The Alexander polynomials of immersed concordant links

Akio Kawauchi

Department of Mathematics, Osaka City University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@sci.osaka-cu.ac.jp

ABSTRACT

When a link L is concordant to a link L' by a concordance of immersed annuli, we relate the number of the double points of the immersed annuli to the beta-ranks $\beta(L), \beta(L')$ and the torsion multi-variable Alexander polynomials $A^T(L), A^T(L')$ of L and L' to obtain a unifying generalization of the speaker's more than 33-year-older result on the Alexander polynomials of concordant links and T. Kanenobu's 18-year-older announcement result on the Alexander polynomial of a link with unlinking number one.

1. Introduction

We consider as a link an oriented link L with components $K_i (i = 1, 2, \ldots, r)$ in the 3-sphere S^3. A link L is *immersed concordant* to a link L' with components $K'_i (i = 1, 2, \ldots, r)$ if there is a smooth proper immersion

$$\alpha : rS^1 \times I \to S^3 \times I, \quad I = [0, 1]$$

such that $\alpha(rS^1 \times 0) = L \times 0$, $\alpha(rS^1 \times 1) = L' \times 1$, and the image $\alpha(S_\alpha)$ of the singular set S_α of α consists of finitely many transversely intersected double points in $S^3 \times (0, 1)$. We understand that the orientations of L and L' are inherited from an orientation of $rS^1 \times I$. We say that the image $\mathcal{A}$ of α consisting of r immersed annuli is called an *immersed concordance* from L to L'. Since L and L' have the same number of components, there is always an immersed concordance from L to L'. The links L and L' are said to be *concordant* if α is an embedding. Let $\mathcal{A} = \cup_{i=1}^r \mathcal{A}_i$ where $\mathcal{A}_i$ is the immersed annulus in $\mathcal{A}$ connecting K_i to K'_i by re-indexing the components K'_i $(i = 1, 2, \ldots, r)$ of L'.[1] The *double point number* $c(\mathcal{A})$ of $\mathcal{A}$ is defined to be the

[1] In an eralier work, "concordant"is called "cobordant".

cardinality of $\alpha(S_\alpha)$. The 4-*dimensional clasp distance* $c^4(L,L')$ between L and L' are defined by

$$c^4(L,L') = \min\{c(\mathcal{A}) |\ \mathcal{A} \text{ is an immersed concordance from } L \text{ to } L'\}$$

By definition, L and L' are concordant if and only if $c^4(L,L') = 0$. The *Gordian distance* $d(L,L')$ between L and L' is the minimal number of crossing changes needed to obtain L' from L. In particular, the *unlinking number* $u(L)$ of L is the Gordian distance $d(L,L')$ with L' a trivial link. If the link L' is obtained from the link L by n crossing changes, then we have an immersed concordance $\mathcal{A}$ from L to L' with $c(\mathcal{A}) = n$ by considering the trace of the link L in $S^3 \times I$ by the crossing changes. Thus, we have

$$d(L,L') \geqq c^4(L,L').$$

Let $E = \mathrm{cl}(S^3 - N)$ be the compact exterior of L for a tubular neighborhood N of L in S^3. The first homology $H_1(E)$ is a free abelian group of rank r and has a meridian basis of L. We explain here the graded multi-variable Alexander polynomials of L ([5, Chapter 7]). Let $\Lambda = Z[t_1^{\pm1}, t_2^{\pm1}, \ldots, t_r^{\pm1}]$ be the integral group ring $Z[H_1(E)]$ where t_i represents the meridian of K_i. The *link module* of L is the Λ-module $H_1(\tilde{E})$ for the universal free ableian covering $\tilde{E} \to E$, namely the regular covering associated with the Hurewicz epimorphism $\pi_1(E) \to H_1(E)$. By the Noetharian property of Λ, the link module $H_L = H_1(\tilde{E})$ is finitely generated over Λ. For a finitely generated Λ-module H, let $\beta(H)$ denote the Λ-rank of H, namely the $Q(\Lambda)$-dimension of the $Q(\Lambda)$-vector space $H \otimes_\Lambda Q(\Lambda)$ for the quotient field $Q(\Lambda)$ of Λ. Let TH be the Λ-torsion part of H, which is also finitely generated over Λ by the Noetharian property of Λ. Let DH be the Λ-submodule of TH consisting of an element x such that $f_i x = 0$ for coprime elements $f_i \in \Lambda$ $(i = 1, 2, \ldots, s)$ for some $s \geqq 2$. Let $BH = H/TH$ be the Λ-torsion-free part of H. For any non-negative integer d, the *d-th characteristic polynomial* $\Delta^{(d)}(H)$ is defined in [5] by using a Λ-presentation matrix of H. It is standard to use the notation $f \doteq f'$ for elements $f, f' \in \Lambda$ which are equal up to the units $\pm t_1^{n_1} t_2^{n_2} \ldots t_r^{n_r}$ $(n_1, n_2, \ldots, n_r \in Z)$ of Λ. Applying [5, 7.2.7] to the the natural short exact sequence

$$0 \to TH \to H \to BH \to 0,$$

we see that $\Delta^{(d)}(H) = 0$ for all $d < \beta$ and

$$\Delta^{(d)}(TH) \doteq \Delta^{(d+\beta(H))}(H)$$

for all $d \geqq 0$. Let $\Delta^T(H) = \Delta^{(0)}(TH) \doteq \Delta^{(\beta(H))}(H)$. By [5, 7.2.7] and the definition of DH, we also have $\Delta^{(0)}(DH) \doteq \pm 1$. Let $\beta(L) = \beta(H_L)$ which is called the *β-rank* of L. Then $0 \leqq \beta(L) \leqq r - 1$. The d-th Alexander polynomial $\Delta_L^{(d)} = \Delta_L^{(d)}(t_1, t_2, \ldots, t_r)$ of L is just the d-th characteristic polynomial $\Delta^{(d)}(H_L)$. Then $\Delta_L^{(d)} = 0$ for all d with $0 \leqq d \leqq \beta(L)$ and the $\beta(L)$-th Alexander polynomial $\Delta_L^{(\beta(L))}$ is equal to the zeroth Alexander polynomial $\Delta^{(0)}(TH_L)$ which is always a non-zero element of Λ, called the *torsion Alexander polynomial* of L and denoted by Δ_L^T. The zeroth Alexander polynomial $\Delta_L^{(0)}$ is usually called the *Alexander polynomial* of L and denoted by Δ_L. By definition, $\Delta_L \neq 0$ if and only if $\beta(L) = 0$. Let $b(L) \in \Lambda$ be the zeroth characteristic polynomial of the torsion Λ-module $H_1(\partial\tilde{E})$ which we call the *boundary polynomial* of L. The *peripheral factor* $p(L) \in \Lambda$ of L is the zeroth characteristic polynomial of the image of the natural Λ-homomorphism $H_1(\partial\tilde{E}) \to H_1(\tilde{E})$ which is a torsion Λ-module. The peripheral factor $p(L)$ of L is a factor of the boundary polynomial $b(L)$ as well as a factor of the torsion Alexander polynomial Δ_L^T. Let $\mathcal{A}$ be an immersed concordance from L to L'. For every i $(i = 1, 2, \ldots, r)$, let

$$c_i = c_i(\mathcal{A}) = \sum_{1 \leqq j \leqq r, j \neq i} c(\mathcal{A}_i, \mathcal{A}_j),$$

where $c(\mathcal{A}_i, \mathcal{A}_j)$ denotes the number of the intersection double points of the immersed annuli $\mathcal{A}_i$ and $\mathcal{A}_j$. We say that the pair $(K_i, L\backslash K_i)$ is *algebraically split* if the linking number $\mathrm{Link}(K_i, K_j) = 0$ for all $j \neq i$ Let

$$\delta_i = \begin{cases} c_i & \text{if neither } (K_i, L\backslash K_i) \text{ nor } (K_i', L'\backslash K_i') \text{ is algebraically split,} \\ c_i - 1 & \text{if either } (K_i, L\backslash K_i) \text{ or } (K_i', L'\backslash K_i') \text{ is algebraically split,} \\ \max\{0, c_i - 2\} & \text{if both } (K_i, L\backslash K_i) \text{ and } (K_i', L'\backslash K_i') \text{ are algebraically split.} \end{cases}$$

We note that

$$c_i \geqq \sum_{1 \leqq j \leqq r, j \neq i} |\mathrm{Link}(K_i, K_j) - \mathrm{Link}(K_i', K_j')|.$$

Thus, if $(K_i, L\backslash K_i)$ is algebraically non-split and $(K_i', L'\backslash K_i')$ is algebraically split, then we have $c_i \geqq 1$. The *peripheral polynomial* of the immersed concordance $\mathcal{A}$ is the element

$$p(\mathcal{A}) = \prod_{i=1}^{r} (t_i - 1)^{\delta_i} \in \Lambda$$

and the *elementary peripheral polynomial* of the immersed concordance $\mathcal{A}$ is the elementary factor $p(L, L') = e(p(\mathcal{A}))$ of $p(\mathcal{A})$ which will be shown later to be independent of a choice of $\mathcal{A}$. For an element $f = f(t_1, t_2, \ldots, t_r) \in \Lambda$, we denote f^* as

$$f^* = f(t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1}) \in \Lambda.$$

It is a classical result due to Blanchfield [1] that $\Delta_L^{(d)} \doteq \Delta_L^{(d)*}$ for all d, so that in particular, $\Delta_L \doteq \Delta_L^*$ and $\Delta_L^T \doteq (\Delta_L^T)^*$. In [4], the author showed the following result.

Link Concordance Theorem. If L and L' are concordant, then we have $\Delta_L^T \Delta_{L'}^T \doteq ff^*$ for an element $f \in \Lambda$.

On the other hand, T. Kanenobu announced the following theorem in [3], where we note that if a two-component link L with $u(L) = 1$ has the linking number $\mathrm{Link}(L) \neq 0$, then we have $\mathrm{Link}(L) = \pm 1$.

Unlinking Number One Theorem. If $u(L) = 1$, then we have $\Delta_L = 0$ for $r > 2$, and for $r = 2$

$$\Delta_L \doteq \begin{cases} ff^* & \text{if } \mathrm{Link}(L) = \pm 1 \\ (t_1 - 1)(t_2 - 1)ff* & \text{if } \mathrm{Link}(L) = 0. \end{cases}$$

We can write any non-zero element $f \in \Lambda$ as

$$f = f(t_1, t_2, \ldots, t_r) = \prod_{i=1}^{r} (t_i - 1)^{n_i} g$$

for some integers $n_i \geqq 0$ $(i = 1, 2, \ldots, r)$ and an element $g \in \Lambda$ which does not have any element $t_i - 1$ $(i = 1, 2, \ldots, r)$ as a factor. Then the *elementary factor* $e(f)$ of f is defined by

$$e(f) = \prod_{i=1}^{r} (t_i - 1)^{\varepsilon_i}$$

where $\varepsilon_i \in \{0, 1\}$ is the modulo 2 reduction of n_i. If $f = 0$, then we take $e(f) = 0$. The following definition. is important to our argument.

Definition. The *gap factor* $g(L, L')$ of a link pair (L, L') is the elementary factor $e(\Delta_L^T \Delta_{L'}^T)$ of the product $\Delta_L^T \Delta_{L'}^T$.

Let $\beta(L, L') = |\beta(L) - \beta(L')| \leqq r - 1$. The following theorem is our main theorem.

Theorem 1.1. We have

$$|u(L) - u(L')| \geqq d(L, L') \geqq c^4(L, L') \geqq \beta(L, L') \geqq 0.$$

Further, if $c^4(L, L') = \beta(L, L')$, then we have

$$\Delta_L^T \Delta_{L'}^T \doteq g(L, L') ff^*$$

for an element $f \in \Lambda$ and

$$g(L, L') = e(p(L)p(L, L')p(L')).$$

In Theorem 1.1, the inequality $|u(L) - u(L')| \geqq d(L, L') \geqq c^4(L, L')$ is obvious by definitions. If L and L' are concordant, then it is shown in [4] that $\beta(L) = \beta(L')$, so that $c^4(L, L') = \beta(L, L') = 0$. Further, let $(\Delta_L^T)_*$ and $(\Delta_{L'}^T)_*$ be the factors of Δ_L^T and $\Delta_{L'}^T$ in Λ obtained from Δ_L^T and $\Delta_{L'}^T$ by removing all the non-unit factors h with $h(1, 1, \ldots, 1) = \pm 1$, respectively. Then we see from [4] that if L and L' are concordant, then we have $(\Delta_L^T)_* \doteq (\Delta_{L'}^T)_*$, so that $g(L, L') = 1$. By using these facts, the link concordance theorem follows direct from Theorem 1.1. To derive the unlinking number one theorem from Theorem 1.1, we need some computations of the boundary polynomial and the peripheral factor of a link, which will be given in §2. In § 3, we give the proof of Theorem 1.1. In § 4, we discuss corollaries to Theorem 1.1 (except Corollary 2.3) and related examples.

2. Computing the boundary polynomial and the peripheral factor of a link the elemtary peripheral factor of a link pair

The following lemma is used to compute the boundary polynomial and the peripheral factor of a link.

Lemma 2.1. We have the following (1)-(3).
(1) The boundary polynomial $b(L)$ of L is given by

$$b(L) = \prod_{i=1}^{r} (t_i - 1)^{\varepsilon_i}$$

where $\varepsilon_i \in \{0, 1\}$ $(i = 1, 2, \ldots, r)$, and $\varepsilon_i = 1$ if and only if the pair $(K_i, L \backslash K_i)$ is algebraically split.
(2) If $\beta(L) = 0$, then $p(L) = b(L)$.
(3) If $\beta(L) = r - 1$, then $p(L) = 1$ and $(K_i, L\backslash K_i)$ is algebraically split for every i so that $b(L) = \prod_{i=1}^{r}(t_i - 1)$.

Proof. To see (1), we note that the lift of the torus component T_i of ∂E around K_i to $\partial \tilde{E}$ consists of the components homeomorphic to $S^1 \times R$ or R^2 according to whether $(K_i, L\backslash K_i)$ is algebraically split or not. Hence we have

$$H_1(\partial \tilde{E}) \cong \oplus_{k=1}^{s} \Lambda/(t_{i_k} - 1)$$

for the members i_k $(k = 1, 2, \ldots, s)$ in $\{1, 2, \ldots, r\}$ with $\varepsilon_{i_k} = 1$ and

$$b(L) = \prod_{k=1}^{s} (t_{i_k} - 1).$$

To see (2), since

$$\beta(L) = 0,\ DH_0(\tilde{E}) = DH_0(\tilde{E}) \cong \Lambda/(t_1 - 1, t_2 - 1, \ldots, t_r - 1),\ DH_1(\partial \tilde{E}) = 0,$$

the Blanchfield duality [1] implies that $H_2(\tilde{E}, \partial\tilde{E})) = DH_2(\tilde{E}, \partial\tilde{E}))$ and the natural homomorphism $H_1(\partial\tilde{E}) \to H_1(\tilde{E})$ is injective. Hence $p(L) = b(L)$. To see (3), let $\Lambda^{(1)} = Z[t, t^{-1}]$ be the t-varianble Laurent polynomial ring, and $\tilde{E}^{(1)} \to E$ the infinite cyclic covering associated with the epimorphism $H_1(E) \to Z$ sending every meridian of L in ∂E to $1 \in Z$. Then $\Lambda^{(1)}$-rank $\beta^{(1)}(L)$ of the $\Lambda^{(1)}$-module $H_1(\tilde{E}^{(1)})$ is $r-1$ (cf. [5, 7.3.12]), which implies that the torsion Alexander polynomial $\Delta^{(1)}$ of the $\Lambda^{(1)}$-module $H_1(\tilde{E}^{(1)})$ has $\Delta^{(1)}(1) = \pm 1$. Thus, the natural homomorphism $H_1(\partial\tilde{E}^{(1)}) = (\Lambda^{(1)}/(t-1))^r \to H_1(\tilde{E}^{(1)})$ must be trivial meaning that the longitude of K_i in ∂E is the boundary of a 2-chain in E and hence $\mathrm{Link}(K_i, L\backslash K_i) = 0$, for every i. By (1), $b(L) = \prod_{i=1}^r (t_i - 1)$. This completes the proof.

The following lemma shows that the elementary peripheral polynomial of an immersed concordance is calculable by the linking numbers modulo 2 of the immersed concordant links and is independent of a choice of an immersed concordance.

Lemma 2.2. The elementary peripheral polynomial $p(L, L')$ between L and L' has the form

$$p(L, L') = \prod_{i=1}^{r} (t_i - 1)^{\varepsilon_i}$$

where $\varepsilon_i \in \{0, 1\}$ is determined by

$$\varepsilon_i = \begin{cases} \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K_i', L'\backslash K_i') \pmod 2 \\ \qquad \text{if neither } (K_i, L\backslash K_i) \text{ nor } (K_i', L'\backslash K_i') \text{ is algebraically split,} \\ \\ \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K_i', L'\backslash K_i') - 1 \pmod 2 \\ \qquad \text{if either } (K_i, L\backslash K_i) \text{ or } (K_i', L'\backslash K_i') \text{ is algebraically split,} \\ \\ 0 \qquad \text{if both } (K_i, L\backslash K_i) \text{ and } (K_i', L'\backslash K_i') \text{ are algebraically split.} \end{cases}$$

Proof. By the well-known relationship between the intersection number and the linking number, we have

$$c_i \equiv \mathrm{Link}(K_i, L\backslash K_i) + \mathrm{Link}(K_i', L'\backslash K_i') \pmod 2.$$

In particular, if both $(K_i, L\backslash K_i)$ and $(K_i', L'\backslash K_i')$ are algebraically split, then c_i is even and $\max\{0, c_i - 2\} \equiv 0 \pmod 2$. This completes the proof.

The following corollary is used to prove the unlinking number one theorem.

Corollary 2.3. Assume that $c^4(L, L') = \beta(L, L') = r - 1$ with $\beta(L) = 0$ and $\beta(L') = r - 1$.

(1) If $(K_i, L\backslash K_i)$ is algebraically split for every i, then we have $g(L, L') = \prod_{i=1}^{r}(t_i - 1)$.
(2) If $\mathrm{Link}(K_i, L\backslash K_i)$ is odd for every i, then we have $g(L, L') = 1$.

Proof. By the assumption of (1), we have $p(L) = \prod_{i=1}^{r}(t_i - 1)$ and $p(L') = 1$ by (1), (2) and (3) of Lemma 2.1, and $p(L, L') = 1$ by Lemma 2.2, obtaining (1) by Theorem 1.1. For the assumption of (2), we have $p(L) = p(L') = 1$ by (1), (2) and (3) of Lemma 2.1, and $p(L, L') = 1$ by Lemma 2.2, obtaining (2) by Theorem 1.1.

Theorem 1.1 and Corollary 2.3 imply the unlinking number one theorem as follows: Let L be a link with $u(L) = c^4((L, L') = 1$ for a trivial link L'. For $r > 2$, we have $1 \geqq \beta(L, L')$ by Thenrem 1.1 and $\beta(L) \geqq \beta(L') - 1 = r - 2 > 0$, implying that $\Delta_L = 0$. For $r = 2$, let $\Delta_L \neq 0$. Then $\beta(L) = 0$ and $c^4(L, L') = \beta(L, L') = 1$. The desired result is direct from Theorem 1.1 and Corollary 2.3.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be done throughout this section. Let $\mathcal{A}$ be an immersed concordance from L to L' with $n = c(\mathcal{A}) = c^4(L, L')$. Let $\beta(L') = \beta(L) + b$ for $b = \beta(L, L') = |\beta(L) - \beta(L')|$. Since every transversely intersected double point in $\mathcal{A}$ is topologically represented by the cone vertex of a Hopf link, we slide the double points in $\mathcal{A}$ into $L' \times 1$. Then we obtain from $\mathcal{A}_i$ a connected oriented proper planar surface F_i in $S^3 \times I$ for every i such that $F_i \cap F_j = \emptyset$ for every $i \neq j$ and the boundary ∂F of the surface $F = \cup_{i=1}^{r} F_i$ is given by $\partial F = (-L) \times 0 \cup L'' \times 1$ where $L'' = L' \#_{j=1}^{s} S_j^H \#_{k=1}^{u} U_k^H$ for Hopf links S_j^H $(j = 1, 2, \ldots, s)$ and U_k^H $(k = 1, 2, \ldots, u)$ with $s + u = n$ such that S_j^H $(j = 1, 2, \ldots, s)$ belong to the same components of F and U_k^H $(k = 1, 2, \ldots, u)$ belong to distinct components of F. See Fig. 1 for an illustration of an immersed concordance $\mathcal{A}$ and Fig. 2 for an illustration of the surface F.

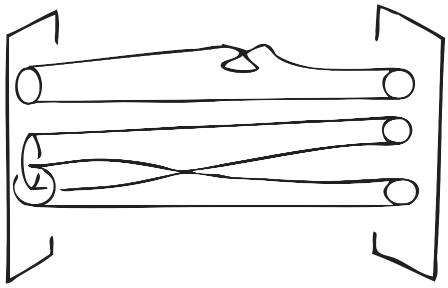

Figure 1: An immersed concordance $\mathcal{A}$

We attach a (± 1)-framed 2-handle h_j^2 to $S^3 \times 1$ to make a (± 1)-twist of S_j^H to change the link L'' into the split link of $(L' \#_{k=1}^{u} U_k^H) \times 1$ and a trivial link $O^s \times 1$ of s components. We assume that the attaching core circle of h_j^2 and S_j^H has the

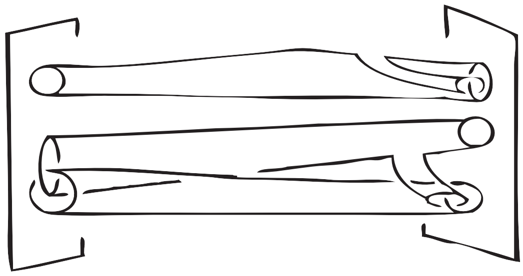

Figure 2: The surface F

linking number 0. By taking mutually disjoint s disks in $S^3 \times 1$ bounded by $O^s \times 1$, the surfaces F_i ($i = 1, 2, \ldots, r$) are changed into planar surfaces F'_i ($i = 1, 2, \ldots, r$) in $X = S^3 \times I \cup_{j=1}^{s} h_j^2$ with $\partial X = S^3 \times 0 \cup S^3 \times 1$ such that $F' = \cup_{i=1}^{r} F'_i$ has $\partial F' = (-L) \times 0 \cup (L' \#_{k=1}^{u} U_k^H) \times 1$. Further, we attach the connected sum bands B_k ($k = 1, 2, \ldots, u$) to $(L' \#_{k=1}^{u} U_k^H) \times 1$ in $S^3 \times 1$ to change the link $(L' \#_{k=1}^{u} U_k^H) \times 1$ into the split link of $L' \times 1$ and the Hopf links $U_k^H \times 1$ ($k = 1, 2, \ldots, u$). The surface $F' \cup_{k=1}^{u} B_k$ is deformed into a proper surface F'' of r components in X such that $\partial F'' = (-L) \times 0 \cup L'_+ \times 1$, where $L'_+ = L' \cup_{k=1}^{u} U_k^H$. Let $Y = \mathrm{cl}(X \backslash N(F''))$ be the compact 4-manifold for a tubular neighborhood $N(F'') = F'' \times D^2$ of F'' in X. For the link L, let $L = L^{(0)} \cup L^{(1)}$ where $L^{(0)}$ is the sublink of L consisting of a component K such that $(K, L \backslash K)$ is algebraically split. and $L^{(1)}$ is the sublink of L consisting of the other components. Applying the same notation to the link L'_+, we have $L'_+ = (L'_+)^{(0)} \cup (L'_+)^{(1)}$ with $(L'_+)^{(0)} = (L')^{(0)}$ and $(L'_+)^{(1)} = (L')^{(1)} \cup_{k=1}^{u} U_k^H$. Let

$$M = \mathrm{cl}(\chi(L^{(0)}; S^3) \backslash N(L^{(1)}))$$

for a tubular neighborhood $N(L^{(1)})$ of $L^{(1)}$ where $\chi(L^{(0)}; S^3)$ denotes the 0-surgery of S^3 along $L^{(0)}$. Similarly, let

$$M' = \mathrm{cl}(\chi((L')^{(0)}; S^3) \backslash N((L')^{(1)}))$$
$$M'_+ = \mathrm{cl}(\chi((L'_+)^{(0)}; S^3) \backslash N((L'_+)^{(1)})).$$

Then we have a connected sum decomposition $M'_+ = M' \#_{k=1}^{u} E(U_k^H)$ for the Hopf link exteriors $E(U_k^H)$ ($k = 1, 2, \ldots, u$). Since ∂Y is a torus sum of link exteriors $E(L)$ and $E(L'_+)$, and the product $F'' \times S^1$, we construct a 4-manifold

$$W = M \times [-1, 0] \cup Y \cup M'_+ \times [1, 2]$$

pasting $M \times [-1, 0]$ and Y along $E(L) \times 0$ and pasting Y and $M'_+ \times [1, 2]$ along $E(L'_+) \times 1$. Let G be a possibly disconnected proper surface in W obtained from F'' by attaching disks to the components of $L^{(0)}$ and $(L'_+)^{(0)} = (L')^{(0)}$. See Fig. 3 for an illustration of the surface G.

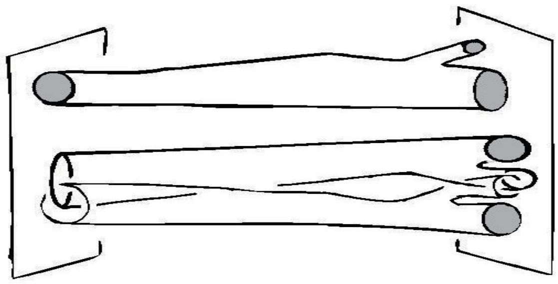

Figure 3: The surface G

Then ∂W is a torus sum of $M \times (-1)$, $M'_+ \times 2$ and the product $G \times S^1$. If G has a 2-sphere component, namely there is an $S^2 \times S^1$ component in $G \times S^1$, then we paste the 4-manifold $B^3 \times S^1$ with B^3 the 3-ball to it and assume that $G \times S^1$ has no $S^2 \times S^1$ components. For simplicity, we take $E = E(L)$, $M = M \times (-1)$, $P = G \times S^1$, and $M'_+ = M'_+ \times 2$. By construction, the maximal free abelian cover $\tilde{E}$ over E extends to a free abelian cover $(\tilde{W}; \tilde{M}, \tilde{P}, \tilde{M}'_+)$ over $(W; M, P, M'_+)$. For a compact submanifold pair (W', W'') of W, the homology $H_*(\tilde{W}', \tilde{W}'')$ of the lift $(\tilde{W}', \tilde{W}'')$ to $\tilde{W}$ forms a finitely generated Λ-modulde. Let $\beta_*(\tilde{W}', \tilde{W}'')$ denote the Λ-rank of $H_*(\tilde{W}', \tilde{W}'')$. We need the following computations of Λ-ranks.

Lemma 3.1.
(1) $\beta_1(\partial\tilde{W}) = 2\beta(L) + b + u$.
(2) $\beta_1(\tilde{W}) \leqq \beta(L)$.
(3) $\beta_1(\partial\tilde{W}) \leqq \beta_2(\tilde{W}, \partial\tilde{W}) + \beta_1(\tilde{W})$.
(4) $\beta_2(\tilde{W}, \partial\tilde{W}) = n + u + \beta_1(\tilde{W})$.

Proof. By construction, $\beta(L) = \beta_1(\tilde{M})$ and $\beta(L'_+) = \beta_1(\tilde{M}'_+) = \beta_1(\tilde{M}') + u = \beta(L') + u = \beta(L) + b + u$. Since ∂W is a torus sum of M, M'_+ and $P = G \times S^1$, we have $\beta_1(\partial\tilde{W}) = \beta_1(\tilde{M}) + \beta_1(\tilde{M}'_+) = 2\beta(L) + b + u$, showing (1). Since $H_1(W, M) = 0$, we see from [4, Lemma 2.1] or [5, 12.3.13] that $\beta_1(\tilde{W}, \tilde{M}) = 0$, which implies that $\beta(L) = \beta_1(\tilde{M}) \geqq \beta_1(\tilde{W})$, showing (2). Using the exact sequence

$$H_2(\tilde{W}, \partial\tilde{W}) \to H_1(\partial\tilde{W}) \to H_1(\tilde{W}),$$

we have (3). To see (4), we note by the Blanchfield duality [1] that $\beta_2(\tilde{W}, \partial\tilde{W}) = \beta_2(\tilde{W})$ and $\beta_1(\tilde{W}, \tilde{M}) = \beta_0(\partial\tilde{W}, \tilde{M}) = 0$ $\beta_3(\tilde{W}) = \beta_1(\tilde{W}, \partial\tilde{W}) = 0$. By the Euler characteristic $\chi(W) = n + u$ and $\beta_d(\tilde{W}) = 0 (d \neq 1, 2)$, we have

$$\beta_2(\tilde{W}) - \beta_1(\tilde{W}) = n + u,$$

showing (4). This completes the proof.

By Lemma 3.1, we have

$$\begin{aligned}&2\beta(L)+b+u=\beta_1(\partial\tilde{W}) \quad \text{by (1)}\\ &\leqq \beta_2(\tilde{W},\partial\tilde{W})+\beta_1(\tilde{W})=2\beta_1(\tilde{W})+n+u \quad \text{by (3) and (4)}\\ &\leqq 2\beta(L)+n+u \quad \text{by (2).}\end{aligned}$$

Hence $\beta(L,L')=b\leqq n=c^4(L,L')$, giving the first half of Theorem 1.1.

Next, assume $c^4(L,L')=n=b=\beta(L,L')$. Then we have

$$(*) \quad \beta_2(\tilde{W},\partial\tilde{W})+\beta_1(\tilde{W})=\beta_1(\partial\tilde{W}).$$

We need the following lemma:

Lemma 3.2 (Exactness Lemma). Under the identity (*), the natural exact sequence

$$H_2(\tilde{W},\partial\tilde{W})\xrightarrow{\partial_*}H_1(\partial\tilde{W})\xrightarrow{i_*}H_1(\tilde{W})$$

induces an exact sequence

$$TH_2(\tilde{W},\partial\tilde{W})\xrightarrow{\partial_*}TH_1(\partial\tilde{W})\xrightarrow{i_*}TH_1(\tilde{W}).$$

This lemma is immediately obtained by noting that the identity (*) implies that the homomorphism $BH_2(\tilde{W},\partial\tilde{W})\to BH_1(\partial\tilde{W})$ induced from ∂_* is injective. Let $H=H_1(\partial\tilde{W})$, and T the image of $\partial'_*: TH_2(\tilde{W},\partial\tilde{W})\to TH_1(\partial\tilde{W})=H$. By [4, Theorem 3.1] we have

$$\Delta^T(H)=\Delta^{(0)}(TH)\doteq\Delta(T)\Delta(T)^*.$$

To calculate $\Delta^T(H)$ we need the following lemma.

Lemma 3.3.
(1) $\Delta^T_L\doteq p(L)\Delta^T(H_1(\tilde{M}))$, $\quad \Delta^T_{L'}\doteq p(L')\Delta^T(H_1(\tilde{M}'))$.
(2) $\Delta^T(H)\doteq p(\mathcal{A})\Delta^T(H_1(\tilde{M})\Delta^T(H_1(\tilde{M}'))$.
(3) $\Delta^T(H_1(\tilde{P}))\doteq p(\mathcal{A})$.

Proof. To see (1), we note that there are exact sequences

$$\begin{aligned}&H_1(\partial E)\to H_1(\tilde{E})\to H_1(\tilde{M})\to 0,\\ &H_1(\partial E')\to H_1(\tilde{E}')\to H_1(\tilde{M}')\to 0.\end{aligned}$$

By [5, 7.2.7], we have (1). To see (2), let $S = \partial M \cup \partial M'_+$. Then we have $H_1(\tilde{S}) = 0$ and $H_0(\tilde{S}) = DH_0(\tilde{S})$. By excision, we have

$$H_1(\partial\tilde{W}, \tilde{S}) \cong H_1(\tilde{M}, \partial\tilde{M}) \oplus H_1(\tilde{P}, \tilde{S}) \oplus H_1(\tilde{M}'_+, \partial\tilde{M}'_+).$$

Since there is a natural exact sequence

$$H_1(\tilde{S}) = 0 \to H_1(\partial\tilde{W}) \to H_1(\partial\tilde{W}, \tilde{S}) \to H_0(\tilde{S}),$$

we have

$$\Delta^T(H) \doteq \Delta^T(H_1(\partial\tilde{W})) \doteq \Delta^T(H_1(\partial\tilde{W}, \tilde{S})).$$

Similarly, we have

$$\begin{aligned}
&\Delta^T(H_1(\tilde{M}, \partial\tilde{M})) \doteq \Delta^T(H_1(\tilde{M})),\\
&\Delta^T(H_1(\tilde{M}'_+, \partial\tilde{M}'_+)) \doteq \Delta^T(H_1(\tilde{M}'_+)) \doteq \Delta^T(H_1(\tilde{M}')),\\
&\Delta^T(H_1(\tilde{P}, \tilde{S})) \doteq \Delta^T(H_1(\tilde{P})).
\end{aligned}$$

Hence

$$\Delta^T(H) \doteq \Delta^T(H_1(\tilde{P}))\Delta^T(H_1(\tilde{M}))\Delta^T(H_1(\tilde{M}')).$$

Assuming (3), we complete the proof of (2). To see (3), we note that the component G_i of G obtained from $\mathcal{A}_\rangle$ is a 2-sphere with $c_i + 2$ holes if $(K_i, L\backslash K_i)$ nor $(K'_i, L'\backslash K'_i)$ is algebraically split, a 2-sphere with $c_i + 1$-holes if either $(K_i, L\backslash K_i)$ or $(K'_i, L'\backslash K'_i)$ is algebraically split, and a 2-sphere with c_i holes if both $(K_i, L\backslash K_i)$ and $(K'_i, L'\backslash K'_i)$ are algebraically split. In the last case, recall that c_i is always even and we omited the case $c_i = 0$. Thus, to complete the proof of (3), it suffices to show the following assertion:

Assertion. Let G be a 2-sphere with $n(\geqq 2)$ holes, and $P = G \times S^1$. Let $x_i (i = 1, 2, \ldots, n)$ be a basis of $H_1(P)$ such that x_1 and $x_i (i = 2, 3, \ldots, n)$ are represented by a loop in $p \times S^1$ $(p \in G)$ and loops in $G \times 1$ $(1 \in S^1)$, respectively. Let $\tilde{P} \to P$ be the covering associated with a homomorphism $H_1(P) to Z^r$ such that x_1 is sent to a basis element e_1 and every element x_i with $i \geqq 2$ is sent to an element which is linearly independent of e_1. Let t_1 be the element corresponding to e_1 in the group ring $\Lambda = Z[Z^r] = Z[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_r^{\pm 1}]$. Then we have the Alexander polynomial $\Delta(H_1(\tilde{P})) \doteq (t_1 - 1)^{n-2}$.

Proof of Assertion. We use the Fox free calculus [2] (see also [5, 7.1.5]). The fundamental group $\pi_1(P)$ has a presentation with generators $x_i (i = 1, 2, \ldots, n)$ and relators $r_1 = x_1 x_1^{-1}$ and $r_i = x_1 x_i x_1^{-1} x_i^{-1}$ $(i = 2, \ldots, n)$. The Jacobian (n, n)-matrix $(\partial r_i / \partial x_j)$ with entries in Λ is given by

$$\begin{pmatrix}
0 & 0 & 0 & \ldots & 0\\
1 - u_2 & t_1 - 1 & 0 & \ldots & 0\\
1 - u_3 & 0 & t_1 - 1 & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots & \ldots\\
1 - u_n & 0 & 0 & \ldots & t_1 - 1
\end{pmatrix},$$

where $u_i (i = 2, 3, \ldots, n)$ are monomials with coefficient $+1$ in $t_i^{\pm 1} (i = 1, 2, \ldots, r)$ distinct from any power of $t_1^{\pm 1}$. This matrix is a presentation matrix of a Λ-module $\mathcal{M}$ admitting a short exact sequence

$$0 \to H_1(\tilde{P}) \to \mathcal{M} \to \varepsilon(\Lambda) \to 0,$$

where $\varepsilon(\Lambda)$ denotes the torsion-free Λ-module of rank one which is the kernel of the homomorphism $\Lambda \to Z$ sending every t_i to 1, namely the Λ-ideal $(t_1 - 1, t_2 - 1, \ldots, t_r - 1)$. Thus, the Alexander polynomial of $H_1(\tilde{P})$ is obtained from the second elementary ideal of the matrix by taking the smallest principal ideal (cf. [5, 7.2.7]). The second elementary ideal is

$$((t_1 - 1)^{n-1}, (1 - u_2)(t_1 - 1)^{n-2}, (1 - u_3)(t_1 - 1)^{n-2}, \ldots, (1 - u_n)(t_1 - 1)^{n-2})$$

and we obtain $\Delta(H_1(\tilde{P})) \doteq (t_1 - 1)^{n-2}$. This completes the proof of Assertion.

This completes the proof of Lemma 3.3.

By this lemma, we have

$$\begin{aligned} &p(\mathcal{A})^2 \Delta_L^T \Delta_{L'}^T \doteq p(\mathcal{A})^2 p(L) p(L') \Delta^T(H_1(\tilde{M})) \Delta^T(H_1(\tilde{M}')) \\ &\doteq p(\mathcal{A}) p(L) p(L') \Delta^T(H) \doteq e(p(L) p(L, L') p(L')) g g* \end{aligned}$$

for some $g \in \Lambda$, so that $g(L, L') = e(p(L)p(\mathcal{A}))p(L')) = e(p(L)p(L, L')p(L'))$ and $\Delta^T(L)\Delta^T(L') \doteq g(L, L') f f*$ for some $f \in \Lambda$. This completes the proof of Theorem 1.1.

4. Corollaries to Theorem 1.1 and related examples

By using an operation on link diagrams, called a *lassoing*, A. Shimizu construted in [10] an algebraically split link L of r components K_i $(i = 1, 2, \ldots, r)$ such that the one-variable Alexander polynomial $\Delta_L(t) \neq 0$ (cf. [5]) and $d(L, K \cdot O) = r - 1$ for the completely split link $K \cdot O$ of a previously given knot K and a trivial link O of $r - 1$ components, where we assume that K_1 and $\cup_{i=2}^r K_i$ are changed into K and O by the crossing changes, respectively. Since $\Delta_L(t) = (t - 1)\Delta_L(t, t, \ldots, t)$, we have $\beta(L) = 0$. As an application of Theorem 1.1, we give some properties of the Alexander polynomials of the components K'_i $(i = 1, 2, \ldots, r)$ of any completely split link L' with $d(L, L') = r - 1$ where we assume that K_i $(i = 1, 2, \ldots, r)$ are changed into K'_i $(i = 1, 2, \ldots, r)$ by the crossing changes, respectively.

Corollary 4.1. We have

$$\Delta_K(t_1)\Delta_{K'_1} \doteq f_1 f_1^*, \quad \Delta_{K'_i} \doteq f_i f_i^* \ (i = 2, 3, \ldots, r)$$

for a t_i-variable element $f_i = f_i(t_i) \in \Lambda$ with $f_i(1) = 1$ for every $i = 1, 2, \ldots, r$.

Proof. We note that $c^4(L, L') = r - 1$, $g(L, L') = p(L) = \prod_{i=1}^{r}(t_i - 1)$. $\Delta_L \Delta^T_{K\cdot O} \doteq g(L, K \cdot O)ff^*$ for some $f \in \Lambda$. By assumption, $c^4(L, L') = \beta(L, L') = r - 1$. Then $\Delta_L \Delta^T_{L'} \doteq g(L, L')gg^*$ for some $g \in \Lambda$. Since $\Delta^T_{K\cdot O} \doteq \Delta_K$, $\Delta^T_{L'} \doteq \Delta_{K'_1}\Delta_{K'_2} \ldots \Delta_{K'_r}$ and $\Delta_L \doteq \Delta^*_L$, we have

$$\Delta_L \Delta^*_L \Delta_K(t_1)\Delta_{K'_1}\Delta_{K'_2} \ldots \Delta_{K'_r} \doteq g(L, K \cdot O)g(L, L')(fg)(fg)^*.$$

This means

$$g(L, L') = g(L, K \cdot O) = p(L), \quad \Delta_K(t_1)\Delta_{K'_1} \doteq f_1 f^*_1, \quad \Delta_{K'_i} \doteq f_i f^*_i \ (i > 1)$$

for some $f_i = f_i(t_i) \in \Lambda$ with $f_i(1) = 1$ $(i = 1, 2, \ldots, r)$. This completes the proof.

The fraction $\hat{\Delta}_L(t) = \Delta_L(t)/(t - 1)^{r-1}$ is known to be an integral polynomial, called the *Hosokawa polynomial* of L (see [5]). We know that the multiplicity of the factor $t - 1$ in $\hat{\Delta}_L(t)$ is always even. In fact, the statement that $\sigma_1(L) \equiv \hat{\kappa}_1(L) \pmod 2$ in Lemma 5.7 of [6] implies this assertion. Thus, the following corollary is obtained from Theorem 1.1.

Corollary 4.2. If the one-variable Alexander polynomial $\Delta_L(t)$ of a link L with $r(\geq 2)$ components is a non-zero polynomial, then $u(L) \geqq c^4(L, O) \geqq g(L, O) = r - 1$. Further if $c^4(L, O) = r - 1$, then the Hosokawa polynomial $\hat{\Delta}_L(t) = \Delta_L(t)/(t - 1)^{r-1}$ has the form $f(t)f(t^{-1})$ for an integral polynomial $f(t)$ in t up to multiplications of $\pm t^i$ $(i \in Z)$.

Recall that the group order of the first homology $H_1(M_L)$ of the double branched covering space M_L of S^3 branched along L coincides with the absolute value $|\Delta_L(-1)|$ by taking the group order of an infinite abelian group to be 0 (see [5]).

Corollary 4.3. If the first homology $H_1(M_L)$ of the double branched covering space M_L of S^3 branched along a link L with $r(\geq 2)$ components is a finite abelian group, then $u(L) \geqq c^4(L, O) \geqq g(L, O) = r - 1$. Further if $c^4(L, O) = r - 1$, then the group order $|H_1(M_L)|$ has the form $2^{r-1}n^2$ for an integer n.

In the case of a link L with $r = 2$ and $u(L) = 1$ (implying $c^4(L, O) = 1$), the latter half of Corollary 3.3 has been observed by P. Kohn [7]. In the following examples, we use the numbering of links in D. Rolfsen[9]. The following example relates to examples on the 4-dimensional clasp number.

Example 4.4.

(1) Let $L = 7^2_1$. Then $\Delta_L \doteq 1-t_1-t_2+(1-t_1-t_2)t_1t_2+(t_1t_2)^2$ which cannot be written as $g(L,O)ff^*$. Hence we have $c^4(L,O) > \beta(L,O) = 1$. Since $c^4(L,O) \leqq d(L,O) \leqq 2$, we have $c^4(L,O) = \beta(L,O) = u(L) = 2$.

(2) Let $L = 7^2_2$. Then $c^4(L,O) = \beta(L,O) = u(L) = 1$ and $\Delta_L \doteq 1 - t_1 - t_2 + (3 - t_1 - t_2)t_1t_2 + (t_1t_2)^2 = (t_1 + t_2 - 1 - 1)(t_1 - 1 + t_2 - 1)$ and $g(L,O) = 1$.

(3) Let $L = 9^3_9$. Then $\Delta_L \doteq (t_1-1)(t_2-1)(t_2-2)(t_12-t_1+1)$, which cannot be written as $g(L,O)ff^*$. Hence $c^4(L,O) > \beta(L,O) = 2$. Since $c^4(L,O) \leqq u(L) = d(L,O) \leqq 3$, we have $c^4(L,O) = u(L) = d(L,O) = 3$.

Let $c_s(\mathcal{A}) = c(\mathcal{A}) - \sum_{i<j} c(\mathcal{A}_i, \mathcal{A}_j)$ be the total self double point number of the immersed annuli $\mathcal{A}_i$ $(i = 1, 2, \ldots, r)$, and $c_{ns}(\mathcal{A}) = c(\mathcal{A}) - c_s(\mathcal{A})$ the total non-self double point number of $\mathcal{A}$. The following example relates to examples on immersed concordances.

Example 4.5.

(1) Let $L = 9^3_{21}$. Then $c^4(L,O) = \beta(L,O) = d(L,O) = 1$ by Theorem 1.1. Hence we have $\beta(L) = 1$ and $\Delta_L = 0$. $\Delta^T(L) = t_1 - 1 = p(L)$ and $p(L,O) = 1$. Every $\mathcal{A}$ with $c(\mathcal{A}) = 1$ has $c_s(\mathcal{A}) = 1$.

(2) Let $L = 8^3_5$. Then there is a split link L' of 4^2_1 and a trivial knot O with $c^4(L,L') = 1$. Then $c^4(L,L') = \beta(L,L') = 1$, and $\Delta_L\Delta^T(L') \doteq g(L,L')(t_1t_2+1)(t_1^{-1}t_2^{-1}+1)$ with $g(L,L') = (t_1-1)(t_2-1)(t_3-1)$, $p(L) = t_1-1$, $p(L,L') = (t_2-1)(t_3-1)$ and $p(L') = 1$. For any split link L' with $c^4(L,L') = 1$, every $\mathcal{A}$ with $c(\mathcal{A}) = 1$ has $c_{ns}(\mathcal{A}) = 1$. In fact, we have $c^4(L,L') = \beta(L,L') = 1$ and $\Delta_L \doteq (t_1 - 1)(t_2 - 1)(t_3 - 1)(t_2t_3 + 1)$. If $c_s(\mathcal{A}) = 1$, then L' is a split link of a 2-component link with linking number ± 1 and a trivial knot. Hence we have $\Delta^T(L'; 1, 1, 1) = \pm 1$. This contradicts the form $\Delta_L\Delta^T(L') \doteq g(L,L')ff*$ for an element $f \in \Lambda$, given by Theorem 1.1.

(3) $L = 9^3_{12}$. Then $c^4(L,O) = d(L,O) = 2$ and $\Delta_L \doteq g(L,O)(t_1 - 1)(t_1^{-1} - 1)$ with $g(L,O) = (t_1 - 1)(t_2 - 1)(t_3 - 1) = p(L)$, and $p(L,O) = 1$. There are two kinds of $\mathcal{A}$ with $c(\mathcal{A}) = 2$, that is, an $\mathcal{A}$ with $c_s(\mathcal{A}) = 2$ and an $\mathcal{A}$ with $c_{ns}(\mathcal{A}) = 2$.

(4) Let $L = 6^3_2$ (the Borromean rings). Then $c^4(L,O) = \beta(L,O) = u(L) = d(L,O) = 2$ and $\Delta_L \doteq g(L,O) = (t_1 - 1)(t_2 - 1)(t_3 - 1) = p(L)$, $p(L,O) = 1$. Every $\mathcal{A}$ with $c(\mathcal{A}) = 2$ has $c_{ns}(\mathcal{A}) = 2$. This is because L is not link-homotopically trivial (see [8]).

References

[1] **R. C. Blanchfield**, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math., 65(1957), 340-356.

[2] **R. H. Crowell and R. H. Fox**, *Introduction to knot theory*, Ginn and Co. (1963); Reissue, Grad. Texts Math., 57(1977), Springer Verlag.

[3] **T. Kanenobu**, *The Alexander polynomials of links with unlinking number one (in Japanese)*, Abstract in the fall term meeting of the Mathematical Society of Japan (1993), 10-11.

[4] **A. Kawauchi**, *On the Alexander polynomials of concordant links*, Osaka Journal of Mathematics, 15(1978), 151-159.

[5] **A. Kawauchi**, *A Survey of Knot Theory*, Birkhauser, 1996.

[6] **A. Kawauchi**, *The quadratic form of a link*, in: Proc. Low Dimension Topology, Contemp. Math., 233(1999),97-116.

[7] **P. Kohn**, *Unlinking two component links*, Osaka J. Math., 30 (1993), 741-752.

[8] **J. W. Milnor**, *Link groups*, Ann. of Math., 59(1954), 177-195.

[9] **D. Rolfsen**, *Knots and Links*, Publish or Perlish, 1976.

[10] **A. Shimizu**, *The complete splitting number of a lassoed link*, Topology Appl. (to appear).