Component-conservative invertibility of links and Samsara 4-manifolds on 3-manifolds

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ABSTRACT

Every Dehn surgery manifold of a component-conservatively invertible link is embedded into a closed oriented 4-manifold with the $Z_{/2}$-homology of $S^1 \times S^3$, where $Z_{/2} = Z[\frac{1}{2}]$ is a subring of Q. This 3-manifold and 4-manifold give a typical example of a closed Samsara 4-manifold on an invertible 3-manifold. After observing that not every closed oriented 3-manifold is a Dehn surgery manifold of a component-conservatively invertible link, we shall construct, for every closed oriented 3-manifold M, an analogous compact oriented 4-manifold but only with as boundary a connected sum of 3-tori, which we call a bounded Samsara 4-manifold on M. This construction leads to an open question asking whether every M is embedded into a compact oriented 4-manifold with the trivial second Q-homology. We shall investigate a relationship between the Z_2-rank of the 2-torsion part of the second homology group of a bounded Samsara 4-manifold on M and the Z_2-rank of the first Z_2-homology group of M. From a closed Samsara 4-manifold on M, we also obtain a closed oriented 4-manifold with the $Z_{/2}$-homology of S^4 where a punctured 3-manifold M^0 of M is embedded, called a reduced closed Samsara 4-manifold on M^0. A reduced bounded Samsara 4-manifold on any punctured 3-manifold is also discussed in parallel.

1. Introduction

An oriented knot K in the 3-sphere S^3 is *invertible* if there is an orientation-preserving self-homeomorphism f of S^3 sending K to the orientation-reversed knot $-K$ of K (see [12] as a general reference of knot theory). Until now, it appears that a topological meaning of an invertible knot has not been enough observed. In this paper, we try to find a meaning in an embedding relationship between a closed oriented 3-manifold and a closed oriented 4-manifold constructed from an invertible knot. The self-homeomorphism $f : S^3 \to S^3$ induces an orientation-preserving and spin-structure-preserving self-homeomorphism $h : M \to M$ such that $h_* = -1 : H_1(M; Z) \to H_1(M; Z)$ for the first homology $H_1(M; Z)$ of the Dehn surgery 3-manifold $M = \chi(K; r)$ with any rational surgery coefficient $r \in Q$. Let Σ be the mapping torus of h: namely,

$$\Sigma = M \times [0,1]/\{(h(x), 0) \sim (x, 1) \mid x \in M\}.$$

We have $H_1(\Sigma; Z) \cong Z \oplus H_1(M; Z_2)$ and $H_1(M; Z) \cong Z_n$ where n is the numerator of the reduced fraction of r.

For a manifold X, the Z-rank ofthe dth homology $H_d(X; Z)$ is denoted by $\beta_d(X; Z)$ and the Z_2-rank of $H_d(X; Z_2)$ by $\beta_d(X; Z_2)$. Then $s = \beta_1(M; Z_2)$ is 0 or 1 according to whether n is odd or even. Since the Euler characteristic $\chi(\Sigma) = 0$, we obtain

$$H_d(\Sigma; Z) \cong \begin{cases} Z & (d = 0, 3, 4) \\ {Z_2}^s & (d = 2) \\ Z \oplus {Z_2}^s & (d = 1) \\ 0 & (\text{others}), \quad \text{where } s = \beta_1(M; Z_2). \end{cases}$$

by Poincaré duality. For an abelian group G and a prime number p, the subgroup $\{x \in G \mid px = 0\}$ of G is denoted by $G^{(p)}$. If G is finite, then we have $G^{(p)} \cong G \otimes Z_p$. By $\beta_d^{(p)}(X; Z)$, we denote the Z_p-rank of $H_d(X; Z)^{(p)}$. Under these notations, we have the following Z_2-*torsion relation*

$$\beta_1(M; Z_2) = \beta_2^{(2)}(\Sigma; Z).$$

Throughout this paper, by a *closed 3-manifold* we mean a closed connected oriented 3-manifold, and by a *punctured 3-manifold* M^0 of a closed 3-manifold M, the bounded 3-manifold $M^0 = \mathrm{cl}(M \backslash D^3)$ for a 3-disk D^3 in M. An embedding[1] $k : M \to Y$ from a closed oriented 3-manifold M into an orientable (possibly non-compact) 4-manifold Y is *of type* 1 or *type* 2 if the complement $Y \backslash k(M)$ is connected or disconnected, respectively. Since we can find a simple closed curve C in Y meeting $k(M)$ with one point transversely, we see that $H_3(Y; Z)$ has a direct summand Z with a generator represented by the 3-submanifold $k(M)$ and $H_1(Y; Z)$ has a direct summand Z with a generator represented by the circle C. Thus, if moreover $H_d(Y; Q) \cong H_d(S^1 \times S^3; Q)$ for $d = 1, 3$, then $H_3(Y; Z) \cong Z$ has a generator represented by the 3-submanifold $k(M)$ and $H_1(Y; Z)/(\text{torsion}) \cong Z$ has a generator represented by the circle C so that the induced homomorphism $k_* : H_1(M; Z) \to H_1(Y; Z)/(\text{torsion})$ is the zero map. To make our argument clear, we use the subring $Z_{/2} = Z[\frac{1}{2}]$ of Q rather than Q itself. Then the 4-manifold Σ

[1] An embedding is assumed to be a smooth or piecewise-linear embedding unless otherwise specified.

is a spin 4-manifold with $Z_{/2}$-homology of $S^1 \times S^3$ and there is a type 1 embedding $k : M \to \Sigma$. Since Σ is an fiber bundle over the circle S^1 with fiber M, we consider a closed 4-manifold $\hat{\Sigma}$ obtained from Σ by a surgery killing a section of S^1, namely

$$\hat{\Sigma} = \mathrm{cl}(\Sigma \backslash S^1 \times D^3) \cup D^2 \times \partial D^3$$

for a regular neighborhood $S^1 \times D^3$ of a section in Σ. The embedding $k : M \to \Sigma$ induces a punctured embedding

$$k^0 : M^0 \to \hat{\Sigma}$$

for a punctured 3-manifold M^0 of M and we have

$$H_d(\hat{\Sigma}; Z) \cong \begin{cases} Z & (d = 0, 4) \\ {Z_2}^s & (d = 1, 2) \\ 0 & (\text{others}), \quad \text{where } s = \beta_1(M; Z_2). \end{cases}$$

In particular, $\hat{\Sigma}$ is a spin 4-manifold with $Z_{/2}$-homology of the 4-sphere S^4 with theZ_2-torsion relation

$$\beta_1(M; Z_2) = \beta_2^{(2)}(\hat{\Sigma}; Z).$$

As a result, we obtain the following observation on a pair of a 3-manifold and a 4-manifold which is constructed from every invertible knot reflecting a topological meaning of an invertible knot.

Observation 1.1. Every Dehn surgery manifold $M = \chi(K; r)$ of an invertible knot K with every rational surgery coefficient $r \in Q$ is embedded in a spin 4-manifold Σ with $Z_{/2}$-homology of $S^1 \times S^3$ and with theZ_2-torsion relation $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$ by a type 1 embedding $k : M \to \Sigma$. Further, there is a punctured embedding $k^0 : M^0 \to \hat{\Sigma}$ for a spin 4-manifold $\hat{\Sigma}$ with $Z_{/2}$-homology of S^4 and with theZ_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) = \beta_1(M; Z_2)$ which is obtained from Σ by a surgery killing a generator of $H_1(\Sigma; Z)/(2\text{-torsion}) = Z$.

Here is a remark showing that a $Z_{/2}$-homology 4-sphere $\hat{\Sigma}$ cannot be always replaced by S^4.

Remark 1.2. One may expect that every punctured manifold M^0 of every Dehn surgery manifold $M = \chi(K; r)$ of an invertible knot K is embeddable in the 4-sphere S^4, but it is not true. For example, the lens space $L(p, q)$ is a Dehn surgery manifold of a trivial knot which is an invertible knot and it is known by Epstein[2] and Zeeman[17] that the punctured manifold $L(p, q)^0$ is embeddable in S^4 if and only if p is odd or $p = 0$. If we consider the 0-surgery manifold $M = \chi(K; 0)$ of the trefoil knot K (known to be invertible), then it is shown in [3] that M^0 is not embeddable in S^4.

The embeddings $M \to \Sigma$ and $M^0 \to \hat{\Sigma}$ both constructed from an invertible knot K are most standard examples of a *closed Samsara manifold Σ on an invertible 3-manifold M* and a *reduced closed Samsara 4-manifold $\hat{\Sigma}$ on a punctured invertible 3-manifold M^0*. Similar examples are constructed from every component-conservatively invertible link and more generally from an invertible 3-manifold. After observing that not every closed 3-manifold is an invertible 3-manifold, we show a main result (Theorem 3.2) that we can have a *bounded* Samsara 4-manifold Σ on every closed 3-manifold and a reduced bounded Samsara 4-manifold $\hat{\Sigma}$ on every punctured 3-manifold with the Z_2-torsion relation. In Remark 3.3, we observe that there are closed 3-manifolds which are not type 2 embeddable in any (closed or bounded) Samsara 4-manifold Σ on any closed 3-manifold M and not embeddable in any (closed or bounded) reduced Samsara 4-manifold $\hat{\Sigma}$ on any punctured 3-manifold M^0.

In Section 2, we generalize Observation 1.1 in the introduction to an observation (Observation 2.1) arising from a component-conservatively invertible link. Then we introduce a concept of an invertible 3-manifold to obtain a similar observation (Observation 2.3). Some non-invertible 3-manifolds are also given here. In Section 3, we define a Samsara 4-manifold on a closed 3-manifold and a reduced Samsara 4-manifold on a punctured 3-manifold . In Observation 3.1, some observations on the Z_2-torsion relation are given. Then we state our main result (Theorem 3.2). In Remark 3.3, non-embedding results relating to Theorem 3.2 are given. In Section 4, we explain an estimate on the signature theorem for an infinite cyclic covering, which is a main tool to prove the Z_2-torsion relation in Theorem 3.2. The proof of Theorem 3.2 is done in Section 5. One remaining open question is also stated there.

Finishing this introduction, we note that the Samsara 4-manifolds on 3-manifolds constructed in this paper are used in [12] to construct a necessary example on an open orientable 4-manifold with every closed 3-manifold type 1 embedded.

2. A generalization to a component-conservatively invertible link and invertible 3-manifolds

An oriented link L with components K_i $(i = 1, 2, ..., n)$ in S^3 is *component-conservatively invertible* if there is an orientation-preserving self-homeomorphism $f : S^3 \to S^3$ sending K_i to $-K_i$ for every i. The self-homeomorphism f induces an orientation-preserving and spin-structure-preserving self-homeomorphism $h : M \to M$ such that $h_* = -1 : H_1(M; Z) \to H_1(M; Z)$ for the Dehn surgery 3-manifold $M = \chi(L; \mathbf{r})$ with every surgery coefficient $\mathbf{r} \in Q^n$. Let Σ be the mapping torus of h. Then we have

$$H_1(\Sigma; Z) \cong Z \oplus H_1(M; Z_2) \cong Z \oplus {Z_2}^s$$

for an integer $s \geqq 0$. Since the Euler characteristic $\chi(\Sigma) = 0$, we obtain by Poincaré duality

$$H_d(\Sigma; Z) \cong \begin{cases} Z & (d = 0, 3, 4) \\ {Z_2}^s & (d = 2) \\ Z \oplus {Z_2}^s & (d = 1) \\ 0 & (\text{others}), \text{where } s = \beta_1(M; Z_2). \end{cases}$$

Thus, there is an embedding $k : M \to \Sigma$ of type 1 and the 4-manifold Σ is a spin 4-manifold with $Z_{/2}$-homology of $S^1 \times S^3$ and with $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$. Since Σ is an fiber bundle over the circle S^1 with fiber M, we consider a closed spin 4-manifold $\hat{\Sigma}$ obtained from Σ by a surgery killing a section of S^1. Then we have

$$H_d(\hat{\Sigma}; Z) \cong \begin{cases} Z & (d = 0, 4) \\ {Z_2}^s & (d = 1, 2) \\ 0 & (\text{others}), \text{where } s = \beta_1(M; Z_2). \end{cases}$$

In particular, there is an embedding $k^0 : M^0 \to \hat{\Sigma}$ and $\hat{\Sigma}$ has the $Z_{/2}$-homology of S^4 with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) = \beta_1(M; Z_2)$. We obtain the following observation similar to Observation 1.1.

Observation 2.1. Every Dehn surgery manifold $M = \chi(L; \mathbf{r})$ for every component-conservatively invertible link L and every surgery coefficient $\mathbf{r} \in Q^n$ is embedded in a spin 4-manifold Σ with $Z_{/2}$-homology of $S^1 \times S^3$ by a type 1 embedding $k : M \to \Sigma$ with $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$. Further, there is a punctured embedding $k^0 : M^0 \to \hat{\Sigma}$ for a spin 4-manifold $\hat{\Sigma}$ with $Z_{/2}$-homology of S^4 and with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) = \beta_1(M; Z_2)$ which is obtained from Σ by killing a generator of $H_1(\Sigma; Z)/(2\text{-torsion}) \cong Z$.

A closed 3-manifold M is *invertible* if there is an orientation-preserving self-homeomorphism $h : M \to M$ such that

$$h_* = -1 : H_1(M; Z) \to H_1(M; Z).$$

Here are some examples of invertible 3-manifolds.

Example 2.2.
(1) Every Dehn surgery manifold $M = \chi(L; \mathbf{r})$ for every component-conservatively invertible link L and every surgery coefficient $\mathbf{r} \in Q^n$ is an invertible 3-manifold.
(2) The double branched cover of S^3 branched along every link is an invertible 3-manifold.
(3) Every closed 3-manifold of Heegaard genus $\leqq 2$ is an invertible 3-manifold. In particular, every lens space is an invertible 3-manifold.

In fact, (1) was discussed in Section 2. To see (2), we consider a tangle decomposition $(D_1^3, T_1) \cup (D_2^3, T_2)$ of every link (S^3, L) such that T_i is a trivial tangle in a 3-disk D_i^3 for $i = 1, 2$. The double branched covering spaces $D_i^3(T_i)_2$ $(i = 1, 2)$ are handlebodies and thus give a Heegaard decomposition of the double branched covering space $M = S^3(L)_2$. The covering transformation t on M gives a hyperelliptic involution on the Heegaard surface $F = \partial B_1^3(T_1)_2 = \partial B_2^3(T_2)_2$ inducing the (-1)-multiple map of $H_1(F; Z)$. This implies that t induces the (-1)-multiple map of $H_1(M; Z)$. (3) follows from the fact that every closed 3-manifold of Heegaard genus $\leqq 2$ is the double covering space of S^3 branched along a link (see Birman-Hilden[1], Viro[16]).

Considering the mapping torus Σ of $h : M \to M$ for an invertible 3-manifold M as it is discussed in Sections 1 and 2, we obtain the following observation similar to Observations 1.1 and 2.1, which has been implicitely used in [7, Theorem 2.5].

Observation 2.3. Every invertible 3-manifold M is embedded in a spin 4-manifold Σ with $Z_{/2}$-homology of $S^1 \times S^3$ and with $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$ by a type 1 embedding $k : M \to \Sigma$. Further, there is a punctured embedding $k^0 : M^0 \to \hat{\Sigma}$ for a spin 4-manifold $\hat{\Sigma}$ with $Z_{/2}$-homology of S^4 and with $\beta_2^{(2)}(\hat{\Sigma}; Z) = \beta_1(M; Z_2)$ which is obtained from Σ by a surgery killing a generator of $H_1(\Sigma; Z)/(2\text{-torsions}) \cong Z$.

An important point in Observation 2.3 is that the embedding $k : M \to \Sigma$ is of type 1, because for every compact oriented 4-manifold W, there is an invertible 3-manifold M which is not embeddable in W by any type II embedding (see [7]). Here are some examples of non-invertible 3-manifolds.

Example 2.4.
(1) Every closed hyperbolic 3-manifold with no symmetry or with only odd symmetries is a non-invertible 3-manifold.
(2) Every closed 3-manifold M admitting 3 elements $u_1, u_2, u_3 \in H^1(M; Z_p)$ for an odd prime $p > 1$ such that $u_1 \cup u_2 \cup u_3 \neq 0$ in $H^3(M; Z_p) \cong Z_p$ is a non-invertible 3-manifold. For example, if M is Z_p-homology cobordant to the connected sum $T^3 \# M'$ of the 3-torus $T^3 = S^1 \times S^1 \times S^1$ and any closed 3-manifold M', then M satisfies this condition and hence is non-invertible.

In fact, (1) follows since every invertible hyperbolic 3-manifold must admit an even order isometry by Mostow rigidity theorem (see Thurston[15]). To see (2), suppose that there is an orientation-preserving self-homeomorphism h of M such that $h_* = -1 : H_1(M; Z) \to H_1(M; Z)$. Then $h^* = -1 : H^1(M; Z_p) \to H^1(M; Z_p)$, so that

$$h^*(u_1 \cup u_2 \cup u_3) = h^*(u_1) \cup h^*(u_2) \cup h^*(u_3) = -(u_1 \cup u_2 \cup u_3).$$

Hence $h^* = -1 : H^3(M; Z_p) \to H^3(M; Z_p)$ meaning that h is an orientation-reversing self-homeomorphism, which is a contradiction. Thus, M is a non-invertible 3-manifold showing (2).

3. Samsara 4-manifold on 3-manifolds

We shall define a closed Samsara 4-manifold and a closed reduced Samsara 4-manifold by the homologies of the closed oriented 4-manifolds constructed from an invertible 3-manifold as follows:

Definition. A *closed Samsara 4-manifold* on a closed 3-manifold M is a spin 4-manifold Σ with $Z_{/2}$-homology of $S^1 \times S^3$ such that there is a type 1 embedding $k : M \to \Sigma$. A *reduced closed Samsara 4-manifold* on a punctured 3-manifold M^0 is a spin 4-manifold $\hat{\Sigma}$ with $Z_{/2}$-homology of S^4 such that there is a punctured embedding $k^0 : M^0 \to \hat{\Sigma}$.

It is unknown whether there is a closed Samusara 4-manifold on every closed 3-manifold. Instead, we shall construct a bounded Samsara 4-manifold on every closed 3-manifold and a bounded reduced Samsara 4-manifold on every punctured 3-manifold as an analogy of the construction from an invertible knot. To define a bounded Samsara 4-manifold, we note that there are three kinds of *4-dimensional solid tori* with boundary the 3-torus T^3, which are defined as follows:

(i) The *solid torus with one meridian disk* is $T^2 \times D^2$.

(ii) The *solid torus with two meridian disks* is the compact exterior $ET^2 = \mathrm{cl}(S^4 \backslash T^2 \times D^2)$ for an unknotted torus-knot T^2 in S^4 where the two loops $S^1 \times 1 \times 1$ and $1 \times S^1 \times (-1)$ of T^3 bound disjoint disks.

(iii) The *solid torus with three medridian disks* is the 4-manifold $D(T^3)$ obtained from the 4-disk D^4 by attaching the 0-framed 2-handles h_i^2 $(i = 1, 2, 3)$ along the components of the Borromean rings L_B, where the three loops $S^1 \times 1 \times 1$, $1 \times S^1 \times (-1)$ and $(-1) \times (-1) \times S^1$ of T^3 bound disjoint disks which are the dual core disks of h_i^2 $(i = 1, 2, 3)$ (see Matumoto [14]).

The solid torus $D(T^3)$ with three medridian disks is a simply connected spin 4-manifold with the following homology:

$$H_d(D(T^3); Z) \cong \begin{cases} Z^3 & (d = 2) \\ Z & (d = 0) \\ 0 & (\text{others}). \end{cases}$$

We see from this calculation that the intersection form on $H_2(D(T^3); Z)$ is the zero form, because the natural homomorphism $H_2(T^3; Z) \to H_2(D(T^3); Z)$ is an isomorphism. The solid torus with one meridian disk or two meridian disks is embedded in

S^4 and obtained from $D(T^3)$ by a surgery killing one sphere or two spheres in $D(T^3)$ respectively. On the other hand, *the solid torus* $D(T^3)$ *with three medridian disks cannot be embedded in* S^4. To see this, suppose that $D(T^3)$ is embedded in S^4. Then the exterior $W = \mathrm{cl}(S^4 \backslash D(T^3))$ has the homology:

$$H_d(W; Z) \cong \begin{cases} Z^3 & (d = 1) \\ Z & (d = 0) \\ 0 & (\text{others}). \end{cases}$$

It is shown in [12, p.192] that T^3 is not the boundary of a compact oriented 4-manifold W with $H_2(W; Z) = 0$, a contradiction. Thus, $D(T^3)$ cannot be embedded in S^4. For $s \geqq 2$, we denote by $D(sT^3)$ the disk sum of s copies of the solid torus $D(T^3)$ with three medridian disks, which is a simply connected spin 4-manifold with $\partial D(sT^3) = \# sT^3$ (the connected sum of s copies of T^3) and with $H_2(D(sT^3); Z) \cong Z^{3s}$ such that the intersection form $\mathrm{Int} : H_2(D(sT^3); Z) \times H_2(D(sT^3); Z) \to Z$ is the zero form. Let

$$\mathbf{\Sigma} = S^1 \times S^3 \# D(sT^3), \quad \text{and} \quad \hat{\mathbf{\Sigma}} = S^4 \# D(sT^3)$$

for a positive integer s. We define a bounded Samsara 4-manifold on a closed 3-manifold and a bounded reduced Samsara 4-manifold on a punctured 3-manifold as follows:

Definition. A *bounded Samsara 4-manifold* on a closed 3-manifold M is a compact oriented spin 4-manifold Σ with $Z_{/2}$-homology of $\mathbf{\Sigma}$ for some s such that there is a type 1 embedding

$$k : M \to \Sigma$$

inducing the trivial homomorphism

$$k_* = 0 : H_2(M; Z_{/2}) \to H_2(\Sigma; Z_{/2}) = {Z_{/2}}^{3s}.$$

A *reduced bounded Samsara 4-manifold* on a punctured 3-manifold M^0 is a compact oriented spin 4-manifold $\hat{\Sigma}$ with $Z_{/2}$-homology of $\hat{\mathbf{\Sigma}}$ for some s such that there is a punctured embedding

$$k^0 : M^0 \to \hat{\Sigma}$$

inducing the trivial homomorphism

$$k^0_* = 0 : H_2(M^0; Z_{/2}) \to H_2(\hat{\Sigma}; Z_{/2}) = {Z_{/2}}^{3s}.$$

In the definition above, $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ for all s are called the *standard Samsara 4-manifolds* on S^3 and the *standard reduced bounded Samsara 4-manifolds* on the punctured 3-sphere $(S^3)^0$, respectively, where we understand that $\mathbf{\Sigma} = S^1 \times S^3$ and $\hat{\mathbf{\Sigma}} = S^4$ for $s = 0$. The following computations are made by using Poincaré duality and the homology exact sequences for $(\Sigma, \partial\Sigma)$ and $(\hat{\Sigma}, \partial\hat{\Sigma})$, which show that there are no $Z_{/2}$-homological differences between $(\Sigma, \partial\Sigma)$ and $(\mathbf{\Sigma}, \partial\mathbf{\Sigma})$ and between $(\hat{\Sigma}, \partial\hat{\Sigma})$ and $(\hat{\mathbf{\Sigma}}, \partial\hat{\mathbf{\Sigma}})$:

$$\begin{aligned}
\partial_* : H_4(\Sigma, \partial\Sigma; Z) \cong H_3(\partial\Sigma; Z) \cong Z, \quad & \partial_* : H_4(\hat{\Sigma}, \partial\hat{\Sigma}; Z) \cong H_3(\partial\hat{\Sigma}; Z) \cong Z, \\
j_* : H_3(\Sigma; Z) \cong H_3(\Sigma, \partial\Sigma; Z) \cong Z, \quad & H_3(\hat{\Sigma}; Z) = H_3(\hat{\Sigma}, \partial\hat{\Sigma}; Z) = 0, \\
i_* : H_2(\partial\Sigma; Z_{/2}) \cong H_2(\Sigma; Z_{/2}), \quad & i_* : H_2(\partial\hat{\Sigma}; Z_{/2}) \cong H_2(\hat{\Sigma}; Z_{/2}), \\
\partial_* : H_2(\Sigma, \partial\Sigma; Z_{/2}) \cong H_1(\partial\Sigma; Z_{/2}), \quad & \partial_* : H_2(\hat{\Sigma}, \partial\hat{\Sigma}; Z_{/2}) \cong H_1(\partial\hat{\Sigma}; Z_{/2}), \\
j_* : H_1(\Sigma; Z_{/2}) \cong H_1(\Sigma, \partial\Sigma; Z_{/2}), \quad & H_1(\hat{\Sigma}; Z_{/2}) = H_1(\hat{\Sigma}, \partial\hat{\Sigma}; Z_{/2}) = 0.
\end{aligned}$$

From the isomorphisms i_*, we have the trivial intersection forms

$$\mathrm{Int} = 0 : H_2(\Sigma; Z) \times H_2(\Sigma; Z) \to Z, \quad \mathrm{Int} = 0 : H_2(\hat{\Sigma}; Z) \times H_2(\hat{\Sigma}; Z) \to Z$$

of every Samsara 4-manifold Σ on any closed 3-manifold and every reduced Samsara 4-manifold $\hat{\Sigma}$ on any punctured 3-manifold. Further, we have the following observations:

Observation 3.1.
(1) Given a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0, we have a (closed or bounded) Samsara 4-manifold Σ on M with $H_2(\Sigma; Z) \cong H_2(\hat{\Sigma}; Z)$. In fact, we have Σ by a surgery of $\hat{\Sigma}$ along the 2-sphere $S^2 = \partial M^0$ since S^2 is null-homologous in $\hat{\Sigma}$. To see this, for a regular neighborhood $S^2 \times D^2$ of S^2 in $\hat{\Sigma}$, let

$$\Sigma = \mathrm{cl}(\hat{\Sigma} \backslash S^2 \times D^2) \cup D^3 \times \partial D^2,$$

which is seen to be a (closed or bounded) Samsara 4-manifold on M. Examining the excision isomorphism

$$H_2(\hat{\Sigma}, S^2 \times D^2; Z) \cong H_2(\Sigma, D^3 \times \partial D^2; Z),$$

we see that $H_2(\Sigma; Z) \cong H_2(\hat{\Sigma}; Z)$. Conversely, given a (closed or bounded) Samsara 4-manifold Σ on M, we obtain a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 with $H_2(\hat{\Sigma}; Z) \cong H_2(\Sigma; Z)$ by a surgery killing a generator of $H_1(\Sigma; Z)/(2\text{-torsion}) \cong Z$.

(2) Given a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0, then for every positive integer n there is a reduced (closed or bounded) Samsara 4-manifold ($\hat{\Sigma}'$ on M^0 with $\beta_2^{(2)}(\hat{\Sigma}'; Z) = \beta_2^{(2)}(\hat{\Sigma}; Z) + n$, where the boundary $\partial\hat{\Sigma} = \# sT^3$ may be unchanged or changed so that the integer s takes any larger integer. Similarly, given a (closed or bounded) Samsara 4-manifold Σ on M, then for every positive integer n there is a (closed or bounded) Samsara 4-manifold Σ' on M with $\beta_2^{(2)}(\Sigma'; Z) =$

$\beta_2^{(2)}(\Sigma; Z) + n$, where the boundary $\partial\Sigma = \#sT^3$ may be unchanged or changed so that the integer s takes any larger integer. These constructions are easily made by taking connected sums or disk sums with some 4-manifolds Y with $Z_{/2}$-homology of S^4 and with $H_1(Y; Z) \cong Z_{2m}$ for any non-zero integers m and some copies of $D(T^3)$.

(3) There are infinitely many closed 3-manifolds M such that there are reduced (closed or bounded) Samsara 4-manifolds $\hat{\Sigma}$ on $M^0$with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) < \beta_1(M; Z_2)$ and (closed or bounded) Samsara 4-manifolds Σ on M with $\beta_2^{(2)}(\Sigma; Z) < \beta_1(M; Z_2)$. For example, we consider any closed 3-manifold M with $\beta_1(M; Z_2) > 0$ whose punctured manifold M^0 is embeddable into S^4. Then S^4 is a reduced closed Samsara 4-manifold on M^0 with $\beta_2^{(2)}(S^4; Z) = 0 < \beta_1(M; Z_2)$. By a surgery of S^4 along the 2-knot $S^2 = \partial M^0$, we obtain a closed Samsara 4-manifold Σ with homology of $S^1 \times S^3$ on M so that $\beta_2^{(2)}(\Sigma; Z) = 0 < \beta_1(M; Z_2)$.

The following theorem is our main theorem.

Theorem 3.2. For every closed 3-manifold M, there is a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) = \beta_1(M; Z_2)$. Further, for every positive integer n, there are infinitely many closed 3-manifolds M such that every reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 has the Z_2-torsion relation

$$\beta_2^{(2)}(\hat{\Sigma}; Z) \geqq \beta_1(M; Z_2) = n.$$

For every closed 3-manifold M, there is a (closed or bounded) Samsara 4-manifold Σ on M with $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$. Further, for every positive integer n, there are infinitely many closed 3-manifolds M such that every (closed or bounded) Samsara 4-manifold Σ on M has

$$\beta_2^{(2)}(\Sigma; Z) \geqq \beta_1(M; Z_2) = n.$$

In the following remark, we discuss some non-embedding results on Theorem 3.2.

Remark 3.3. There are closed 3-manifolds which are not type 2 embeddable in any (closed or bounded) Samsara 4-manifold Σ on every closed 3-manifold and which are not embeddable in any reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on every punctured 3-manifold. There are punctured 3-manifolds which are not embeddable in the standard reduced Samsara 4-manifold $\hat{\mathbf{\Sigma}}$ on the punctured 3-sphere and in the standard Samsara 4-manifold $\mathbf{\Sigma}$ on S^3 for any s.

To see (1) of Remark 3.3, we need the following assertion.

(3.3.1) Let Y be a compact connected oriented 4-manifold with connected boundary ∂Y. Assume that $H_1(Y;Q) = 0$ and the natural homomorphism $H_2(\partial Y;Q) \to H_2(Y;Q)$ is an isomorphism. Let p be a prime number such that $H_1(Y;Z)^{(p)} = H_1(\partial Y;Z)^{(p)} = 0$. If a closed 3-manifold M with $H_1(M;Q) = 0$ is embedded in Y, then the linking form

$$\ell_p : H_1(M;Z)_p \times H_1(M;Z)_p \to Q/Z$$

on the p-primary component $H_1(M;Z)_p$ of $H_1(M;Z)$ is hyperbolic.

First, by assuming (3.3.1), we show (1) of Remark 3.3. Since $H_1(\hat{\varSigma};Q) = 0$ and $H_1(\hat{\varSigma};Z)^{(p)} = H_1(\partial\hat{\varSigma};Z)^{(p)} = 0$ for every odd prime p, the lens space $L(n,m)$ for any odd $n \neq \pm 1$ cannot be embedded in $\hat{\varSigma}$ by (3.3.1). If a closed 3-manifold M with $H_1(M;Q) = 0$ is type 2 embedded in $\varSigma$, then we can find a cirlce C in $\varSigma$ representing a generator of $H_1(\varSigma;Z)/(\text{torsions}) \cong Z$ and not meeting M. In fact, M splits $\varSigma$ into two 4-manifolds W_1 and W_2. Then, we may have $H_1(W_1;Q) \cong Q$ and $H_1(W_2;Q) = 0$. By Mayer-Vietoris sequence, we have a natural isomorphism

$$H_1(W_1;Z)/(\text{torsions}) \oplus H_1(W_2;Z)/(\text{torsions}) \to H_1(\varSigma;Z)/(\text{torsions}).$$

Since $H_1(W_2;Z)/(\text{torsions}) = 0$, we have a natural isomorphism

$$H_1(W_1;Z)/(\text{torsions}) \cong H_1(\varSigma;Z)/(\text{torsions}).$$

Any embedded circle C in $\text{int}W_1$ representing a generator of $H_1(W_1;Z)/(\text{torsions})$ is a desired one. Then the 3-manifold M is embedded in a reduced Samsara 4-manifold $\hat{\varSigma}$ obtained from $\varSigma$ by a surgery killing C. Thus, the lens space $L(n,m)$ for any odd $n \neq \pm 1$ cannot be embedded in $\varSigma$.

If a punctured 3-manifold M^0 with $H_1(M;Q) = 0$ is embedded in the standard reduced Samsara 4-manifold $\hat{\mathbf{\Sigma}}$, then $M^0 \times [-1,1]$ is embedded in $\hat{\mathbf{\Sigma}}$ and hence the closed 3-manifold $DM^0 = \partial(M^0 \times [-1,1])$ is embedded in $\hat{\mathbf{\Sigma}}$. By (3.3.1) the linking form on DM^0 is hyperbolic, for $H_1(\hat{\mathbf{\Sigma}};Z)^{(p)} = H_1(\partial\hat{\mathbf{\Sigma}};Z)^{(p)} = 0$ for all primes p. In particular, the linking form on the 2-primary component $H_1(M;Z)_2$ is isomorphic to a block sum of some linking forms in E_0^k ($k \geqq 1$) and E_1^k ($k \geqq 2$) described in [11]. In particular, $H_1(M;Z)_2$ is a direct double. Thus, the punctured lens space $L(m,n)^0$ for any non-zero even integer n is not embeddedable in $\hat{\mathbf{\Sigma}}$ and also in $\mathbf{\Sigma}$ from which $\hat{\mathbf{\Sigma}}$ is obtained by a surgery killing a generator of $H_1(\mathbf{\Sigma};Z) = Z$.

Proof of (3.3.1). Since $H_1(Y;Q) = 0$, we see that M splits Y into two compact connected 4-manifolds Y', Y'' where we take $\partial Y'' \supset \partial Y$. Since $H_3(Y,\partial Y;Z) = H_2(Y,\partial Y;Z)^{(p)} = 0$, the natural isomorphism $i_* : H_2(\partial Y;Q) \to H_2(Y;Q)$ implies that the natural homomorphism $i_* : H_2(\partial Y;Z) \to H_2(Y;Z)$ is injective whose cokernel is a p-torsion-free finite abelian group. Thus, in the homology with coefficients in the p-local ring

$$Z_{(p)} = \{\frac{m}{n} \in Q |\, m, n \in Z,\ (n, p) = 1\},$$

we have an isomorphism

$$i_* : H_2(\partial Y; Z_{(p)}) \cong H_2(Y; Z_{(p)}).$$

Then we see that the natural homomorphisms $i''_* : H_2(\partial Y; Z_{(p)}) \to H_2(Y''; Z_{(p)})$ and $j''_* : H_2(Y''; Z_{(p)}) \to H_2(Y; Z_{(p)})$ are isomorphisms. In fact, i_* factors through i''_* and j''_*, the composite $\tau = (i_*)^{-1} j''_* : H_2(Y''; Z_{(p)}) \to H_2(\partial Y; Z_{(p)})$ is the left inverse of $i''_* : H_2(\partial Y; Z_{(p)}) \to H_2(Y''; Z_{(p)})$. Since $H_2(M; Z) = 0$, we see from the Mayer-Vietoris sequence on $(Y', Y''; M)$ that

$$j'_* + j''_* : H_2(Y'; Z_{(p)}) \oplus H_2(Y''; Z_{(p)}) \to H_2(Y; Z_{(p)})$$

is injective. By Poincaré duality, $H_2(Y; Z_{(p)})$ is $Z_{(p)}$-free. Hence $H_2(Y'; Z_{(p)})$ and $H_2(Y''; Z_{(p)})$ are $Z_{(p)}$-free, so that i''_*, j''_* and $j'_* + j''_*$ are all isomorphisms on $Z_{(p)}$-free modules. In particular, $H_2(Y'; Z_{(p)}) = 0$, by which we have the short exact sequence

$$0 \to H_2(Y', M; Z_{(p)}) \stackrel{\partial'_*}{\to} H_1(M; Z_{(p)}) \stackrel{(i'_M)_*}{\to} H_1(Y'; Z_{(p)}) \to 0.$$

This implies the following short exact sequence

$$0 \to H_2(Y', M; Z)_p \stackrel{\partial'_*}{\to} H_1(M; Z)_p \stackrel{(i'_M)_*}{\to} H_1(Y'; Z)_p \to 0.$$

By Poincaré duality, this means that the linking form $\ell_p : H_1(M; Z)_p \times H_1(M; Z)_p \to Q/Z$ has

$$\ell_p(\ker(i'_M)_*, \ker(i'_M)_*) = 0 \quad \text{and} \quad |H_1(M; Z)_p| = |H_1(Y'; Z)_p|^2.$$

Using that $j'_* + j''_*$ is onto and $H_1(Y; Z_{(p)}) = 0$, we have a natural isomorphism

$$(i'_M)_* + (i''_M)_* : H_1(M; Z_{(p)}) \to H_1(Y'; Z_{(p)}) \oplus H_1(Y''; Z_{(p)}),$$

implying that

$$(i'_M)_* + (i''_M)_* : H_1(M; Z)_p \cong H_1(Y'; Z)_p \oplus H_1(Y''; Z)_p.$$

Thus, we have $|H_1(M; Z)_p| = |H_1(Y'; Z)_p||H_1(Y''; Z)_p|$ and hence $|H_1(Y'; Z)_p| = |H_1(Y''; Z)_p|$. The homology sequence of the pair $(Y'', M \cup \partial Y)$ induces the following exact sequence:

$$0 \to H_2(Y'', M \cup \partial Y; Z)_p \stackrel{\bar{\partial}''_*}{\to} H_1(M \cup \partial Y; Z)_p \stackrel{\bar{i}''_*}{\to} H_1(Y''; Z)_p \to 0,$$

because we have $|H_1(M;Z)_p| = |H_1(Y'';Z)_p|^2$, a natural isomorphism $H_1(M;Z)_p \cong H_1(M \cup \partial Y;Z)_p$ implied by $H_1(\partial Y;Z)^{(p)} = 0$, and an isomorphism $H_2(Y'', M \cup \partial Y;Z)_p \cong H_1(Y'';Z)_p$ obtained by Poincaré duality. For the linking form $\bar{\ell}_p : H_1(M \cup \partial Y;Z)_p \times H_1(M \cup \partial Y;Z)_p \to Q/Z$, we have

$$\bar{\ell}_p(\mathrm{ker}\bar{i}''_*, \mathrm{ker}\bar{i}''_*) = 0.$$

Since $(i'_M)_* : H_1(M;Z)_p \to H_1(Y'';Z)_p$ is identical to $\bar{i}''_* : H_1(M \cup \partial Y;Z)_p \to H_1(Y'';Z)_p$ and the linking form $\ell_p : H_1(M;Z)_p \times H_1(M;Z)_p \to Q/Z$ is identical to $\bar{\ell}_p$, we see that

$$\ell_p(\mathrm{ker}(i'_M)_*, \mathrm{ker}(i'_M)_*) = 0.$$

Since $H_1(M;Z)_p \cong \mathrm{ker}(i'_M)_* \oplus \mathrm{ker}(i''_M)_*$, we see that the linking form $\ell_p : H_1(M;Z)_p \times H_1(M;Z)_p \to Q/Z$ is hyperbolic, completing the proof of (3.3.1).

The following remark concerns the definitions of a Samsara 4-manifold and a reduced Samsara 4-manifold.

Remark 3.4. We replace $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ by the 4-manifolds

$$\mathbf{\Sigma}^* = S^1 \times S^3 \# sD(T^3) \quad \text{and} \quad \hat{\mathbf{\Sigma}}^* = S^4 \# sD(T^3)$$

whose boundaties are respectively the disjoint union of s copies of T^3 to define a "new bounded Samsara 4-manifold" Σ^* on M and a "new reduced bounded Samsara 4-manifold" $\hat{\Sigma}^*$ on M^0 by the following three conditions:

(1) $H_*(\Sigma^*;Z_{/2}) \cong H_*(\mathbf{\Sigma}^*;Z_{/2})$ and $H_*(\hat{\Sigma}^*;Z_{/2}) \cong H_*(\hat{\mathbf{\Sigma}}^*;Z_{/2})$,
(2) $\partial \Sigma^* = \partial \mathbf{\Sigma}^*$ and $\partial \hat{\Sigma}^* = \partial \hat{\mathbf{\Sigma}}^*$,
(3) There is a type 1 embedding $M \to \Sigma^*$ and there is an embedding $M^0 \to \hat{\Sigma}^*$.

Then the same results (Observation 3.1 and Theorem 3.2) hold for Σ^* and $\hat{\Sigma}^*$ in place of Σ and $\hat{\Sigma}$ except in (4) of Observation 3.1 where a type 2 embedding $M \to \Sigma$ must be understood as a type 2 embedding $M \to \Sigma^*$ with $[M] = 0$ in $H_3(\Sigma^*;Z)$.

4. An estimate on the signature theorem for an infinite cyclic covering

We need an estimate on the signature theorem in [6] and [7].

Let Y be a compact connected oriented 4-manifold with boundary a closed 3-manifold B. For the intersection form $\mathrm{Int} : H_2(Y;Z) \times H_2(Y;Z) \to Z$, let

$$O_2(Y;Z) = \{x \in H_2(Y;Z) \mid \mathrm{Int}(x, H_2(Y;Z)) = 0\}.$$

Then the quotient group $\hat{H}_2(Y;Z) = H_2(Y;Z)/O_2(Y;Z)$ is a free abelian group of finite rank, whose rank is denoted by $\hat{\beta}_2(Y;Z)$. For example, we have

$$\hat{\beta}_2(\Sigma;Z) = \hat{\beta}_2(\hat{\Sigma};Z) = 0$$

for every (closed or bounded) Samsara 4-manifold Σ on any M and every reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on any M^0. Assume that there is an epimorphism $\gamma : H_1(Y; Z) \to Z$. We take the infinite cyclic covering $(\tilde{Y}, \tilde{B})$ of (Y, B) associated with γ. Then $H_2(\tilde{Y}; Q)$ is a finitely generated Γ-module for the Laurent polynomial ring $\Gamma = Q[t, t^{-1}]$. We consider the Γ-intersection form

$$\mathrm{Int}_\Gamma : H_2(\tilde{Y}; Q) \times H_2(\tilde{Y}; Q) \to \Gamma$$

defined by

$$\mathrm{Int}_\Gamma(x, y) = \sum_{m=-\infty}^{+\infty} \mathrm{Int}_Q(x, t^{-m}y)t^m$$

for $x, y \in H_2(\tilde{Y}; Q)$, where Int_Q denotes the ordinary intersection pairing on $H_2(\tilde{Y}; Q)$. Then we have the identity

$$\mathrm{Int}_\Gamma(f(t)x, g(t)y) = f(t^{-1})g(t)\mathrm{Int}_\Gamma(x, y)$$

for all $x, y \in H_2(\tilde{Y}; Q)$. Let

$$O_2(\tilde{Y}; Q)_\Gamma = \{x \in H_2(\tilde{Y}; Q) \mid \mathrm{Int}_\Gamma(x, H_2(\tilde{Y}; Q)) = 0\}.$$

Then the quotient Γ-module $\hat{H}_2(\tilde{Y}; Q)_\Gamma = H_2(\tilde{Y}; Q)/O_2(\tilde{Y}; Q)_\Gamma$ is a free Γ-module of finite rank, whose rank is denoted by $\hat{\beta}_2(\tilde{Y}; Q)_\Gamma$. Let $A(t)$ be a Γ-Hermitian matrix representing the Γ-intersection form Int_Γ on $\hat{H}_2(\tilde{Y}; Q)_\Gamma$. For $a, x \in (-1, 1)$, we define

$$\tau_{a\pm 0}(\tilde{Y}) = \lim_{x \to a\pm 0} \mathrm{sign} A(\omega_x)$$

for the complex number $\omega_x = x + \sqrt{1 - x^2}i$ of norm one. By the quadratic form

$$b : \mathrm{Tor}_\Gamma H_1(\tilde{B}; Q) \times \mathrm{Tor}_\Gamma H_1(\tilde{B}; Q) \to Q$$

on the Γ-torsion module $\mathrm{Tor}_\Gamma H_1(\tilde{B}; Q)$ of $H_1(\tilde{B}; Q)$, we have the signature invariants $\sigma_a(\tilde{B})$ ($a \in [-1, 1]$) of $\tilde{B}$, taking the value 0 except a finite number of a (see [12, 5]). We set

$$\begin{aligned}
\sigma_{[a,1]}(\tilde{B}) &= \sum_{x \in [a,1]} \sigma_x(\tilde{B}), \\
\sigma_{(a,1]}(\tilde{B}) &= \sum_{x \in (a,1]} \sigma_x(\tilde{B}), \\
\sigma_{(-1,1]}(\tilde{B}) &= \lim_{a \to -1+0} \sigma_{(a,1]}(\tilde{B}).
\end{aligned}$$

Let $\kappa_1(\tilde{B})$ denote the Q-dimension of the kernel of the homomorphism $t-1$: $H_1(\tilde{B};Q) \to H_1(\tilde{B};Q)$. The following signature theorem is given in [6] and contains an estimate of the signature explained in [7, Theorem 1.6] (see also [12] for a non-compact version):

Signature Theorem. We have the following identities.

$$\tau_{a-0}(\tilde{Y}) - \mathrm{sign}Y = \sigma_{[a,1]}(\tilde{B}),$$
$$\tau_{a+0}(\tilde{Y}) - \mathrm{sign}Y = \sigma_{(a,1]}(\tilde{B}).$$

For every $a \in (-1,1)$, we have the inequalities

$$|\sigma_{(a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \leqq \hat{\beta}_2(Y;Z) + |\mathrm{sign}(Y)| \leqq 2\hat{\beta}_2(Y;Z).$$

5. Proof of Theorem 3.2.

We are in a position to prove Theorem 3.2.

5.1 Proof of Theorem 3.2. If M is invertible, then there is a reduced closed Samsara 4-manifold $\hat{\Sigma}$ on M^0 by Observation 2.3. If M is not invertible, then M is the Dehn surgery manifold $\chi(L,0)$ with all zero coefficients for a component-conservatively non-invertible link L in S^3 of some r components (see for example [10]). By a result of Murakami-Nakanishi in [13], the link $-L$ which is the same link as L but with the orientation reversed is obtained by a fusion of of a link which is a split union of L and some copies of the Borromean rings L_{B_i} $(i=1,2,...,s)$. Thus, there is a proper oriented surface F consisting of punctured annuli in $S^3\times[0,1]$ such that $\partial F = (L \cup_{i=1}^{s} L_{B_i}) \times 0 \cup (-L)\times 1$. By attaching 2-handles $D^2\times D^2_i$ with 0 framing $(i=1,2,...,3s)$ to $S^3\times 0$ along the components of the sublink $\cup_{i=1}^{3s} L_{B_i}\times 0$, the surface F extends to the union A of r proper annuli $S^1\times[0,1]_i$ $(i=1,2,...,r)$ with $\partial A = L\times 0\cup(-L)\times 1$ in the connected sum $Y = S^3\times[0,1]\#D(sT^3)$. Then the zero framings on the components of L in S^3 extend to a trivial normal disk bundle $A\times D^2$ on A in Y. Replacing $S^1\times[0,1]_i\times D^2$ with $D^2\times[0,1]_i\times\partial D^2$ for every i, we obtain a spin 4-manifold Σ' with $\partial\Sigma' = \#sT^3\cup M\cup -M$. We identify M and $-M$ by considering them as the copies of M to obtain from Σ' a bounded Samsara 4-manifold Σ on M with $\partial\Sigma = \#sT^3$. Let $\mathbf{\Sigma} = S^1\times S^3\#D(sT^3)$ be the standard Samsara 4-manifold on S^3 obtained from Y by identifying $S^3\times 0$ with $S^3\times 1$. Σ is also obtained from $\mathbf{\Sigma}$ by replacing a normal disk bundle $N(K_B)$ of the union K_B of r Klein bottles in $\mathbf{\Sigma}$ obtained from the normal disk bundle $A\times D^2$ on A in Y with the union of r twisted S^1-bundles over the solid Klein bottles. To confirm that Σ is a bounded Samsara 4-manifold on M, we have the following properties on Σ.

(1) Since Y is simply connected, every element of $H_1(Y\backslash A;Z)$ is generated by meridians of A in Y, so that the natural map

$$H_1(S^3\times 0\backslash L\times 0;Z)\to H_1(Y\backslash A;Z)$$

is onto and hence the natural map $H_1(M\times 0;Z)\to H_1(\Sigma';Z)$ is onto.

(2) By (1) and from construction, there is a natural exact sequence

$$H_1(M;Z)\xrightarrow{k_*}H_1(\Sigma;Z)\to Z\to 0$$

for the embedding $k: M\subset\Sigma$ arising from the inclusion $M\subset\Sigma'$, where the image $\mathrm{Im}(k_*)\subset H_1(\Sigma;Z)$ is generated by order 2 elements, meaning $H_1(\Sigma;Z_{/2})\cong Z_{/2}$.

(3) There is a Z-basis $x_i\in H_2(Y;Z)$ $(i=1,2,\ldots,3s)$ with $\mathrm{Int}(x_i,x_j)=0$ for all i,j such that every x_i is represented by an embedded surface S_i disjoint from A.

(4) There is a Z-basis $y_i\in H_2(Y,\#sT^3;Z)$ $(i=1,2,\ldots,3s)$ with $\mathrm{Int}(x_i,y_j)=\delta_{ij}$ for all i,j such that every y_i is represented by an embedded proper 2-disk D_i meeting S_i transversely with one point and $D_i\cap D_j=\emptyset$ and $D_i\cap S_j=\emptyset$ for distinct i and j. Also, every D_i meets A transversely with one point in Y.

(5) The loops ∂D_i $(i=1,2,\ldots,3s)$ form a Z-basis of $H_1(\#sT^3;Z)$.

(6) From D_i and its parallel D'_i, we can construct an annulus $A(D_i)$ in $\mathbf{\Sigma}$ disjoint from K_B with

$$[\partial A(D_i)]=[\partial D_i+\partial D'_i]=2[\partial D_i]\in H_1(\#sT^3;Z)$$

by piping D_i and D'_i along an arc on K_B. Let $y_i^*=\frac{1}{2}[A(D_i)]\in H_2(\Sigma,\partial\Sigma;Z_{/2})$. Then the boundary operator

$$\partial_*: H_2(\Sigma,\partial\Sigma;Z_{/2})\to H_1(\partial\Sigma;Z_{/2})$$

is onto, because $\partial_*(y_i^*)=[\partial D_i]$ $(i=1,2,\ldots,3s)$ form a basis for $H_1(\partial\Sigma;Z_{/2})$. Since $\partial\Sigma$ is connected, the natural map

$$j_*: H_1(\Sigma;Z_{/2})\to H_1(\Sigma,\partial\Sigma;Z_{/2})$$

is an isomorphism. By (2) and Poincaré duality, we obtain $H_3(\Sigma;Z_{/2})\cong Z_{/2}$, where M represents a generator.

(7) $\chi(\Sigma)=\chi(\mathbf{\Sigma})=3s-1$ implies $\dim_Q H_2(\Sigma;Q)=3s$. By Poincaré duality, we have $H_2(\Sigma;Z_{/2})\cong {Z_{/2}}^{3s}$. Regarding $x_i\in H_2(\Sigma;Z_{/2})$ $(i=1,2,\ldots,3s)$, we have $\mathrm{Int}_{Z_{/2}}(x_i,y_i^*)=\delta_{ij}$ for all i,j. Hence x_i $(i=1,2,\ldots,3s)$ form a $Z_{/2}$-basis of $H_2(\Sigma;Z_{/2})$ and by Poincaré duality y_i^* $(i=1,2,\ldots,3s)$ form a $Z_{/2}$-basis of $H_2(\Sigma,\partial\Sigma;Z_{/2})$. In particular, we have $H_*(\Sigma;Z_{/2})\cong H_*(\mathbf{\Sigma};Z_{/2})$.

(8) Since M is disjoint from a 2-cycle representing $y_i^* \in H_2(\Sigma, \partial\Sigma; Z_{/2})$ and x_i ($i = 1, 2, \ldots, 3s$) form a $Z_{/2}$-basis of $H_2(\Sigma; Z_{/2})$, we see from the identity $\mathrm{Int}_{Z_{/2}}(x_i, y_j^*) = \delta_{ij}$ that $k_* = 0 : H_2(M; Z_{/2}) \to H_2(\Sigma; Z_{/2})$.

From (1)-(8) we see that Σ is a bounded Samsara 4-manifold on M. By Observation 3.1 (1), we have a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 with $H_2(\hat{\Sigma}; Z) \cong H_2(\Sigma; Z)$ by a surgery killing a generator of $H_1(\Sigma; Z)/(2\text{-torsion}) \cong Z$.

By the property (2), we have the Z_2-torsion relation $\beta_1(M; Z_2) \geqq \beta_1^{(2)}(\hat{\Sigma}; Z)$. Since $\partial\hat{\Sigma}$ is connected, the group $H_1(\hat{\Sigma}, \partial\hat{\Sigma}; Z)$ is a quotient torsion group of $H_1(\hat{\Sigma}; Z)$, so that by Poincaré duality the torsion part of $H_2(\hat{\Sigma}; Z)$ is isomorphic to $H_1(\hat{\Sigma}, \partial\hat{\Sigma}; Z)$. Thus, we have the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) \leqq \beta_1(M; Z_2)$. If $\beta_2^{(2)}(\hat{\Sigma}; Z) < \beta_1(M; Z_2)$, then we can obtain a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}'$ on M^0 with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}'; Z) = \beta_1(M; Z_2)$ by Observation 3.1 (2). Counting Observation 3.1 (1), we also have a (closed or bounded) Samsara 4-manifold Σ on M with $\beta_2^{(2)}(\Sigma; Z) = \beta_1(M; Z_2)$. This completes the proof of the existence part of Theorem 3.2.

To prove the inequality part of Theorem 3.2, we need a computation of the signature invariant on an infinite cyclic covering of the double $DM^0 = \partial(M^0 \times [-1, 1])$ of a punctured 3-manifold M^0.

For every positive integer n, we take n knots K_i ($1 \leqq i \leqq n$) whose signatures $\sigma(K_i)$ ($1 \leqq i \leqq n$) have the condition that $|\sigma(K_1)| > 0$ and

$$|\sigma(K_i)| > \left|\sum_{j=1}^{i-1} \sigma(K_j)\right| \qquad (i = 2, 3, \ldots, n).$$

Let $M_i = \chi(K_i, 0)$ and $M = M_1 \# M_2 \# \ldots \# M_n$. We call M an *efficient 3-manifold of rank* n. We note here that for any n there are infinitely many efficient 3-manifolds M with $\beta_1(M; Z) = \beta_1(M; Z_2) = n$.

A homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$ is *Z_2-asymmetric* if the Z_2-reduction $\dot{\gamma}_2 : H_1(DM^0; Z) \to Z_2$ is not invariant under the reflection α on the double DM^0. From construction and [7, Lemma 1.3], we see that

$$\sigma_{(-1,1]}(\widetilde{DM^0}) \neq 0$$

for every efficient 3-manifold M of any rank n and every Z_2-asymmetric homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$. Also, from construction, we see that

$$\kappa_1(\widetilde{DM^0}) = 0$$

for every homomorphism $\dot{\gamma} : H_1(DM^0; Z) \to Z$.

For every efficient 3-manifold M of any rank $n \geqq 1$ and every reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0, we shall show the Z_2-torsion relation

$$\beta_2^{(2)}(\hat{\Sigma}; Z) \geqq \beta_1(M; Z_2) = n.$$

Suppose that there is a reduced (closed or bounded) Samsara 4-manifold $\hat{\Sigma}$ on M^0 with the Z_2-torsion relation $\beta_2^{(2)}(\hat{\Sigma}; Z) < \beta_1(M; Z_2) = n$. For any punctured embedding $k^0 : M^0 \subset \hat{\Sigma}$, we note that the kernel of the induced homomorphism $k^0_* : H_2(M^0; Z) \to H_2(\hat{\Sigma}; Z))$ is a free abelian subgroup H of the free abelian group $G = H_2(M^0; Z)$ with the same rank n, because the image of $k^0_* : H_2(M^0; Z) \to H_2(\hat{\Sigma}; Z))$ is a torsion group. With a suitable basis x_i $(i = 1, 2, \ldots, n)$ of G, there are positive integers r_i $(i = 1, 2, \ldots, n)$ such that the elements $r_i x_i$ $(i = 1, 2, \ldots, n)$ form a basis of H. Since $\beta_2^{(2)}(\hat{\Sigma}; Z) < n$, we can find an index i such that the integer r_i is odd. Then we see that the element $x_i \in H_2(M^0; Z)$ is sent to an odd order element of $H_2(\hat{\Sigma}; Z))$ by the homomorphism k^0_*. Since x_i is indivisible, we see from [9] that there is a closed *connected* oriented surface S in M^0 representing x_i.

We take a bi-collar $M^0 \times [-1, 1]$ of M^0 in $\hat{\Sigma}$ with $M^0 = M^0 \times 0$ and a regular neighborhood $N_S = S \times D^2$ of $S \subset M = M \times 0$ in $M^0 \times [-1, 1]$. Let $E = \mathrm{cl}(\hat{\Sigma} \backslash N_S)$. Using the excision isomorphism $H_3(E, \partial N_S; Z) \cong H_3(\hat{\Sigma}, N_S; Z)$, we see that there is an indivisible element $z \in H_3(E, \partial N_S; Z)$ with $\partial_*(z) = r[S'] \in H_2(\partial N_S; Z)$ for a section S' of the S^1-bundle ∂N_S over S and some odd number r under the boundary operator $\partial_* : H_2(E, \partial N_S; Z) \to H_2(\partial N_S; Z)$. By Poincaré duality $H_3(E, \partial N_S; Z) \cong H^1(E, \partial \hat{\Sigma}; Z)$ and transverse regularity, the indivisible element z in $H_3(E, \partial N_S; Z)$ is represented by a compact oriented 3-manifold V such that the boundary ∂V is a union of $r^*(> 0)$-paralells of a closed connected oriented surface S^* in ∂N_S with

$$[\partial V] = r^*[S^*] = r[S] \in H_2(\partial N_S; Z).$$

By replacing V with a suitable connected component of V, we may assume that V is connected. For the submanifold $E_M = \mathrm{cl}(\hat{\Sigma} \backslash M^0 \times [-1, 1])$ of E, we have a composite homomorphism

$$\gamma : H_1(E_M; Z) \xrightarrow{i_*} H_1(E; Z) \xrightarrow{\mathrm{Int}_V} Z$$

where i_* is a natural homomorphism and Int_V is a homomorphism defined by the identity $\mathrm{Int}_V(x) = \mathrm{Int}(x, V)$ for $x \in H_1(E; Z)$. By construction, i_* is onto. Since $E \backslash V$ is connected, we have an element $x \in H_1(E; Z)$ with $\mathrm{Int}(V, x) = \mathrm{Int}_V(x) = 1$ and hence Int_V is onto, so that γ is onto.

Let $\partial E_M = DM^0 \cup M^*$ for $M^* = \partial \hat{\Sigma}$. We show that the restriction $\dot{\gamma} : H_1(DM^0; Z) \to Z$ of γ to DM^0 is Z_2-asymmetric. To see this, we note that the meridian $m(S)$ of S in $M^0 \times [-1, 1]$ is deformed into a simple loop m' in DM^0 such that $\alpha(m') = -m'$. Then we have

$$\dot{\gamma}([m']) = \mathrm{Int}_V([m']) = \mathrm{Int}(V, m') = r.$$

We note that m' is written as a connected sum $m''\#\alpha(m'')$ for a simple loop m'' in $M^0 \times 1$ and the reflection image $\alpha(m'')$. Then we have

$$\dot{\gamma}([m'']) + \dot{\gamma}([\alpha(m'')]) = \dot{\gamma}([m']) = r.$$

Since r is odd, we see that $\dot{\gamma}$ is Z_2-asymmetric. Let $(\tilde{E}_M; \widetilde{DM^0}, \tilde{M}^*)$ be the infinite cyclic covering of the triad $(E_M; DM^0, M^*)$ associated with γ. The covering $\tilde{E}_M \to E_M$ is a restriction of the infinite cyclic covering $\tilde{E} \to E$ associated with the epimorphism $\mathrm{Int}_V : H_1(E; Z) \to Z$. Since M^* does not meet V in E, the covering $\tilde{M}^* \to M^*$ is the trivial covering. Thus, we see that $\sigma_{[a,1]}(\tilde{M}^*) = 0$ for every $a \in [-1, 1]$. Using that $\kappa_1(\widetilde{DM^0}) = 0$, we also see that $\kappa_1(\partial\tilde{E}_M) = 0$. Because $\hat{\beta}_2(E_M; Z) = \hat{\beta}_2(\hat{\Sigma}; Z) = 0$ and $\kappa_1(\partial\tilde{E}_M) = 0$, we see from the signature theorem that

$$\sigma_{(a,1]}(\partial\tilde{E}_M) = \sigma_{(a,1]}(\widetilde{DM^0}) + \sigma_{(a,1]}(\tilde{M}^*) = 0$$

for every $a \in (-1, 1)$. Thus, we have

$$\sigma_{(-1,1]}(\widetilde{DM^0}) = 0$$

which contradicts that $\sigma_{(-1,1]}(\widetilde{DM^0}) \neq 0$. Counting Observation 3.1 (1), we also have the inequality for a (closed or bounded) Samsara 4-manifold Σ on M. This completes the proof of the inequality part and hence the full proof of Theorem 3.2.

Here is a concluding remark.

Remark 5.2. For the standard type 1 embedding $f : M \to M \times S^1 = W$, we see that if $H_1(M; Q) \neq 0$, then the intersection form $\mathrm{Int} : H_2(W; Z) \times H_2(W; Z) \to Z$ is non-trivial and the image $\mathrm{im}[f_* : H_2(M; Q) \to H_2(W; Q)] \neq 0$. On the other hand, in Theorem 3.2 we construct a type 1 embedding $k : M \to \Sigma$ for every closed 3-manifold M and a compact oriented 4-manifold Σ such that the intersection form $\mathrm{Int} : H_2(\Sigma; Z) \times H_2(\Sigma; Z) \to Z$ is trivial and $\mathrm{im}[k_* : H_2(M; Q) \to H_2(\Sigma; Q)] = 0$. This leads to the following open question:

Is every closed 3-manifold type 1 embeddable in a compact connected oriented 4-manifold with the trivial second Q-homology ?

We note that there are lots of closed 3-manifolds M which cannot be type II embedded in any compact connected oriented 4-manifold X with $H_2(X; Q) = 0$. For example, the 0-surgery manifold $M = \chi(K; 0)$ of the trefoil knot K gives such an example. To see this, suppose that M is type II embedded in X. Using $H_2(X; Q) = 0$, we can assume further that $H_1(\partial X; Q) = 0$, if neceaasry, by attaching suitable 2-handles to ∂X. Let Y and Y' be the 3-manifolds obtained from X by splitting along M. Since the natural homomorphism $H_1(Y; Q) \oplus H_1(Y'; Q) \to H_1(X; Q)$ is onto and there are isomorphisms $H^1(X;Q) \cong H^1(X,\partial X;Q) \cong H_3(X;Q)$, we can assume $H_1(X;Q) = H_2(X;Q) = 0$ by surgeries not meeting M which kill a Q-basis for $H_1(X;Q)$. Then one of Y or Y', say, Y has $H_1(Y;Q) \cong Q$ and $H_2(Y;Q) = 0$. Let $(\tilde{Y}, \tilde{M})$ be the infinite cyclic covering of (Y, M) associated with an epimorphism $\gamma : H_1(Y;Z) \to Z$. Since $\kappa_1(\tilde{M}) = 0$, we see from the signature theorem that $\sigma_{(a,1]}(\tilde{M}) = 0$ for every $a \in (-1,1)$. By [7, Corollary 1.4], we find an $a \in (-1,1)$ with $\sigma_{(a,1]}(\tilde{M}) = \pm 2$, which is a contradiction. Thus, M cannot be type II embedded in X.

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