ON LINKING SIGNATURE INVARIANTS OF SURFACE-KNOTS

Akio KAWAUCHI
Department of Mathematics, Osaka City University
Sumiyoshi-ku, Osaka 558-8585, Japan
kawauchi@sci.osaka-cu.ac.jp

ABSTRACT

We show that the linking signature of a closed oriented 4-manifold with infinite cyclic first homology is twice the Rochlin invariant of an exact leaf with a spin support if such a leaf exists. In particular, the linking signature of a surface-knot in the 4-sphere is twice the Rochlin invariant of an exact leaf of an associated closed spin 4-manifold with infinite cyclic first homology. As an application, we characterize a difference between the spin structures on a homology quaternion space in terms of closed oriented 4-manifolds with infinite cyclic first homology, so that we can obtain examples showing that some different punctured embeddings into $S^4$ produce different Rochlin invariants for some homology quaternion spaces.

Keywords : quadratic function, linking signature, surface-knot, Rochlin invariant, exact leaf, spin structure, homology quaternion space

0. Introduction

A quadratic function on a finite abelian group $G$ is a function

$$q : G \longrightarrow \mathbb{Q}/\mathbb{Z}$$

such that $q(-x) = q(x)$ for all $x \in G$ and the pairing $\ell : G \times G \rightarrow \mathbb{Q}/\mathbb{Z}$ defined by the identity $\ell(x, y) = q(x + y) - q(x) - q(y)$ is a non-singular symmetric bilinear form which we call the linking induced from $q$. We note that $2q(x) = \ell(x, x)$ and $q(2^m x) = 2^m q(x)$ for every integer $m \geq 1$ and $x \in G$. A quadratic function $q : G \times G \rightarrow \mathbb{Q}/\mathbb{Z}$ is said to be isomorphic to a quadratic function $q' : G' \times G' \rightarrow \mathbb{Q}/\mathbb{Z}$ if there is an isomorphism $f : G \cong G'$ such that $q = q' f$. The linking signature $\sigma(q)$ of $q$ is a rational number modulo one which is defined by the Gauss sum identity

$$GS(q) = \sum_{x \in G} \exp(2\pi \sqrt{-1} \cdot q(x)) = \sqrt{|G|} \exp(2\pi \sqrt{-1} \cdot \sigma(q))$$

(see [8]). The linking signature $\sigma(q) \in \mathbb{Q}/\mathbb{Z}$ is an invariant of a quadratic function $q$ up to isomorphisms and has $8\sigma(q) = 0$ in $\mathbb{Q}/\mathbb{Z}$ in general. A closed connected oriented 4-manifold $M$ with $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ is simply called a $Z$-manifold. By a result of [7], the $Z$-manifold $M \# n(S^2 \times S^2)$ admits an exact leaf $V$ if $n$ is greater than a constant depending on $M$. In this paper, we consider the torsion quadratic function

$$q : G_M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

of the infinite cyclic covering space $M$ over a $Z$-manifold $M$ belonging to a generator $\gamma \in H^1(M; \mathbb{Z})$. This quadratic function $q$ was first defined in [8], although the induced torsion linking

$$\ell : G_M \times G_M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

had been defined in [4] (see also [6]). We denote the linking signature $\sigma(q)$ by $\sigma(M) = \sigma(\gamma)(M)$. For our purpose, we consider a $Z$-manifold $M$ containing an
exact leaf $V$ with a spin support $M^*$ in $M$, that is, a compact spin 4-manifold neighborhood $M^*$ of $V$ in $M$ such that $H_1(\partial M^*; Z)$ is a free abelian group. For a spin structure $\iota$ on $V$ induced from any spin structure on $M^*$, we take the Rochlin invariant $\mu(V, \iota)$, arranged to have the value in $Q/Z$. Then, as the main result of this paper, we shall show that the linking signature $\sigma^\iota(M)$ is equal to $2\mu(V, \iota)$. This theorem will be exactly stated in §2 and proved in §3. In §1, some facts of the torsion quadratic function, the torsion linking, and the linking signature are explained.

As a concrete object, we consider an oriented surface-knot $F$ in a closed connected oriented 4-manifold $M_1$ with $H_1(M_1; Z) = 0$ such that $F$ admits a Seifert hypersurface in $M_1$ (in other words, such that $[F] = 0 \in H_2(M_1; Z)$). For standard examples, we take $M_1 = S^4$. By [8] we still have the torsion quadratic function

$$q_F : G_F \longrightarrow Q/Z$$

and the torsion linking

$$\ell_F : G_F \times G_F \longrightarrow Q/Z$$

of the surface-knot $F$. This torsion linking $\ell_F$ is a natural generalization of the Farber-Levine linking of an $S^2$-knot in $S^4$ (see Farber [2], Levine [13]). We denote the linking signature $\sigma(q_F) \in Q/Z$ by $\sigma(F)$. Let $E_F$ be the compact exterior of $F$ in $M_1$, i.e., $E_F = \text{cl}(M_1 - N_F)$ for a trivial normal bundle $N_F$ of $F$ in $M_1$. We have the first homology $H_1(E_F; Z) \cong Z$ with a unique meridian generator. We choose a trivialization $N_F = F \times D^2$ so that the natural composite

$$H_1(F \times 1; Z) \longrightarrow H_1(\partial E_F; Z) \longrightarrow H_1(E_F; Z) \cong Z$$

is the zero map under the identification $\partial E_F = \partial N_F = F \times S^1$. Let $V_0$ be the handlebody such that $\partial V_0 = F$. Let $M_{\phi, n}$ be the closed 4-manifold obtained from $E_F$ and $V \times S^1$ by attaching the boundaries by a homeomorphism $\phi : \partial E_F = F \times S^1 \to \partial V_0 \times S^1$ which preserves the $S^1$-factor. Then $M_{\phi, n}$ is a $Z^{H^1}$-manifold. We refer the $Z^{H^1}$-manifold $M_{\phi, n} = M_{\phi, n}(S^2 \times S^2)$ as a $Z^{H^1}$-manifold associated with the surface-knot $F$. Choosing $\phi$ carefully, we can make $M_{\phi, n}$ spin. Let $V$ be an exact leaf of the spin $Z^{H^1}$-manifold $M_{\phi, n}$ for a large integer $n$. Let $\iota$ be a spin structure on $V$ induced from any spin structure on $M_{\phi, n}$. In this case, our main theorem implies the identity

$$\sigma(F) = 2\mu(V, \iota).$$

We shall also note in Remark 2.6 that the Rochlin invariant $\mu(V, \iota)$ itself is not an invariant for a high genus surface-knot $F$, although it is an invariant when $F$ is an $S^2$-knot in $S^4$ by a result of Ruberman [14]. Our main result is applied in §4 to characterize a difference between the spin structures $\iota$ on a homology quaternion space $V$ by constructing $Z^{H^1}$-manifolds $M$ such that $V$ is an exact leaf of $M$ with a spin support $M^*$ in $M$ and $\iota$ is a spin structure on $V$ induced from a spin structure on $M^*$. Using this characterization, we shall obtain examples showing that some different punctured embeddings into $S^4$ produce different Rochlin invariants for the quaternion space and a homology quaternion space.

Finally, the torsion linking and the torsion quadratic function of a surface-link are also defined in [8], and the surface-link version of this paper will be discussed in [9].
1. The torsion linking, the torsion quadratic function, and the linking signature

Let $\Lambda = Z[Z] = Z[t, t^{-1}]$. Let $W$ be a compact connected oriented 4-manifold which admits an infinite cyclic connected covering $p : \tilde{W} \to W$ belonging to an indivisible element $\gamma \in H^1(W; Z)$. Let $A$ and $A'$ be $\emptyset$ or compact 3-submanifolds of $\partial W$ such that $A' = \text{cl}(\partial W - A)$. For a subspace $W'$ of $W$, let $\widetilde{W}' = p^{-1}(W')$. For a finitely generated $\Lambda$-module $H$, let $DH$ be the maximal finite $\Lambda$-submodule of $H$ (see [4;§3]), $tH$ the $Z$-torsion part of $H$, and $TH$ the $\Lambda$-torsion part of $H$. Let $BH = H/TH$. Let $E^q(H) = \text{Ext}^q_{\Lambda}(H, \Lambda)$. By an argument of [4] we have a $t$-anti epimorphism

$$\theta_{A, A'} : DH_1(\tilde{W}, \tilde{A}; Z) \to E^1(BH_2(\tilde{W}, \tilde{A}'; Z))$$

which is an invariant of $(\tilde{W}, \tilde{A}, \tilde{A}')$ or $(W, A, A', \gamma)$. We denote the kernels of $\theta_{A, A'}$ and $\theta_{A', A}$ by $DH_1(\tilde{W}, \tilde{A}; Z)^0$ and $DH_1(\tilde{W}, \tilde{A}'; Z)^0$, respectively. The second duality of [4] then says that there is a $t$-isometric non-singular bilinear form

$$\ell : DH_1(\tilde{W}, \tilde{A}; Z)^0 \times DH_1(\tilde{W}, \tilde{A}'; Z)^0 \to Q/Z$$

which is an invariant of $(\tilde{W}, \tilde{A}, \tilde{A}')$ or $(W, A, A', \gamma)$. By taking $A = \emptyset$ and $A' = \partial W$, let $\tilde{DH}_1(\tilde{W}; Z)^0$ denote the following quotient finite $\Lambda$-module:

$$\tilde{DH}_1(\tilde{W}; Z)^0 / \text{Im}(i_* : H_1(\partial \tilde{W}; Z) \to H_1(\tilde{W}; Z)) \cap DH_1(\tilde{W}; Z)^0,$$

where $i_*$ denotes the natural homomorphism. Then we have the following lemma (see [8]):

**Lemma 1.1.** The bilinear form $\ell$ induces a $t$-isometric linking

$$\hat{\ell} : \hat{DH}_1(\tilde{W}; Z)^0 \times \hat{DH}_1(\tilde{W}; Z)^0 \to Q/Z.$$ 

The linking $\hat{\ell}$ is an invariant of $\tilde{W}$ or $(W, \gamma)$ and called the **torsion linking** of $\tilde{W}$ or $(W, \gamma)$. We say that $H$ is $(t - 1)$-divisible if $t - 1$ is an automorphism of $H$. For a finitely generated $(t - 1)$-divisible $\Lambda$-module $H$, it is well-known that the $Z$-torsion part $tH$ of $H$ is equal to $DH$, originally due to M. A. Kervaire [12] (cf. [4;§3]). For $DH$, let

$$D_0 H = \cap_{n=1}^{\infty} (t - 1)^n DH,$$

$$D_1 H = \{ x \in DH | \exists n \geq 1, (t - 1)^n x = 0 \}.$$

Then we have a natural splitting $DH = D_0 H \oplus D_1 H$, so that $D_0 H$ is a unique maximal $(t - 1)$-divisible finite $\Lambda$-submodule of $H$ (see [8]). We denote by $G(\tilde{W})$ the unique maximal $\Lambda$-submodule $D_0(\tilde{DH}_1(\tilde{W})^0)$ of $\tilde{DH}_1(\tilde{W})^0$. The restriction $\ell_G$ of $\hat{\ell}$ to $G(\tilde{W})$ induces a $t$-isometric linking

$$\ell_G : G(\tilde{W}) \times G(\tilde{W}) \to Q/Z,$$

which we call the $(t - 1)$-divisible **torsion linking** of $\tilde{W}$ or $(W, \gamma)$, which leads to the following definition (see [8]):
Definition 1.2. The torsion quadratic function of $\tilde{W}$ or $(W, \gamma)$ is the function
\[ q : G(\tilde{W}) \rightarrow Q/Z \]
defined by $q(x) = \ell_G(x, (1-t)^{-1}x)$. ■

It is direct to see that $q$ is an invariant of $\tilde{W}$ or $(W, \gamma)$. We have
\[ q(-x) = q(x) \quad \text{and} \quad q(x + y) - q(x) - q(y) = \ell_G(x, y). \]
Thus, $q$ is a quadratic function inducing $\ell_G$. The linking signature $\sigma(q) \in Q/Z$ is given by the Gauss sum identity
\[ GS(q) = \sum_{x \in G(\tilde{W})} \exp(2\pi \sqrt{-1} \cdot q(x)) = \sqrt{|G(\tilde{W})|} \exp(2\pi \sqrt{-1} \cdot \sigma(q)), \]
which is denoted by $\sigma(\tilde{W}) = \sigma^q(W)$ and called by the linking signature of $\tilde{W}$ or $(W, \gamma)$. For every prime $p$, let $G(\tilde{W})_p$ be the $p$-torsion subgroup of $G(\tilde{W})$. Then we see that the linking $\ell_G : G(\tilde{W}) \times G(\tilde{W}) \rightarrow Q/Z$ is the unique orthogonal sum of the linkings $\ell_p : G(\tilde{W})_p \times G(\tilde{W})_p \rightarrow Q/Z$ induced from $\ell_G$ for all primes $p$. The restricted function $q_p : G(\tilde{W})_p \rightarrow Q/Z$ of $q$ is a quadratic function inducing $\ell_p$. We denote $\sigma(q_p)$ by $\sigma_p(\tilde{W})$ and call it the $p$-local linking signature. Then we have the identity
\[ \sigma(\tilde{W}) = \sum_p \sigma_p(\tilde{W}) \]
where the summation $\sum_p$ ranges over all primes $p$. Further, the $p$-primary component $G(\tilde{W})_p$ has a homogeneous orthogonal splitting $\oplus_{i=1}^{\infty} G(\tilde{W})^i_p$ with respect to $\ell_p$ where $G(\tilde{W})^i_p$ is a direct sum of copies of $Z_{p^i}$. The restricted function $q^i_p : G(\tilde{W})^i_p \rightarrow Q/Z$ of $q_p$ is a quadratic function inducing the linking $\ell^i_p : G(\tilde{W})^i_p \times G(\tilde{W})^i_p \rightarrow Q/Z$ induced from $\ell_p$. We denote the linking signature $\sigma(q^i_p)$ by $\sigma^i_p(\tilde{W})$ and call it the $i$th $p$-local signature of $\tilde{W}$ or $(W, \gamma)$. By definition,
\[ \sigma^i_p(\tilde{W}) = \sum_{i=1}^{\infty} \sigma^i_p(\tilde{W}). \]
It turns out that $\sigma^i_p(\tilde{W})$ is an invariant of $\tilde{W}$ and takes a value in $Q/Z$ as follows:
\[ \sigma^i_p(\tilde{W}) = \begin{cases} 0 & \text{if } p \text{ is any prime and } i \text{ is even} \\ 0 \text{ or } \frac{1}{2} & \text{if } p = 2 \text{ and } i \text{ is odd} \\ 0, \frac{1}{2}, \text{ or } \pm \frac{1}{4} & \text{if } p \text{ and } i \text{ are odd}. \end{cases} \]
Let $F$ be a surface-knot in a closed connected oriented 4-manifold $M_1$ such that $H_1(M_1; Z) = 0$ and $|F| = 0 \in H_2(M_1; Z)$, and $E_F$ the compact exterior of $F$ in $M_1$. The torsion linking
\[ \ell_F : G_F \times G_F \rightarrow Q/Z \]
of this surface-knot $F$ is the torsion linking $\ell_G : G(\tilde{E}_F) \times G(\tilde{E}_F) \to Q/Z$ for the infinite cyclic covering space $\tilde{E}_F$ over $E_F$ belonging to the element $\gamma \in H^1(E_F; Z)$ sending the meridian of $F$ to 1, and the torsion quadratic function

$$q_F : G_F \to Q/Z$$

of $F$ is the torsion quadratic function $q : G(\tilde{E}_F) \to Q/Z$. We define

$$\sigma(F) = \sigma(\tilde{E}_F), \quad \sigma_p(F) = \sigma_p(\tilde{E}_F), \quad \sigma^i_p(F) = \sigma^i_p(\tilde{E}_F).$$

Let $\tilde{M}_{\phi,n} \to M_{\phi,n}$ be the infinite cyclic covering belonging to $\gamma$ under the identification $H^1(\tilde{E}_F; Z) = H^1(M_{\phi,n}; Z)$, which extends the infinite cyclic covering $\tilde{E}_F \to E_F$. As observed in [8;Proposition 4.4], the torsion quadratic function $q_F : G_F \to Q/Z$ is isomorphic to the torsion quadratic function $q : G(\tilde{M}_{\phi,n}) \to Q/Z$ for all $n$, so that

$$\sigma(F) = \sigma(\tilde{M}_{\phi,n}), \quad \sigma_p(F) = \sigma_p(\tilde{M}_{\phi,n}), \quad \sigma^i_p(F) = \sigma^i_p(\tilde{M}_{\phi,n})$$

for all primes $p$ and all positive integers $i$.

2. Identifying the linking signature with twice the Rochlin invariant

A leaf of a $\tilde{Z}^{H_1}$-manifold $M$ is a bicollared 3-submanifold $V$ of $M$ such that $V$ represents a generator of $H_3(M; Z) \cong H^1(M; Z) \cong Z$. The following definition of exact leaf is found in [7] together with two equivalent definitions:

**Definition 2.1.** A leaf $V$ of a $\tilde{Z}^{H_1}$-manifold $M$ is *exact* if the natural semi-exact sequence

$$0 \to tH_2(\tilde{M}, \tilde{V}; Z) \xrightarrow{\partial} tH_1(\tilde{V}; Z) \xrightarrow{i^*} tH_1(\tilde{M}; Z)$$

induced from the homology exact sequence of the pair $(\tilde{M}, \tilde{V})$ is exact. ■

Further, we say that a $\tilde{Z}^{H_1}$-manifold $M$ is *exact* if there is an exact leaf $V$ of $M$. The following lemma is proved in [7]:

**Lemma 2.2.** For every $\tilde{Z}^{H_1}$-manifold $M$, we have a non-negative integer $n$ such that the connected sum $M \# n(S^2 \times S^2)$ is exact. ■

For a closed oriented 3-manifold $V$, we have a linking form

$$\ell_V : tH_1(V; Z) \times tH_1(V; Z) \to Q/Z$$

on the $Z$-torsion part $tH_1(V; Z)$ of $H_1(V; Z)$ defined by the Poincaré duality. Given a spin structure $\iota$ on $V$, we have a unique quadratic function

$$q_V^\iota : tH_1(V; Z) \to Q/Z,$$

such that

$$q_V^\iota(x + y) - q_V^\iota(x) - q_V^\iota(y) = \ell_V(x, y)$$

(see [8;Lemma 1.1]). By [8;Corollary 1.4], the linking signature $\sigma(q_V^\iota) \in Q/Z$ given by the Gauss sum identity

$$GS(q_V^\iota) = \sum_{x \in tH_1(V; Z)} \exp(2\pi i \cdot q_V^\iota(x)) = \sqrt{|tH_1(V; Z)|} \exp(2\pi i \cdot \sigma(q_V^\iota))$$
has $8\sigma(q_V) = 0 \in Q/Z$. We call this invariant the spin linking signature of $(V, \iota)$ and denote it by $s(V, \iota)$. The Rochlin invariant $\mu(V, \iota) \in Q/Z$ of $(V, \iota)$ is defined by the identity

$$\mu(V, \iota) = -\text{sign}(U)/16 \in Q/Z$$

for any smooth spin 4-manifold $(U, \iota_U)$ bounded by $(V, \iota)$. By [8;Lemma 1.3], we have

$$s(V, \iota) = 2\mu(V, \iota).$$

To state our main theorem, we generalize the concept of a spin $Z^{H_4}$-manifold as follows: A leaf $V$ of a $Z^{H_4}$-manifold $M$ admits a spin support $M^*$ in $M$ if $M^*$ is a compact spin 4-manifold neighborhood of $V$ in $M$ such that $tH_1(\partial M^*; Z) = 0$. For example, let $V$ be a leaf of a spin $Z^{H_4}$-manifold $M'$, and $M$ the connected sum of $M'$ and any closed non-spin 4-manifold $W$ with $H_1(W; Z) = 0$. Then $V$ is a leaf of the non-spin $Z^{H_4}$-manifold $M$ with a spin support in $M$. Using this concept, we state our main theorem (proved in §3) as follows:

**Theorem 2.3.** Let $V$ be an exact leaf of a $Z^{H_4}$-manifold $M$ with a spin support $M^*$ in $M$. For any spin structure $\iota$ on $V$ induced from any spin structure on $M^*$, we have

$$\sigma(M) = s(V, \iota) = 2\mu(V, \iota).$$

Let $F$ be a surface-knot in a closed spin 4-manifold $M_1$ such that $H_1(M_1; Z) = 0$ and $F$ admits a Seifert hypersurface in $M_1$. By our choice of a trivialization $N_F = F \times D^2$, the surface $F \times 1$ bounds a bicollared 3-submanifold $V_F$ in $E_F$. By Poincaré duality over $Z$, we have a $Z_2$-symplectic basis $x_i, y_i$ ($i = 1, 2, \ldots, m$) for $H_1(F \times 1; Z_2)$ whose $Z_2$-intersection numbers have $x_i \cdot x_j = y_i \cdot y_j = 0$ and $x_i \cdot y_j = \delta_{i,j}$ for all $i, j = 1, 2, \ldots, m$ and such that $x_i$ bounds a $Z_2$-chain in $V_F$ for all $i$. We represent $x_i$ and $y_i$ by circles $S^*_i$ and $S^*_i$ embedded in $F \times 1$ such that $S^*_i \cap S^*_j = S^*_i \cap S^*_j = S^*_i \cap S^*_j = \emptyset$ for all $i, j$ with $i \neq j$ and $S^*_i \cap S^*_j = \emptyset$ for all $i$. We choose a homeomorphism $\phi: \partial E_F = F \times S^1 \to \partial V_0 \times S^1$ preserving the $S^1$-factor such that $\phi(S^*_i)$ is a meridian disk in $V_0 \times 1$. Then we have the following lemma:

**Lemma 2.4.** The $Z^{H_4}$-manifold $M_\phi$ is spin. ■

**Proof.** We consider the following part

$$H_2(E_F; Z_2) \to H_2(M_\phi; Z_2) \to H_2(M_\phi, E_F; Z_2) \xrightarrow{\vartheta} H_1(E_F; Z_2)$$

of the exact sequence of the pair $(M_\phi, E_F)$. Using the excision isomorphism

$$H_2(M_\phi, E_F; Z_2) \cong H_2(V_0 \times S^1, F \times S^1; Z_2),$$

we see that $H_2(M_\phi; Z_2)$ is generated by $Z_2$-cycles $C$ in $H_2(E_F; Z_2)$ and $Z_2$-cycles $C'_i$ ($i = 1, 2, \ldots, m$) in $M_\phi$ such that $C'_i$ is the sum of a $Z_2$-chain in $V_F$ bounded by $S^*_i$ and a meridian disk in $V_0$ bounded by $\phi(S^*_i)$. Since $E_F$ is spin, we have the $Z_2$-intersection number $C \cdot C = 0$ for every $Z_2$-cycle in $E_F$. By construction, we also have the $Z_2$-intersection number $C'_i \cdot C'_i = 0$ for all $i$. These mean that $M_\phi$ is spin. □

Combining Lemma 2.4 with Theorem 2.3, we obtain the following corollary from the identity $\sigma(F) = \sigma(\hat{M}_{\phi,n})$: 6
Corollary 2.5. Let $M_{\phi, n}$ be any spin $Z^{H_1}$-manifold associated with any surface-knot $F$ in $M_1$ which admits an exact leaf $V$, and $\iota$ a spin structure on $V$ induced from any spin structure on $M_{\phi, n}$. Then we have

$$\sigma(F) = s(V, \iota) = 2\mu(V, \iota).$$

We consider an $S^2$-knot $K$ in $S^4$. Let $V$ be a closed oriented 3-manifold obtained from a Seifert hypersurface $V_K$ for $K$ in $S^4$ by adding a 3-ball, and $\iota$ the spin structure on $V$ induced from $S^4$. Then Ruberman [14] showed that the Rochlin invariant $\mu(V, \iota) \in \mathbb{Q}/\mathbb{Z}$ is independent of a choice of $V_K$ and hence an invariant of $K$. A geometric proof of this fact is also easily derived from the fact in [5] that any two Seifert hypersurfaces for $K$ are connected by a surgery sequence on embedded 1-handles or 2-handles, because the surgery trace of every embedded 1-handle or 2-handle on a Seifert hypersurface $V_K$ is in $S^4$ and hence has the signature zero. By definition, $V$ is an exact leaf of the (unique) spin $Z^{H_1}$-manifold $M_{\phi}$ associated with $K$. Hence we have

$$\sigma(K) = s(V, \iota) = 2\mu(V, \iota)$$

by Corollary 2.5. By this evidence, one may expect that $\mu(V, \iota)$ itself is an invariant for a positive genus surface-knot $F$. However, the following remark shows that this is not true for a high genus surface-knot:

Remark 2.6. It is well-known that every homology 3-sphere $V$ can embedded smoothly into the connected sum $\# n(S^2 \times S^2)$ for a positive integer $n$. For our purpose, we take any $V$ such that $\mu(V, \iota_V) = \frac{1}{2}$, where $\iota_V$ denotes the unique spin structure on $V$. We note that the $Z^{H_1}$-manifold $M = S^1 \times S^3 \# n(S^2 \times S^2)$ is a $Z^{H_1}$-manifold associated with a trivial surface-knot of genus $n$. Since $V$ separates $\# n(S^2 \times S^2)$ into two submanifolds, we see that $V$ is a leaf of the $Z^{H_1}$-manifold $M$. The factor $S^3$ of the connected summand $S^1 \times S^3$ of $M$ gives a leaf of $M$. Since $H_1(S^3; \mathbb{Z}) = H_1(V; \mathbb{Z}) = 0$, we see from Lemma 4.2 later that $S^3$ and $V$ are exact leaves of $M$. However, we have $\mu(S^3, \iota_{S^3}) = 0$ and $\mu(V, \iota_V) = \frac{1}{2}$. ■

In spite of this example, we can re-use the Rochlin invariant as an invariant of a positive-genus surface-knot $F$ together with a self-orthogonal $\Lambda$-submodule $X$ of $BH_2(\overline{E}_F; \mathbb{Z})$ (see [10]).

3. Proof of Theorem 2.3

Let $\tilde{V}$ be a leaf of a $Z^{H_1}$-manifold $\tilde{M}$. Let $\mu \in H_3(\tilde{M}; \mathbb{Z})$ be the fundamental class of the covering $p : \tilde{M} \to M$, that is a homology class represented by a lift of the leaf $V$ to $\tilde{M}$ (see [5]). Unless a confusion might occur, this lift is also denoted by $V$. Let $\tau H^2(\tilde{M}; \mathbb{Z})$ be the image of the Bockstein homomorphism $\delta_{\mathbb{Q}/\mathbb{Z}} : H^1(\tilde{M}; \mathbb{Q}/\mathbb{Z}) \to H^2(\tilde{M}; \mathbb{Z})$. Let $tH_1(\tilde{V}; \mathbb{Z})^0$ be the subgroup of $tH_1(\tilde{V}; \mathbb{Z})$ given by $(\cap [V])i^* (\tau H^2(\tilde{M}; \mathbb{Z}))$ in the following commutative diagram:

$$
\begin{array}{ccc}
\tau H^2(\tilde{M}; \mathbb{Z}) & \xrightarrow{\cap \mu} & tH_1(\tilde{M}; \mathbb{Z}) \\
\downarrow i^* & & \downarrow i_* \\
tH^2(\tilde{V}; \mathbb{Z}) & \xrightarrow{\cap [V]} & tH_1(\tilde{V}; \mathbb{Z}).
\end{array}
$$
It is shown in [6:Lemma 4.1 and Theorem 4.2] that if $V$ is an exact leaf of $M$, then there is an orthogonal splitting $tH_1(V; Z) = tH_1(V; Z)^\theta \oplus \text{Ker}i_*$ with respect to the linking $\ell_V$. Further, we have the following (1) and (2):

1. The map $i_*$ induces an isomorphism from the restricted linking
   $\ell_V|_{tH_1(V; Z)^\theta} : tH_1(V; Z)^\theta \times tH_1(V; Z)^\theta \longrightarrow Q/Z$
   to the torsion linking
   $\ell_G : G(\tilde{M}) \times G(\tilde{M}) \longrightarrow Q/Z$
   of $\tilde{M}$ with $G(\tilde{M}) = tH_1(\tilde{M}; Z)^\theta$.

2. We have $\text{Ker}i_* = K_+ \oplus K_-$ and $\ell_V(K_+, K_+) = \ell_V(K_-, K_-) = 0$ for

   $$K_\pm = \text{Im}(\partial : H_2(\tilde{M}_\pm, \tilde{V}_\pm; Z) \rightarrow tH_1(\tilde{V}_\pm; Z)) \cap tH_1(V; Z)$$

   where $\tilde{M}_\pm$ are 4-submanifolds obtained by $\tilde{M}$ splitting along $V$ such that $\tilde{M} = \tilde{M}_+ \cup \tilde{M}_-$, $\tilde{M}_+ \cap \tilde{M}_- = V$, $\tilde{M}_\pm \cap \tilde{V}_\pm$, and $\tilde{V}_\pm = \tilde{M}_\pm \cap \tilde{V}$.

We are now in a position of the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Let $V$ be an exact leaf of a $Z^{H_1}$-manifold $M$ with a spin support $M^*$ in $M$. Let $\iota$ be the spin structure on $V$ induced from any spin structure on $M^*$. Let $\tilde{M}^*_\pm = \tilde{M}_\pm \cap M^*$. We first show the following:

1. The linking signature of the restricted quadratic function $q_V^i|_{\text{Ker}i_*}$ is 0.

   Let $k$ be a 1-knot in $V$ with $[k] \in K_+$. Then there is a 2-chain $C$ in $\tilde{M}_+$ with $\partial C = k$ and $[C] \in tH_2(\tilde{M}_+, \tilde{V}_+; Z)$. Then $C$ meets $\tilde{M}^*_+$ in a 2-chain $C^*$ such that $c = \partial C^* - k$ is a torsion cycle in $\partial M^*$. Since $\partial M^*$ is a trivial lift of $\partial M^*$ and $tH_1(\partial M^*_+; Z) = 0$, we have $tH_1(\partial M^*_+; Z) = 0$. Hence $c$ is null-homologous in $\partial M^*_+$. Let $\hat{C}$ be a 2-chain in $\tilde{M}^*_+$ with $\partial \hat{C} = k$ obtained from the 2-chain $C^*$ by adding a 2-chain in $\partial \tilde{M}^*_+$ with boundary $-c$. Let $k'$ be a longitude of $k$ in $V$ given by the spin structure $\iota$. For a 2-chain $\hat{C}'$ in $\tilde{M}^*_+$ with $\partial \hat{C}' = k'$ obtained from $\hat{C}$ by moving $k$ to $k'$ locally, the $Z$-intersection number $s(\hat{C}, \hat{C}')$ in $\tilde{M}^*_+$ is even by using a property of the spin structure on $M^*$. Let $c_Q(\hat{C})$ and $c_Q(\hat{C}')$ be rational 2-cycles in $\tilde{M}^*_+$ obtained from $\hat{C}$ and $\hat{C}'$ by adding rational 2-chains $c_Q$ and $c_Q'$ in $V$ with $\partial c_Q = -k$ and $\partial c_Q' = -k'$, respectively. Since $c_Q(\hat{C})$ and $c_Q(\hat{C}')$ are rationally homologous (in $\tilde{M}^*$) to rational 2-cycles in $\tilde{V}_+ \cup \partial \tilde{M}^*_+$, we see that the $Q$-intersection number $s_Q(c_Q(\hat{C}), c_Q(\hat{C}')) = s_Q(c_Q(\hat{C}), c_Q(\hat{C}')) = 0$. This means that the $Z$-intersection number $s(\hat{C}, \hat{C}') = -\text{Link}_Q(k, k')$. Therefore, we have

   $$q_V^i([k]) = \frac{\text{Link}_Q(k, k')}{2} \quad (\text{mod } 1)$$

   $$= -\frac{s(\hat{C}, \hat{C}')}{2} \quad (\text{mod } 1) = 0.$$
Thus, \( q^*_t(K_+) = 0 \). Similarly, \( q^*_t(K_-) = 0 \). By a result of hyperbolic quadratic function in [8; Corollary 2.5], the linking signature of the quadratic function \( q^*_t |_{\text{Ker}_*} \) is 0, as desired. Next, we show the following:

(2) For any elements \( x \in G(\hat{M}) \) and \( y \in tH_1(V;Z)^\theta \) with \( i_*(y) = x \), we have

\[
q^*_t(y) = \ell_G(x, (1-t)^{-1}x) = q(x).
\]

As a result, we see that the linking signature of the restricted quadratic function \( q|_{tH_1(V;Z)^\theta} \) is equal to \( \sigma(\hat{M}) \).

Let \( z = (1-t)^{-1}x \in G(\hat{M}) \). Let \( k \) and \( k_t \) be 1-knots in \( V \) with \( [k], [k_t] \in tH_1(V;Z)^\theta \) such that \( i_*(k) = z \) and \( i_*(k_t) = tz \). Since \( i_*(k - k_t) = (1-t)z = x \), we have \( y = [k - k_t] \). Let \( U \) be the compact manifold obtained from \( M \) by splitting along \( V \) which we identify with a canonical lift to \( \hat{M} \) such that \( \partial U = V - tV \). Using that \( V \) is an exact leaf and \( \partial_t[k_t - tk] = 0 \), we have a 2-chain \( \hat{C} \) in \( tH_2(\hat{M}, \hat{V}; Z) \) such that \( \partial \hat{C} = k_t - tk \). Considering the intersection of \( \hat{C} \) with \( U \), we have a 2-chain \( C \) in \( tH_2(U, \partial U; Z) \) such that \( \partial C = (k_t + k_-) - (tk + tk_+) \) for some 1-knots \( k_\pm \) in \( V \) with \( [k_\pm] \in K_\pm \) and \( k_t \cap k_- = k_\cap k_+ = \emptyset \). Let \( U^* \) be the compact manifold obtained from \( M^* \) by splitting along \( V \). Then \( C \) meets \( U^* \) in a 2-chain \( C^* \) such that \( c = \partial C^* - \partial C \) is a torsion and hence null-homologous 1-cycle in \( \partial M^* \). Let \( \hat{C} \) be a 2-chain in \( U^* \) with \( \partial \hat{C} = \partial C \) obtained from the 2-chain \( C^* \) by adding a 2-chain in \( \partial M^* \) with boundary \( c \). Let \( k'_t, k'_-, k', k'_+ \) be longitudes of \( k_t, k_-, k, k_+ \) given by the spin structure \( \iota \), respectively. Then for a 2-chain \( \hat{C}' \) in \( U^* \) with \( \partial \hat{C}' = (k'_t + k'_-) - (tk' + tk'_+) \) obtained from \( \hat{C} \) by moving \( k_t, k_-, k, k_+ \) to \( k'_t, k'_-, k', k'_+ \) locally, respectively, we have that \( s(\hat{C}, \hat{C}') \) is an even integer. Let \( \text{cl}_Q(\hat{C}) \) and \( \text{cl}_Q(\hat{C}') \) be rational 2-cycles in \( U^* \) obtained from \( \hat{C} \) and \( \hat{C}' \) by adding rational 2-chains in \( V \cup tV \) with boundaries \( -(k_t + k_-) + (tk + tk_+) \) and \( -(k'_t + k'_-) - (tk' + tk'_+) \), respectively. Since \( \text{cl}_Q(\hat{C}) \) and \( \text{cl}_Q(\hat{C}') \) are rationally homologous (in \( U^* \)) to rational 2-cycles in \( V \cup tV \cup \partial M^* \), we see that the \( Q \)-intersection number \( s_Q(\text{cl}_Q(\hat{C}), \text{cl}_Q(\hat{C}')) = s_Q(\text{cl}_Q(\hat{C}), \hat{C}') = 0 \). This means that

\[
-s(\hat{C}, \hat{C}') = \text{Link}_Q(k_t + k_- + k'_t + k'_-) - \text{Link}_Q(k + k_+ + k'_t + k'_+)
\]

\[
= \text{Link}_Q(k_t, k'_t) + 2\text{Link}_Q(k_t, k_-) + \text{Link}_Q(k_-, k'_-)
- \text{Link}_Q(k, k') - 2\text{Link}_Q(k, k_+) - \text{Link}_Q(k_+, k'_+).
\]

Since \( \frac{s(\hat{C}, \hat{C}')}{2}, \text{Link}_Q(k_t, k_-), \frac{\text{Link}_Q(k_-, k'_-)}{2}, \text{Link}_Q(k, k_+), \) and \( \frac{\text{Link}_Q(k_+, k'_+)}{2} \) are all 0 (mod 1), it follows that

\[
\frac{\text{Link}_Q(k_t, k'_t)}{2} = \frac{\text{Link}_Q(k, k')}{2} \pmod{1}.
\]
Then
\[
q^V_\iota(y) = \frac{\text{Link}_Q(k - k_t, k' - k'_t)}{2} \mod 1
\]
\[
= \frac{\text{Link}_Q(k, k')}{2} - \text{Link}_Q(k_t, k) + \frac{\text{Link}_Q(k_t, k')}{2} \mod 1
\]
\[
= \ell_V([k], [k]) - \ell_V([k_t], [k])
\]
\[
= \ell_V([k] - [k_t], [k])
\]
\[
= \ell_G(x, z)
\]
\[
= \ell_G(x, (1-t)^{-1}x)
\]
\[
= q(x).
\]

By (1) and (2) and the identity
\[
s(V, \iota) = \sigma(q^V_\iota|_{Ker_i}) + \sigma(q^V_\iota|_{H_1(V; Z)^0}),
\]
the identity $s(V, \iota) = \sigma(\tilde{M})$ is obtained. \qed

4. **An application to spin structures on a homology quaternion space**

A **homology quaternion space** is a closed connected oriented 3-manifold $V$ such that $H_1(V; \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the linking $\ell_V : H_1(V; \mathbb{Z}) \times H_1(V; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is hyperbolic. Then we have $\ell_V(x, x) = \ell_V(y, y) = 0$ and $\ell_V(x, y) = \frac{1}{2}$ for every $\mathbb{Z}_2$-basis $x, y$ for $H_1(V; \mathbb{Z})$. By [8;Corollary 2.5], the spin linking signature $s(V, \iota) = 0$ or $\frac{1}{2}$ for every spin structure $\iota$ on $V$. We represent $x$ and $y$ by disjoint knots $k_x$ and $k_y$ in $V$, respectively. By [3;Lemma 1.1], we have unique longitudes $L_x$ and $L_y$ on the tubular neighbourhoods $T(k_x)$ and $T(k_y)$ such that two parallels (with the same orientation) of $L_x$ and $L_y$ bound compact oriented surfaces $F_{2x}$ and $F_{2y}$ in $\text{cl}(V - T(k_x))$ and $\text{cl}(V - T(k_y))$, respectively. Using $\ell_V(x, y) = \frac{1}{2}$, we can assume that $k_x$ meets $F_{2y}$ transversely in one point and $k_y$ meets $F_{2x}$ transversely in one point. There are four spin structures on a homology quaternion space $V$ which we can specify for a $\mathbb{Z}_2$-basis $x, y$ of $H_1(V; \mathbb{Z})$ as follows:

Let $\iota_{00}$ be the spin structure on $V$ such that $L_x$ and $L_y$ are the $\mathbb{Z}_2$-longitudes on $k_x$ and $k_y$ on this spin structure, respectively.

Let $\iota_{01}$ be the spin structure on $V$ such that $L_x$ is the $\mathbb{Z}_2$-longitude on $k_x$ and $L_y$ is not the $\mathbb{Z}_2$-longitude on $k_y$ on this spin structure.

Let $\iota_{10}$ be the spin structure on $V$ such that $L_x$ is not the $\mathbb{Z}_2$-longitude on $k_x$ and $L_y$ is the $\mathbb{Z}_2$-longitude on $k_y$ on this spin structure.

Let $\iota_{11}$ be the spin structure on $V$ such that $L_x$ and $L_y$ are the non-$\mathbb{Z}_2$-longitudes on $k_x$ and $k_y$ on this spin structure, respectively.

For the spin linking signature $s(V, \iota)$ of a spin homology quaternion space $(V, \iota)$, we have the following characterization result:

**Theorem 4.1.** For a homology quaternion space $V$ and a spin structure $\iota$ on $V$, the following statements are mutually equivalent:

1. $s(V, \iota) = 0$.
2. $\iota = \iota_{00}$, $\iota_{01}$, or $\iota_{10}$ for any $\mathbb{Z}_2$-basis $x, y$ for $H_1(V; \mathbb{Z})$.
3. There is a $\mathbb{Z}_2$-basis $x, y$ for $H_1(V; \mathbb{Z})$ on which $\iota = \iota_{00}$.
There is a $Z^{H_1}$-manifold $M$ with $H_1(M;\mathbb{Z}) = 0$ which contains $V$ as an exact leaf with a spin support $M^*$ in $M$, and $\iota$ is induced from a spin structure on $M^*$.

$\mu(V, \iota) = 0, \frac{1}{2}$.

The following statements are also mutually equivalent:

1. $s(V, \iota) = \frac{1}{2}$.
2. $\iota = \iota_{11}$ for any $\mathbb{Z}_2$-basis $x, y$ for $H_1(V; \mathbb{Z})$.
3. There is a $Z^{H_1}$-manifold $M$ with $H_1(M;\mathbb{Z}) \neq 0$ which contains $V$ as a leaf with a spin support $M^*$ in $M$, and $\iota$ is induced from a spin structure on $M^*$.
4. There is a $Z^{H_1}$-manifold $M$ with $H_1(M;\mathbb{Z}) \neq 0$ which contains $V$ as an exact leaf with a spin support $M^*$ in $M$, and $\iota$ is induced from a spin structure on $M^*$.
5. $\mu(V, \iota) = \pm \frac{1}{4}$. ■

**Proof.** Because $s(V, \iota) = 0$ if and only if there is a $\mathbb{Z}_2$-basis $x, y$ of $H_1(V; \mathbb{Z})$ such that $q_V^x(y) = q_V^y(x) = 0$ (see [8;Corollary 2.5]), we have $(1) \iff (2) \iff (3)$ and $(1') \iff (2')$. Using $s(V; \iota) = 2\mu(V, \iota)$, we see that $(5) \iff (1)$ and $(5') \iff (1')$. Thus, it suffices to show that

$$ (3) \Rightarrow (4) \Rightarrow (5), $$
$$ (2') \Rightarrow (3') \Rightarrow (4') \Rightarrow (5'), $$

To see that $(3) \Rightarrow (4)$, we use the fact that there is a $\mathbb{Z}_2$-basis $x, y$ of $H_1(V; \mathbb{Z})$ such that $q_V^x(y) = q_V^y(x) = 0$. This means that $L_x$ and $L_y$ are $\mathbb{Z}_2$-longitudes of $k_x$ and $k_y$ on $\iota$. By taking homeomorphisms $f_{\pm 1} : D^2 \times D^2 \to h_{\pm 1}$, we construct a 4-manifold

$$ W = V \times [-1, 1] \cup h_{-1} \cup h_1 $$

where we identify $T(k_x) \times (-1)$ with $f_{-1}((\partial D^2) \times D^2)$ and $T(k_y) \times 1$ with $f_1((\partial D^2) \times D^2)$ so that $L_x \times (-1)$ and $L_y \times 1$ correspond to $f_{-1}(\partial D^2 \times p)$ and $f_1(\partial D^2 \times p)$ for a point $p \in \partial D^2$. Then $W$ is a spin 4-manifold with $H_1(W; \mathbb{Z}) = 0$ and $\partial W$ is the disjoint union of two closed 3-manifolds $V_{-1}$ and $V_1$ such that $H_1(V_{\pm 1}; \mathbb{Z}) \cong \mathbb{Z}$ where $k_y \times (-1)$ and $k_x \times 1$ represent generators of $H_1(V_{-1}; \mathbb{Z})$ and $H_1(V_1; \mathbb{Z})$, respectively. Let $M$ be the double of $W$. Then $M$ is a spin $Z^{H_1}$-manifold with $H_1(M;\mathbb{Z}) = 0$. We show that $V = V \times 0$ of a copy of $W$ in $M$ is an exact leaf of $M$. Let $M_V$ be the 4-manifold obtained from $M$ by splitting along $V$. By construction, we have $H_1(M_V; \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the boundary operator $\partial' : H_2(M, M_V; \mathbb{Z}) \to H_1(M_V; \mathbb{Z})$ is onto, for $H_1(M; \mathbb{Z}) = 0$. Since

$$ H_2(M, M_V; \mathbb{Z}) \cong H_2(V \times I, V \times \partial I; \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 $$

by excision, we see that $H_2(M, M_V; \mathbb{Z})$ and $H_1(M_V; \mathbb{Z})$ are $\Lambda$-isomorphic to the same $\Lambda$-module $\Lambda_2 \oplus \Lambda_2$ with $\Lambda_2 = \mathbb{Z}_2 \otimes \Lambda$, so that the $\Lambda$-epimorphism $\partial' : H_2(M, M_V; \mathbb{Z}) \to H_1(M_V; \mathbb{Z})$ is an isomorphism by a Noetherian ring property. By [7;Theorem 2.1], this means that $V$ is an exact leaf of $M$. Thus, we have $(3) \Rightarrow (4)$. The assertion $(4) \Rightarrow (5)$ is direct from Theorem 2.3, because $\sigma(M) = 0$.

To see that $(2') \Rightarrow (3')$, we take knots $k_x, k_y$, and $k_{x+y}$ in $V$ representing $x, y, x+y$. By the assumption of $(2')$, the longitudes $L_x$ on $T(k_x), L_y$ on $T(k_y), \ldots$
and $L_{x+y}$ on $T(k_{x+y})$ are non-$Z_2$ longitudes on $i$, respectively. We construct an orientable 4-manifold $W'$ from $V \times [0,1]$ by identifying $T(k_x) \times 0$ with $T(k_y) \times 1$ and $T(k_y) \times 0$ with $T(k_{x+y}) \times 1$ so that $L_x \times 0$ and $L_y \times 0$ coincide with $L_y \times 1$ and $L_{x+y} \times 1$, respectively. Then $W'$ is a spin 4-manifold. To calculate $H_1(\partial W'; Z)$, let $V(x,y) = cl(V - T(k_x) \cup T(k_y))$ and $V(y,x+y) = cl(V - T(k_y) \cup T(k_{x+y}))$. Examining the relations between $k_x$ and $F_{2y}$ and $k_y$ and $F_{2x}$, we see that the meridians of $T(k_x)$ and $T(k_y)$ are homologous to $2L_y$ and $2L_x$ in $V(x,y)$, respectively. Then we have $H_1(V(x,y); Z) \cong Z \oplus Z$ with a basis represented by $L_x$ and $L_y$. Similarly, we have $H_1(V(y,x+y); Z) \cong Z \oplus Z$ with a basis represented by $L_y$ and $L_{x+y}$. Thus, $H_1(\partial W'; Z) \cong Z \oplus Z \oplus Z$ with a basis represented by $L_x \times 0$, $L_y \times 0$, and a simple loop $L$ such that $L \cap V(x,y) \times 0$ is an arc and hence $L \cap V(y,x+y) \times 1$ is also an arc. Attaching a 2-handle $D^2 \times D^2$ to $W'$ along a tubular neighborhood $T(L)$ of $L$ in $\partial W'$ with a $Z_2$-longitude given by a spin structure on $W'$, we obtain a spin 4-manifold $W^*$ such that $H_1(W^*; Z) \cong Z$ and $H_1(\partial W^*; Z) \cong Z \oplus Z$ with a basis represented by $L_x \times 0$ and $L_y \times 0$. To examine the homology $H_1(\tilde{W}^*; Z)$, let $W^*_\pm$ be a 4-manifold obtained from $W^*$ by splitting along $V = V \times \frac{1}{2}$, and $V_\pm$ the two copies of $V$ in $W^*_\pm$. From construction, the natural homomorphisms $H_1(V_\pm; Z) \to H_1(W^*_\pm; Z)$ are isomorphisms. Thus, the natural homomorphism $H_1(V; Z) \to H_1(\tilde{M}^*; Z)$ is an isomorphism. Since the 3-dimensional bordism group

$$\Omega_3(S^1 \times S^1) \cong \oplus_{p+q=3} H_p(S^1 \times S^1; Z) \otimes \Omega_q = 0$$

by Conner-Floyd [1], there is a compact orientable (not necessarily spin) 4-manifold $X$ such that $\partial X = \partial W^*$ and the natural homomorphism $H_1(\partial X; Z) \to H_1(X; Z)$ is an isomorphism. Then $M = M^* \cup X$ is a $Z^{H_1}$-manifold such that $V$ is a leaf of $M$ with a spin support $M^*$ in $M$. From construction, the natural homomorphism $H_1(M^*; Z) \to H_1(\tilde{M}; Z)$ is an isomorphism, and hence $H_1(\tilde{M}; Z) \neq 0$, showing that $(2') \Rightarrow (3')$. To see that $(3') \Rightarrow (4')$, we use the following two lemmas proved later:

**Lemma 4.2.** For every leaf $V$ of a $Z^{H_1}$-manifold $M$ and every field $\mathbb{F}$, the natural homomorphism $i_* : H_1(V; \mathbb{F}) \to H_1(\tilde{M}; \mathbb{F})$ is an epimorphism. In particular, if $H_1(V; Z)$ is finite, then the natural homomorphism $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an epimorphism. ■

**Lemma 4.3.** A leaf $V$ of a $Z^{H_1}$-manifold $M$ is exact if the natural homomorphism $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is a monomorphism. ■

By lemma 4.2, $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an epimorphism for a homology quaternion space $V$. If $i_*$ is not an isomorphism, then we must have $H_1(\tilde{M}; Z) \cong Z_2$, because $H_1(V; Z) \cong Z_2 \oplus Z_2$ and $H_1(\tilde{M}; Z) \neq 0$. However, this is impossible since $H_1(\tilde{M}; Z)$ is $(t-1)$-divisible. Thus, $i_*$ is an isomorphism and by Lemma 4.3 $V$ is exact and the assertion that $(3') \Rightarrow (4')$ is proved. To show that $(4') \Rightarrow (5')$, we may assume by the preceding argument that $i_* : H_1(V; Z) \to H_1(\tilde{M}; Z)$ is an isomorphism. By Theorem 2.3, we have $2\mu(V, i) = \sigma(V, i) = \sigma(\tilde{M})$. To calculate $\sigma(\tilde{M})$ directly, we note that the elements $x, tx$ for any non-zero element $x \in H_1(\tilde{M}; Z)$ form a $Z_2$-basis for $G(\tilde{M}) = H_1(\tilde{M}; Z)$, for $G(\tilde{M})$ is $(t-1)$-divisible. Further, we see that $\ell_G(x, x) = \ell_G(tx, tx) = 0$, $\ell_G(x, tx) = 1/2$, and $(1-t)^{-1}e = te$, 12.
which imply that \( q(x) = q(tx) = q(x + tx) = \frac{1}{2} \). Hence we have 

\[
GS(q) = -2 = 2\exp(2\pi \sqrt{-1} \cdot \frac{1}{2})
\]

and \( \sigma(\tilde{M}) = \sigma(q) = \frac{1}{2} \), showing \((4') \Rightarrow (5')\). \(\square\)

**Proof of Lemma 4.2.** By [11], the natural homomorphism

\[
i^F_* : H_1(V; F) \to H_1(\tilde{M}; F)
\]

is onto for every field \( F \). Taking \( F = Q \), we see from \( H_1(V; Q) = 0 \) that \( H_1(\tilde{M}; Z) \)

is a \( Z \)-torsion \( \Lambda \)-module. Hence \( H_1(\tilde{M}; Z) \) is finite because it is \((t - 1)\)-divisible. If \( i_* : H_1(V; Z) \to H_1(\tilde{M}; Z) \) is not onto, then the cokernel \( \text{coker}(i_*) \) is a non-trivial finite abelian group and we have a prime \( p \) such that \( \text{coker}(i_*) \otimes Z_p \neq 0 \). Then the homomorphism

\[
i_* \otimes 1 : H_1(V; Z) \otimes Z_p \to H_1(\tilde{M}; Z) \otimes Z_p
\]

which is identical to \( i_*^{Z_p} : H_1(V; Z_p) \to H_1(\tilde{M}; Z_p) \) is not onto. Thus, we have a contradiction. \( \square \)

**Proof of Lemma 4.3.** Let \( M_V \) be the manifold obtained from \( M \) by splitting along \( V \). As a part of the exact sequence of the pair \((\tilde{M}, M_V)\), we have the following exact sequence:

\[
H_2(\tilde{M}, M_V; Z) \xrightarrow{\partial'} H_1(\tilde{M}_V; Z) \xrightarrow{i_*} H_1(\tilde{M}; Z).
\]

Further, by excision we have an isomorphism

\[
H_2(\tilde{M}, M_V; Z) \cong H_2(\tilde{V} \times I, \tilde{V} \times \partial I; Z)(\cong H_1(\tilde{V}; Z)).
\]

Using that \( i_* : H_1(V; Z) \to H_1(\tilde{M}; Z) \) is injective, we see that the boundary operator \( \partial' : H_2(\tilde{M}, M_V; Z) \to H_1(\tilde{M}_V; Z) \) is injective, and thus the exact sequence above implies that the following semi-exact sequence

\[
0 \to tH_2(\tilde{M}, M_V; Z) \xrightarrow{\partial'} tH_1(\tilde{M}_V; Z) \xrightarrow{i_*} tH_1(\tilde{M}; Z)
\]

is exact. By [7;Theorem 2.1], this means that \( V \) is an exact leaf of \( M \). \( \square \)

Here are two examples showing that some different punctured embeddings into \( S^4 \) produce different Rochlin invariants for the quaternion space and a homology quaternion space.

**Example 4.4.** Let \( V \) be the quaternion space, which is the boundary \( \partial N \) of a tubular neighborhood \( N \) of the real projective plane \( P^2 \) embedded smoothly in \( S^4 \). Then a punctured 3-manifold \( V^o \) of \( V \) is the interior of a Seifert hypersurface of a trivial 2-knot \( K_0 \) in \( S^4 \). Let \( t_0 \) be the spin structure on \( V \) induced from the inclusion \( V \subset S^4 \). By a 2-handle surgery along \( K_0 \), we see that \( V \) is a leaf of the spin \( Z^H \)-manifold \( M_0 = S^1 \times S^3 \). Since \( BH_2(\tilde{M}_0; Z) = 0 \), we see that \( V \) is
necessarily an exact leaf of $M_0$ by [7]. By construction, the spin structure $\iota_0$ on $V$ coincides with the one induced from any spin structure on $M_0$. Since the spin 3-manifold $(V, \iota_0)$ is the boundary of a compact spin 4-submanifold of $S^4$ (which has zero signature), we have $\mu(V; \iota_0) = 0$. Using $H_1(M_0; Z) = 0$, we see from Theorem 4.1 that there is a $Z_2$-basis $x, y$ for $H_1(V; Z)$ with $\iota_0 = \iota_{00}$. On the other hand, a punctured 3-manifold $V^0$ of the quaternion space $V$ is a fiber of a fibered $S^2$-knot $K$ in $S^4$ (for example, consider the 3-twist spun trefoil by E. C. Zeeman [15]). Let $\iota_1$ be the spin structure on $V$ determined by the inclusion $V^0 \subset S^4 - K \subset S^4$. The quaternion space $V$ is a fiber of a spin fiber bundle $M_K$ over $S^4$ with $H_*(M_K; Z) = H_*(S^1 \times S^3; Z)$, obtained from $S^4$ by a 2-handle surgery along $K$, and $V$ is an exact leaf of $M_K$ by [7] since $BH_2(\tilde{M}_K; Z) = 0$. By construction, the spin structure $\iota_1$ on $V$ coincides with the one induced from any spin structure on $M_K$. Since $H_1(\tilde{M}_K; Z) \cong H_1(V; Z) \neq 0$, we see from Theorem 4.1 that $\mu(V, \iota_1) = \pm \frac{1}{4}$ and $\iota_1 = \iota_{11}$ on any $Z_2$-basis $x, y$ for $H_1(V; Z)$.

Example 4.5. Let $V^P$ be the homology quaternion space $V \# P$ where $V$ is the quaternion space and $P$ is the Poincaré homology 3-sphere with $\mu(P, \iota_P) = \frac{1}{2}$ for the unique spin structure $\iota_P$ on $P$. A punctured 3-manifold $P^0$ of $P$ is a fiber of a fibered 2-knot $K_P$ in $S^4$ (for example, consider the 5-twist spun trefoil by [15]). A punctured 3-manifold $(V^P)^0$ of $V^P$ is the interior of a Seifert hypersurface for the $S^2$-knot $K_0 \# K_P = K_P$ in $S^4$. Let $\iota_0^P$ be the spin structure on $V^P$ determined by the inclusion $(V^P)^0 \subset S^4 - K_P \subset S^4$, which is equal to the spin structure determined from $\iota_0$ in Example 4.4 by construction. The homology quaternion space $V^P$ is a leaf of a spin $Z^H_1$-manifold $M_0^P$ with $H_*(M_0^P; Z) = H_*(S^1 \times S^3; Z)$, obtained from $S^4$ by a 2-handle surgery along $K_P$, and $V^P$ is an exact leaf of $M_0^P$ by [7] since $BH_2(M_0^P; Z) = 0$. We note that the spin structure $\iota_0^P$ is the one induced from any spin structure on $M_0^P$ and $H_1(M_0^P; Z) = 0$. In this case, we see that $\mu(V^P; \iota_0^P) = \frac{1}{2}$, and there is a $Z_2$-basis $x, y$ for $H_1(V^P; Z)$ with $\iota_0^P = \iota_{00}$ as it is shown in Theorem 4.1. On the other hand, a punctured manifold $(V^P)^0$ of $V^P$ is a fiber of a fibered $S^2$-knot $K \# K_P$ in $S^4$, where $K$ denotes the $S^2$-knot in Example 4.4. Let $\iota_1^P$ be the spin structure on $V^P$ determined by the inclusion $(V^P)^0 \subset S^4 - K \# K_P \subset S^4$, which is equal to the spin structure determined from $\iota_1$ in Example 4.4 by construction. The homology quaternion space $V^P$ is a fiber of a spin fiber bundle $M_K^P$ over $S^4$ with $H_*(M_K^P; Z) = H_*(S^1 \times S^3; Z)$, obtained from $S^4$ by a 2-handle surgery along $K \# K_P$, and $V^P$ is an exact leaf of $M_K^P$ by [7] since $BH_2(M_K^P; Z) = 0$. We note that the spin structure $\iota_1^P$ is the one induced from any spin structure on $M_K^P$ and $H_1(M_K^P; Z) \cong H_1(V^P; Z) \neq 0$. In this case, we see that

$$\mu(V^P, \iota_1^P) = \mu(V, \iota_1) + \frac{1}{2} = \pm \frac{1}{4},$$

and $\iota_1^P = \iota_{11}$ on any $Z_2$-basis $x, y$ for $H_1(V^P; Z)$ as it is shown in Theorem 4.1.

References


