illustration of an immersed concordance $\mathcal{A}$ and Fig. 2 for an illustration of the surface F):

(1) $s + u = n$.

(2) The connected sum connects one component of L' and one component of S_j^H for every j and one component of L' and one component of U_k^H for every k.

(3) The components of every Hopf link S_j^H except the arc used for the connected sum belong to the same component of F and the components of every Hopf link U_k^H except the arc used for the connected sum belong to distinct components of F.

We attach a (± 1)-framed 2-handle h_j^2 to $S^3 \times 1$ to make a (± 1)-twist on S_j^H producing a two-component trivial link. Then the surgery of $S^3 \times 1$ on the 2-handles h_j^2 $(j = 1, 2, \dots, s)$ change the link L'' into the split link of the links $(L' \#_{k=1}^u U_k^H) \times 1$ and an s-component trivial link $O^s \times 1$. We assume that the attaching core circle of the 2-handle h_j^2 has linking number 0 with S_j^H. By adding s mutually disjoint disks in $S^3 \times 1$ bounded by $O^s \times 1$ to the surfaces F_i $(i = 1, 2, \dots, r)$, we obtain planar surfaces F_i' $(i = 1, 2, \dots, r)$ in a 4-manifold $X = S^3 \times I \cup_{j=1}^s h_j^2$ with boundary $\partial X = S^3 \times 0 \cup S^3 \times 1$ such that the surface $F' = \cup_{i=1}^r F_i'$ has boundary $\partial F' = (-L) \times 0 \cup (L' \#_{k=1}^u U_k^H) \times 1$. Further, we add the connected sum bands B_k $(k = 1, 2, \dots, u)$ used for the connected sum $(L' \#_{k=1}^u U_k^H) \times 1$ in $S^3 \times 1$ to the surface F'. The resulting surface $F'' = F' \cup_{k=1}^u B_k$ in X has r components such that the boundary $\partial F''$ is given by $\partial F'' = (-L) \times 0 \cup L'_+ \times 1$, where $L'_+ \times 1$ is a link in $S^3 \times 1$ split into the link $L' \times 1$ and the Hopf links $U_k^H \times 1$ $(k = 1, 2, \dots, u)$. After deforming F'' into a proper surface in X, let $Y = \mathrm{cl}(X \backslash N(F''))$ be the compact 4-manifold for a tubular neighborhood $N(F'') = F'' \times D^2$ of F'' in X. For the link L, let $L = L^{(0)} \cup L^{(1)}$, where $L^{(0)}$ is the sublink of L consisting of all components K such that $(K, L \backslash K)$ is algebraically split and $L^{(1)}$ is the sublink of L consisting of the other components. Applying the same notation to the link L'_+, we have $L'_+ = (L'_+)^{(0)} \cup (L'_+)^{(1)}$ with $(L'_+)^{(0)} = (L')^{(0)}$ and $(L'_+)^{(1)} = (L')^{(1)} \cup_{k=1}^u U_k^H$. Let M_0 be the 0-surgery manifold of S^3 along $L^{(0)}$, and

$$M = \mathrm{cl}(M_0 \backslash N(L^{(1)}))$$

for a tubular neighborhood $N(L^{(1)})$ of $L^{(1)}$. Similarly, let M_0' be the 0-surgery manifold of S^3 along $L^{(0)} = (L'_+)^{(0)}$, and

$$M' = \mathrm{cl}(M_0' \backslash N((L')^{(1)})), \quad M'_+ = \mathrm{cl}(M_0' \backslash N((L'_+)^{(1)})).$$

Then we have a connected sum decomposition $M'_+ = M' \#_{k=1}^u E(U_k^H)$ for the Hopf link exteriors $E(U_k^H)$ $(k = 1, 2, \dots, u)$. Since ∂Y is a torus sum of link exteriors $E(L)$ and $E(L'_+)$, and the product $F'' \times S^1$, we construct a 4-manifold

$$W = M \times [-1, 0] \cup Y \cup M'_+ \times [1, 2]$$

pasting $M \times [-1, 0]$ and Y along $E(L) \times 0$ and pasting Y and $M'_+ \times [1, 2]$ along $E(L'_+) \times 1$. Let G be a possibly disconnected proper surface in W obtained from a push off of F'' by attaching disks to the components of $L^{(0)}$ and $(L'_+)^{(0)} = (L')^{(0)}$. See Fig. 3 for an illustration of the surface G.

Then the boundary ∂W of the 4-manifold W is a torus sum of $M \times (-1)$, $M'_+ \times 2$ and the product $G \times S^1$. If G has a 2-sphere component, namely if there is an $S^2 \times S^1$ component in $G \times S^1$, then we paste the 4-manifold $B^3 \times S^1$ with B^3 the 3-ball to it and hence assume that $G \times S^1$ has no $S^2 \times S^1$ components. For simplicity, we take $M = M \times (-1)$, $P = G \times S^1$, and $M'_+ = M'_+ \times 2$. By construction, the maximal free abelian covering $\tilde{E}(L)$ over $E(L)$ extends to a free abelian covering

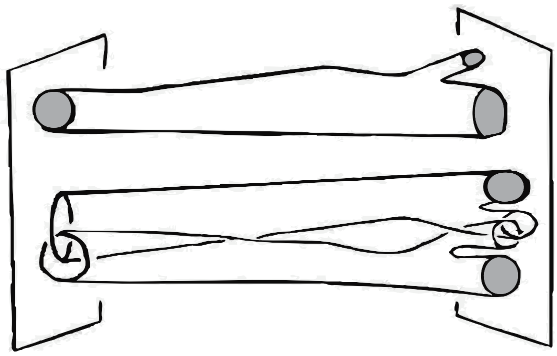

Figure 3: The surface G

$(\tilde{W};\tilde{M},\tilde{P},\tilde{M}'_+)$ over $(W;M,P,M'_+)$. For a compact submanifold pair (W',W'') of W, the homology $H_*(\tilde{W}',\tilde{W}'')$ of the lift $(\tilde{W}',\tilde{W}'')$ to $\tilde{W}$ forms a finitely generated Λ-module. Let $\beta_*(\tilde{W}',\tilde{W}'')$ denote the Λ-rank of $H_*(\tilde{W}',\tilde{W}'')$. At first, we note that $H_1(\tilde{P})$ is a finitely generated torsion Λ-module, which is seen from the following lemma, because every component P_1 of P satisfies the assumption of the lemma after a suitable permutation of the indexes of t_j $(j=1,2,\ldots,r)$.

Lemma 3.1 Let G_1 be a 2-sphere with $n(\geq 2)$ holes, and $P_1 = G_1 \times S^1$. Let the fundamental group $\pi_1(P_1)$ have a presentation with generators x_i $(i=1,2,\ldots,n)$ and relators $r_1 = x_1x_1^{-1}$ and $r_i = x_1x_ix_1^{-1}x_i^{-1}$ $(i=2,\ldots,n)$ where x_1 and $x_i(i=2,3,\ldots,n)$ are represented by a loop $p\times S^1$ $(p\in G_1)$ and loops in $G_1\times 1$ $(1\in S^1)$, respectively. Let $\tilde{P}_1 \to P_1$ be the covering associated with a homomorphism $\gamma: \pi_1(P_1)\to Z^r$ such that $\gamma_*(x_1)=t_1$ and $\gamma_*(x_i)=u_i$ is a monomial with coefficient $+1$ in t_j $(j=1,2,\ldots,r)$ such that $u\neq t_1^k$ for any integer k. Then we have the Alexander polynomial $\Delta(H_1(\tilde{P}_1)) \doteq (t_1-1)^{n-2}$. In particular, $H_1(\tilde{P}_1)$ is a finitely generated torsion Λ-module.

Proof We use the Fox free calculus [3] (see also [6, 7.1.5]). The Jacobian (n,n)-matrix $(\partial r_i/\partial x_j)^{\gamma_*}$ with entries in Λ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1-u_2 & t_1-1 & 0 & \ldots & 0 \\ 1-u_3 & 0 & t_1-1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1-u_n & 0 & 0 & \ldots & t_1-1 \end{pmatrix}.$$

This matrix is a presentation matrix of a Λ-module $\mathcal{M}$ admitting a short exact sequence

$$0 \to H_1(\tilde{P}) \to \mathcal{M} \to \varepsilon(\Lambda) \to 0,$$

where $\varepsilon(\Lambda) = (t_1 - 1, t_2 - 1, \ldots, t_r - 1)$ denotes the fundamental ideal which is the torsion-free Λ-module of rank one obtained as the kernel of the homomorphism $\Lambda \to Z$ sending every t_i to 1. By Fact (2), the Alexander polynomial $\Delta(H_1(\tilde{P}_1)) \doteq \Delta^{(1)}(\mathcal{M})$, which is obtained from the second elementary ideal of the Jacobian matrix by taking the smallest principal ideal. The second elementary ideal is

$$((t_1 - 1)^{n-1},\ (1 - u_i)(t - 1)^{n-2}\ (i = 1, 2, 3, \ldots, n)).$$

Hence we have $\Delta(H_1(\tilde{P})) \doteq (t_1 - 1)^{n-2}$, completing the proof of Lemma 3.1.

We need the following computations of Λ-ranks, where the identity $s + u = n$ is noted.

Lemma 3.2
(1) $\beta_1(\partial\tilde{W}) = 2\beta(L) + b + u$.
(2) $\beta_1(\tilde{W}) \leq \beta(L)$.
(3) $\beta_1(\partial\tilde{W}) \leq \beta_2(\tilde{W}, \partial\tilde{W}) + \beta_1(\tilde{W})$.
(4) $\beta_2(\tilde{W}, \partial\tilde{W}) = n + u + \beta_1(\tilde{W})$.

Proof Since every solid torus $S^1 \times D^2$ attaching to $E(L)$ or $E(L'_+)$ used to construct M or M'_+ lifts to the disjoint union of copies of $R^1 \times D^2$ in $\tilde{M}$ or $\tilde{M}'_+$, we have $\beta(L) = \beta_1(\tilde{M})$ and

$$\beta(L'_+) = \beta_1(\tilde{M}'_+) = \beta_1(\tilde{M}') + u = \beta(L') + u = \beta(L) + b + u.$$

Since ∂W is a torus sum of M, M'_+ and $P = G \times S^1$ and $H_1(\tilde{P})$ is a finitely generated torsion Λ-module, we have

$$\beta_1(\partial\tilde{W}) = \beta_1(\tilde{M}) + \beta_1(\tilde{M}'_+) = 2\beta(L) + b + u,$$

showing (1). Since $H_1(W, M) = 0$, we see from Fact (4) that $\beta_1(\tilde{W}, \tilde{M}) = 0$, which implies that $\beta(L) = \beta_1(\tilde{M}) \geq \beta_1(\tilde{W})$, showing (2). Using the exact sequence

$$H_2(\tilde{W}, \partial\tilde{W}) \to H_1(\partial\tilde{W}) \to H_1(\tilde{W}),$$

we have (3). To see (4), we note by Blanchfield duality [1] that $\beta_2(\tilde{W}, \partial\tilde{W}) = \beta_2(\tilde{W})$ and $\beta_3(\tilde{W}) = \beta_1(\tilde{W}, \partial\tilde{W})$. We see that $\beta_1(\tilde{W}, \partial\tilde{W}) = 0$ by the exact sequence

$$H_1(\tilde{W}, \tilde{M}) \to H_1(\tilde{W}, \partial\tilde{W}) \to H_0(\partial\tilde{W}, \tilde{M})$$

of the triad $(\tilde{W}, \partial\tilde{W}, \tilde{M})$ and $\beta_1(\tilde{W}, \tilde{M}) = \beta_0(\partial\tilde{W}, \tilde{M}) = 0$. The Euler characteristic $\chi(W)$ of W is given by $\chi(W) = n + u$. In fact, W is a union of $M \times [-1, 0]$, Y, $M'_+ \times [1, 2]$ and some copies of $B^3 \times S^1$ pasting along $E(L) \times 0$, $E(L'_+) \times 1$ and some copies of $S^2 \times S^1$ whose Euler characteristics are 0 except Y. Hence we have

$\chi(W) = \chi(Y) = \chi(X) - \chi(F'')$, for $Y = \mathrm{cl}(X \backslash N(F''))$. Since X is obtained from $S^3 \times I$ by attaching the 2-handles h_j^2 $(j = 1, 2, \ldots, s)$, we have $\chi(X) = s$. Because the surface F'' is obtained from F by attaching s disks along the boundaries and by spanning u bands where F is homeomorphic to r annuli with n open disks removed, we have $\chi(F'') = -n + s + u - 2u = -2u$. Thus, we have $\chi(W) = s + 2u = n + u$ as desired. Then since $\beta_d(\tilde{W}) = 0\,(d \neq 1, 2)$, it follows that

$$\beta_2(\tilde{W}) - \beta_1(\tilde{W}) = n + u,$$

showing (4). This completes the proof.

By Lemma 3.2, we have the following inequalities:

$$\begin{aligned} 2\beta(L) + b + u \;\; &= \beta_1(\partial\tilde{W}) \quad (\text{by}(1)) \\ &\leq \beta_2(\tilde{W}, \partial\tilde{W}) + \beta_1(\tilde{W}) \\ &= 2\beta_1(\tilde{W}) + n + u \quad (\text{by}(3)\text{and}(4)) \\ &\leq 2\beta(L) + n + u \quad (\text{by}(2)). \end{aligned}$$

This means the inequality $\beta(L, L') = b \leq n = c^4(L, L')$, giving the first half of Theorem 1.1. Further, if this inequality is replaced by the equality, namely, if $\beta(L, L') = b = n = c^4(L, L')$, then we have the identity:

$$(*) \quad \beta_2(\tilde{W}, \partial\tilde{W}) + \beta_1(\tilde{W}) = \beta_1(\partial\tilde{W}).$$

We need the following lemma:

Lemma 3.3 (Exactness Lemma) Under the identity (*), the natural exact sequence

$$H_2(\tilde{W}, \partial\tilde{W}) \xrightarrow{\partial_*} H_1(\partial\tilde{W}) \xrightarrow{i_*} H_1(\tilde{W})$$

induces an exact sequence

$$TH_2(\tilde{W}, \partial\tilde{W}) \xrightarrow{\partial'_*} TH_1(\partial\tilde{W}) \xrightarrow{i'_*} TH_1(\tilde{W}).$$

This lemma is shown by the observation that the identity (*) implies that the Λ-homomorphism

$$BH_2(\tilde{W}, \partial\tilde{W}) \to BH_1(\partial\tilde{W})$$

induced from ∂_* is a monomorphism, where we note that the Λ-module BH induced from every Λ-module H is a torsion-free Λ-module by definition. Let $H = H_1(\partial\tilde{W})$, and T' the image of $\partial'_* : TH_2(\tilde{W}, \partial\tilde{W}) \to TH_1(\partial\tilde{W}) = TH$. By [5, Theorem 3.1], we have

$$\Delta^T(H) = \Delta(TH) \doteq \Delta(T')\Delta(T')^*.$$

To calculate $\Delta^T(H)$, we need the following lemma.

Lemma 3.4
(1) $\Delta^T_L \doteq p(L)\Delta^T(H_1(\tilde{M}))$, $\quad \Delta^T_{L'} \doteq p(L')\Delta^T(H_1(\tilde{M}'))$.
(2) $\Delta^T(H) \doteq p(\mathcal{A})\Delta^T(H_1(\tilde{M}))\Delta^T(H_1(\tilde{M}'))$.
(3) $\Delta(H_1(\tilde{P})) \doteq p(\mathcal{A})$.

Proof To see (1), let V be the solid tori used to construct M from $E(L)$. Since $H_1(\tilde{V}) = 0$, the homology exact sequences of the pairs $(\tilde{E}(L), \partial\tilde{E}(L))$, $(\tilde{M}, \tilde{V})$ connected by the excision isomorphisms $H_1(\tilde{E}(L), \partial\tilde{E}(L)) \cong H_1(\tilde{M}, \tilde{V})$ induce the following short exact sequence

$$0 \to \mathrm{im}H_1(\partial\tilde{E}(L)) \to H_1(\tilde{E}(L)) \to H_1(\tilde{M}) \to 0.$$

Similarly, we have the following short exact sequence.

$$0 \to \mathrm{im}H_1(\partial\tilde{E}(L')) \to H_1(\tilde{E}(L')) \to H_1(\tilde{M}') \to 0.$$

Applying Fact (1) to the short exact sequences restricted to the torsion parts of these short exact sequences, we have (1). To see (2), let $S = \partial M \cup \partial M'_+$ which is a union of tori. Then we have $H_1(\tilde{S}) = 0$. By excision, we have

$$H_1(\partial\tilde{W}, \tilde{S}) \cong H_1(\tilde{M}, \partial\tilde{M}) \bigoplus H_1(\tilde{P}, \tilde{S}) \bigoplus H_1(\tilde{M}'_+, \partial\tilde{M}'_+).$$

Since there is a natural exact sequence

$$H_1(\tilde{S}) = 0 \to H_1(\partial\tilde{W}) \to H_1(\partial\tilde{W}, \tilde{S}) \to H_0(\tilde{S})$$

and $H_0(\tilde{S}) = DH_0(\tilde{S})$, we see from Facts (1) and (3) that

$$\Delta^T(H) = \Delta^T(H_1(\partial\tilde{W})) \doteq \Delta^T(H_1(\partial\tilde{W}, \tilde{S})).$$

Similarly, we have

$$\Delta^T(H_1(\tilde{M}, \partial\tilde{M})) \doteq \Delta^T(H_1(\tilde{M})),$$
$$\Delta^T(H_1(\tilde{M}'_+, \partial\tilde{M}'_+)) \doteq \Delta^T(H_1(\tilde{M}'_+)) \doteq \Delta^T(H_1(\tilde{M}')).$$

By lemma 3,1, note that $H_1(\tilde{P}, \tilde{S}) = TH_1(\tilde{P}, \tilde{S})$ and $H_1(\tilde{P}) = TH_1(\tilde{P})$. By the exact sequence of the pair $(\tilde{P}, \tilde{S})$, we have $\Delta(H_1(\tilde{P}, \tilde{S})) \doteq \Delta(H_1(\tilde{P}))$. Thus, we have

$$\Delta^T(H) \doteq \Delta(H_1(\tilde{P}))\Delta^T(H_1(\tilde{M}))\Delta^T(H_1(\tilde{M}')).$$

Assuming (3), we complete the proof of (2). To see (3), we note that the component G_i of G obtained from $\mathcal{A}_i$ is a 2-sphere with $c_i + 2$ holes if $(K_i, L\backslash K_i)$ nor $(K'_i, L'\backslash K'_i)$

is algebraically split, a 2-sphere with c_i+1 holes if either, but not both, of $(K_i, L\backslash K_i)$ or $(K_i', L'\backslash K_i')$ is algebraically split, and a 2-sphere with c_i holes if both $(K_i, L\backslash K_i)$ and $(K_i', L'\backslash K_i')$ are algebraically split. In the last case, recall that c_i is always even and we omitted the case $c_i = 0$. Then, (3) is confirmed from Lemma 3.1. This completes the proof of Lemma 3.4.

By Lemmas 3.3 and 3.4, we have

$$\begin{aligned}
&p(\mathcal{A})^2\Delta_L^T\Delta_{L'}^T \\
&\doteq p(\mathcal{A})^2 p(L)p(L')\Delta^T(H_1(\tilde{M}))\Delta^T(H_1(\tilde{M}')) \\
&\doteq p(\mathcal{A})p(L)p(L')\Delta^T(H) \\
&\doteq p(\mathcal{A})p(L)p(L')\Delta(T')\Delta(T')^* \\
&\doteq e(p(L)p(L,L')p(L'))gg^*
\end{aligned}$$

for some non-zero element $g \in \Lambda$, so that

$$\Delta^T(L)\Delta^T(L') \doteq g(L,L')ff^*,$$
$$g(L,L') = e(p(L)p(\mathcal{A})p(L')) = e(p(L)p(L,L')p(L'))$$

for some $f \in \Lambda$. This completes the proof of Theorem 1.1.

4. Corollaries to Theorem 1.1 and related examples

We consider a pair (L, L') of an r-component algebraically split link L with $\Delta_L \neq 0$ and an r-component completely split link L' with components K_i' $(i = 1, 2, \ldots, r)$ such that $c^4(L, L') \leq r-1$. Since $\beta(L, L') = \beta(L') = r-1$, we see from Theorem 1.1 that $c^4(L, L') = \beta(L, L') = \beta(L') = r-1$. For every $r \geq 2$, there are lots of such pairs (L, L'). For example, from any given knot K and any $r \geq 2$, A. Shimizu constructed in [12] an algebraically split link L of r components, called a *lassoed link* associated with K, such that $\Delta_L \neq 0$ and $u(L, L') = r-1$ for the completely splittable link L' consisting of K and the $(r-1)$-component trivial link, where we have $u(L, L') = c^4(L, L') = \beta(L, L') = r-1$ by Theorem 1.1. In the following corollary, we will concern with a relationship between the Alexander polynomials of such a link pair (L, L').

Corollary 4.1 Let (L, L') be a pair of an r-component algebraically split link L with $\Delta_L \neq 0$ and an r-component completely split link L' with components K_i' $(i = 1, 2, \ldots, r)$ such that $c^4(L, L') \leq r-1$. Let $\delta_{K_i'}$ be the product of all mutually distinct prime factors a of $\Delta_{K_i'}$ such that $a^* \doteq a$ and the exponent of a in the prime decomposition of $\Delta_{K_i'}$ is odd. Then we have $c^4(L, L') = \beta(L, L') = \beta(L') = r-1$ and

$$\Delta_L \doteq \prod_{i=1}^{r}(t_i - 1)\prod_{j=1}^{r}\delta_{K_j'}ff^*$$

for an element $f \in \Lambda$, where the Alexander polynomial $\Delta_{K'_i}$ is regarded as an integral Laurent polynomial in t_i for every i.

Proof By Theorem 1.1, we have the identities $c^4(L, L') = \beta(L, L') = \beta(L') = r - 1$ and hence $\Delta_L \Delta^T_{L'} \doteq g(L, L')gg^*$ for an element $g \in \Lambda$. By Lemmas 2.1 and 2.2, $p(L) = \prod_{i=1}^r (t_i - 1)$, $p(L') = 1$ and $p(L, L') = 1$, so that $g(L, L') = p(L) = \prod_{i=1}^r (t_i - 1)$. Using that $\Delta^*_{K'_i} \doteq \Delta_{K'_i}$, we see that $\Delta_{K'_i} \doteq \delta_{K'_i} d_i d_i^*$ for an element $d_i \in \Lambda$ and hence

$$\Delta^T_{L'} \doteq \prod_{j=1}^r \Delta_{K'_j} \doteq \delta_{L'} dd^*$$

for $\delta_{L'} = \prod_{j=1}^r \delta_{K'_j}$ and $d = \prod_{j=1}^r d_j$. If aa^* divides bb^* for non-zero elements $a, b \in \Lambda$, then the quotient bb^*/aa^* is written as the form cc^* for an element $c \in \Lambda$, which can be easily shown by taking the prime decompositions of a and b. Since $\Delta^T_{L'}(1, 1, \ldots, 1) = \pm 1$, we see that $\Delta^T_{L'}$ is coprime with $g(L, L')$ and hence $\Delta^T_{L'}$ divides gg^*. Let hh^* be the quotient gg^*/dd^* for an element $h \in \Lambda$. Then $\Delta_L \delta_{L'} \doteq g(L, L')hh^*$. For every prime factor a of $\delta_{L'}$, the product a^2 divides hh^* because a divides hh^* and satisfies $a^* \doteq a$. Hence $\delta^2_{L'} \doteq \delta_{L'}\delta^*_{L'}$ divides hh^*. The quotient is denoted by the product ff^* for an element $f \in \Lambda$. Then we have $\Delta_L \doteq g(L, L')\delta_{L'} ff^*$, completing the proof.

For a link L with r components, the fraction

$$\hat{\Delta}_L(t) = \Delta_L(t)/(t-1)^{r-1} = \Delta_L(t, t, \ldots, t)/(t-1)^{r-2}$$

is known to be an integral polynomial, called the *Hosokawa polynomial* of L (see [6]). We know that the multiplicity of the factor $t - 1$ in $\hat{\Delta}_L(t)$ is always even. In fact, the statement that $\sigma_1(L) \equiv \hat{\kappa}_1(L) \pmod 2$ in Lemma 5.7 of [8] implies this assertion. Then the following corollary is obtained from Theorem 1.1.

Corollary 4.2 If the one-variable Alexander polynomial $\Delta_L(t)$ of a link L with $r(\geq 2)$ components is a non-zero polynomial, then

$$u(L) \geq c^4(L) \geq \beta(L, O) = r - 1.$$

Further if $c^4(L) = r - 1$, then the Hosokawa polynomial $\hat{\Delta}_L(t)$ has the form

$$\hat{\Delta}_L(t) \doteq f(t)f(t^{-1})$$

for an integral polynomial $f(t)$ in t up to multiplications of $\pm t^i$ $(i \in Z)$.

Recall that the group order of the first homology $H_1(M_L)$ of the double branched covering space M_L of S^3 branched along L coincides with the absolute value $|\Delta_L(-1)|$ by taking the group order of an infinite abelian group to be 0 (see [6]).

Corollary 4.3 If the first homology $H_1(M_L)$ of the double branched covering space M_L of S^3 branched along a link L with $r(\geq 2)$ components is a finite abelian group, then

$$u(L) \geq c^4(L) \geq \beta(L, O) = r - 1.$$

Further if $c^4(L) = r - 1$, then the group order $|H_1(M_L)|$ has the form

$$|H_1(M_L)| = 2^{r-1}n^2$$

for an integer n.

In the case of a link L with $r = 2$ and $u(L) = 1$ (implying $c^4(L) = 1$), the latter half of Corollary 4.3 has been observed by P. Kohn [9]. The following example concerns a computation on the unlinking number, the 4-dimensional clasp number and the peripheral polynomials for the links illustrated in Fig. 4 together with the notation of "linkinfo"[2] in the bracket, whose Alexander polynomials are given in [11].

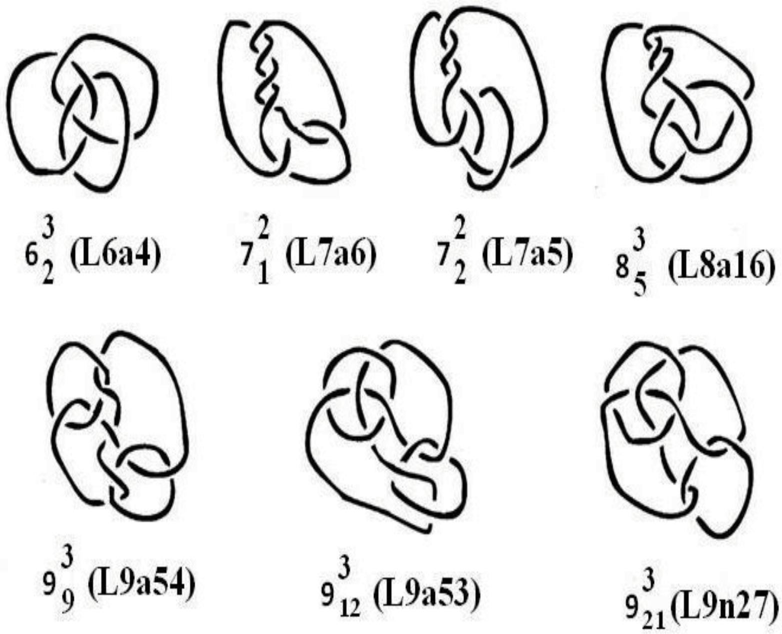

Figure 4: Links used in Example 4.4

Example 4.4 (1) Let $L = 6^3_2\,(L6a4)$ (the Borromean rings), and O the 3-component trivial link. Since $u(L) \leq 2$, $\beta(L) = 0$ and $\beta(O) = 2$, we have $u(L) = c^4(L) = \beta(L, O) = 2$ by the first half of Theorem 1.1. We have $p(O) = 1$. Since L is

[2]http://www.indiana.edu/ linkinfo/.

algebraically split, we see from Lemma 2.2 that $p(L,O)=1$. The latter half assertion of Theorem 1.1 is confirmed as follows:

$$\Delta_L \doteq g(L,O) = (t_1-1)(t_2-1)(t_3-1) = p(L).$$

We can also see $u(L)=c^4(L)=2$ from the inequality $u(L)\leq 2$ and the following fact that the link L is algebraically split, but not link-homotopically trivial (see [10]).

(2) Let $L=7_1^2\,(L7a6)$, and O the 2-component trivial link. Then we have

$$\Delta_L \doteq 1-t_1-t_2+(1-t_1-t_2)t_1t_2+(t_1t_2)^2$$

which cannot be written as $g(L,O)ff^*$. Hence by Theorem 1.1 we obtain

$$u(L)\geq c^4(L) > \beta(L,O)=1.$$

Since $u(L)\leq 2$, we have $u(L)=c^4(L,O)=2$.

(3) Let $L=7_2^2\,(L7a5)$, and O the 2-component trivial link. Then we have

$$\Delta_L \doteq 1-t_1-t_2+(3-t_1-t_2)t_1t_2+(t_1t_2)^2.$$

Since $u(L)\leq 1$, $\beta(L)=0$ and $\beta(O)=1$, we see from the first half of Theorem 1.1 that $u(L)=c^4(L)=\beta(L,O)=1$. The latter half assertion of Theorem 1.1 is confirmed as follows:

$$\Delta_L \doteq (t_1+t_2^{-1}-1)(t_1^{-1}+t_2-1), \quad g(L,O)=p(L)=p(O)=p(L,O)=1.$$

(4) Let $L=8_5^3\,(L8a16)$, and O the 3-component trivial link. Then we have $u(L,L^0)=1$ for the split link L^0 consisting of the unoriented $(2,4)$-torus link $T_{2,4}$ and a trivial knot. Note that $\beta(L)=0$, $\beta(L^0)=1$ and

$$\Delta_{L^0}^T \doteq t_1t_2+1, \quad \Delta_L=(t_1-1)(t_2-1)(t_3-1)(t_1t_2+1),$$

where t_1 and t_2 are represented by the meridians of the sublink $T_{2,4}$ of L^0. Then we have $u(L,L^0)=c^4(L,L^0)=\beta(L,L^0)=1$ by the first half of Theorem 1.1 and the latter half assertion of Theorem 1.1 is confirmed as follows:

$$\Delta_L\Delta_{L^0}^T \doteq g(L,L^0)(t_1t_2+1)(t_1^{-1}t_2^{-1}+1)$$

where $g(L,L^0)=e(p(L)p(L,L^0)p(L^0))=(t_1-1)(t_2-1)(t_3-1)$. Since the sublink $T_{2,4}$ of L^0 has the linking number ± 2 and the corresponding sublink L_T in L has the linking number ± 1, and $\mathrm{Link}(K,L_T)=0$ for the component $K=L\backslash L_T$, we see from Lemma 2.2 that $p(L,L^0)=(t_1-1)(t_2-1)$. Using that $b(L^0)=t_3-1$ and the longitude of O bounds a disk in the exterior $E(L^0)$, we see that $p(L^0) = 1$, so that $p(L) = t_3 - 1$. From the linking number of $T_{2,4}$, we obtain that $u(L^0) = 2$ and hence $u(L) \leq 3$. On the other hand, we have $\beta(L, O) = \beta(O) = 2$. Examining the form of Δ_L, we see from Theorem 1.1 that $c^4(L) > \beta(L, O) = 2$. Thus, we have that $u(L) = c^4(L) = 3$.

(5) Let $L = 9^3_9\,(L9a54)$, and O the 3-component trivial link. Then we have

$$\Delta_L \doteq (t_1 - 1)(t_2 - 1)(t_3 - 1)(t_1^2 - t_1 + 1),$$

which cannot be written as $g(L, O)ff^*$. Hence by Theorem 1.1 we have

$$u(L) \geq c^4(L) > \beta(L, O) = 2.$$

Since $u(L) \leq 3$, we have $u(L) = c^4(L) = 3$.

(6) $L = 9^3_{12}\,(L9a53)$ which is an algebraically split link, and O the 3-component trivial link. Since $u(L) \leq 2$ and $\beta(L, O) = 2$, we have $u(L) = c^4(L) = \beta(L, O) = 2$ and

$$\Delta_L \doteq g(L, O)(t_1 - 1)(t_1^{-1} - 1)$$

where $g(L, O) = (t_1 - 1)(t_2 - 1)(t_3 - 1) = p(L)$ and $p(L, O) = p(O) = 1$.

(7) Let $L = 9^3_{21}\,(L9n27)$ which is an algebraically split link, and O the 3-component trivial link. Then we have $\Delta_L = 0$, $u(L) \leq 1$, $\beta(O) = 2$, $\beta(L) = 1$, $\Delta^T(L) \doteq t_1 - 1$ and $p(O) = 1$. By the first half of Theorem 1.1, we have $u(L) = c^4(L) = \beta(L, O) = 1$. The latter half of Theorem 1.1 is confirmed as follows:

$$\Delta_L^T \doteq g(L, O) = e(p(L)p(L, O)p(O)) = t_1 - 1.$$

By Lemma 2.2, $p(L, O) = 1$, so that $p(L) = t_1 - 1$.

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