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CLASSIFICATION OF PRETZEL KNOTS

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A *pretzel knot* is a knot given by a knot diagram consisting of a row of 2-strand braids. Fig. 1 shows a pretzel knot with a row of braids of q_1-, q_2-,..., q_m-half twists, which we denote by $k(q_1, q_2,..., q_m)$. We assume that $q_i \neq 0$, $i=1, 2,..., m$. Let $q_{j_1}, q_{j_2},..., q_{j_n} (j_1 < j_2 < \cdots < j_n)$ be the non-unit integers in the q_j's. Let $p_i = q_{j_i}$, $i=1, 2,..., n$. Let $b=\sum_{j=1}^{m} q_j - \sum_{i=1}^{n} p_i$. By turning, if necessary, the braids of p_i-half twists, we can deform $k(q_1, q_2,..., q_m)$ into a knot with diagram, illustrated in Fig. 2, which we denote by $k(-b; p_1, p_2,..., p_n)$. Since it is a knot, only the following two cases occur:

(1) All of the p_i's and $n+b$ are odd and $n \geq 0$,

(2) Exact one of the p_i's is even and b is arbitrary and $n \geq 1$.

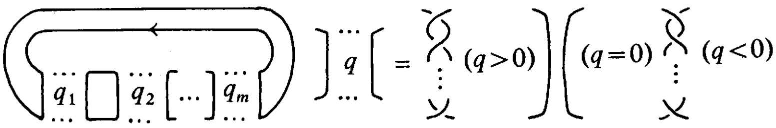

Fig. 1

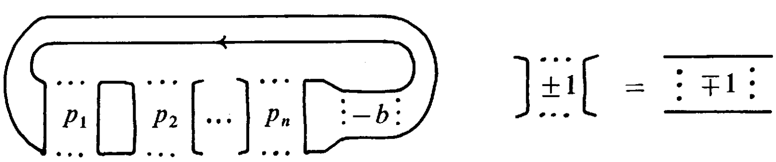

Fig. 2

We say that $k(-b; p_1, p_2,..., p_n)$ is *odd* (or *even*, resp.) if it is in the case (1) (or (2), resp.). Two oriented knots k, k' are *equivalent* and denoted by $k \cong k'$, if there is an orientation-preserving auto-homeomorphism of S^3 sending k to k' orientation-preservingly. We orient $k(-b; p_1, p_2,..., p_n)$ by the orientation indicated in Fig. 2. When $(p'_1, p'_2,..., p'_n)$ is a cyclic translation of $(p_1, p_2,..., p_n)$, we write $(p'_1, p'_2,..., p'_n) \cong (p_1, p_2,..., p_n)$. Then we have easily $k(-b; p'_1, p'_2,..., p'_n) \cong k(-b; p_1, p_2,..., p_n)$. The inverse, the reflection and the reflected inverse of $k(-b; p_1, p_2,..., p_n)$ are equivalent to $k(-b; p_n,..., p_2, p_1)$, $k(b; -p_1, -p_2,..., -p_n)$ and $k(b; -p_n,..., -p_2, -p_1)$, respectively. For even pretzel knots, one can show that $k(-b; p_n,..., p_2, p_1) \cong k(-b; p_1, p_2,..., p_n)$. Fig. 3

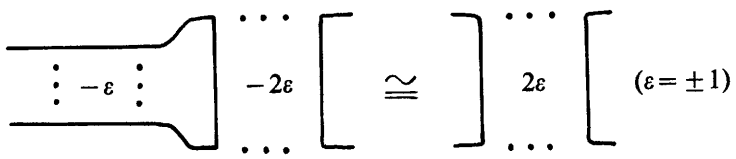

Fig. 3

also shows that $k(-b;\, p_1,\dots, p_i,\dots, p_n)\cong k(-b';\, p_1,\dots, p'_i,\dots, p_n)$ if for some i, $|p_i|=|p'_i|=2$ and $\varepsilon(p_i)(b'-b)=\varepsilon(p'_i)(b-b')=1$, where $\varepsilon(p)=p/|p|$. Then according to if $|b|<|b'|$ or $|b'|<|b|$, $k(-b;\, p_1,\dots, p_i,\dots, p_n)$ or $k(-b';\, p_1,\dots, p'_i,\dots, p_n)$ is said to have a *minimal presentation. Unless otherwise stated, only pretzel knots with minimal presentations will be considered for pretzel knots with braids of* ± 2*-half twists.* We define the *Euler number* $e(k)(\neq 0)$ and for $n\leq 2$ the *character* $c(k)(\neq 0)$ of $k=k(-b;\, p_1,\, p_2,\dots, p_n)$ by

$$e(k) = b + \textstyle\sum_{i=1}^{n} 1/p_i, \quad \text{and}$$

$$c(k) = -1/e(k) (\text{if } n\leq 1) \quad \text{or} \quad (bp_1+1)/p_1p_2e(k)\ (\text{if } n=2).$$

Non-zero rational numbers x, x' are *equivalent* and denoted by $x\cong x'$, if the irreducible fractions q/p, q'/p' $(p,\, p'>0)$ of x, x' have $p=p'$ and $q^{\pm 1}\equiv q'$ (mod p). A knot k is *simple* if the exterior $E(k)=S^3\text{-Int}\, N(k)$, $N(k)$ being the regular neighborhood of k, has no incompressible imbedded torus that is not boundary-parallel (cf. [J]). In this note, we shall prove the following three theorems:

THEOREM I. *The pretzel knots* $k=k(-b;\, p_1,\, p_2,\dots, p_n)$ *and* $k'=k(-b';\, p'_1,\, p'_2,\dots, p'_{n'})$ *are equivalent if and only if one of the following cases occurs:*
(1) *Both* n *and* n' *are* ≤ 2 *and* $c(k)\cong c(k')$,
(2) *Both* k *and* k' *are odd,* $n=n'\geq 3$, $b=b'$ *and* $(p'_1,\, p'_2,\dots, p'_n)\cong(p_1,\, p_2,\dots, p_n)$,
(3) *Both* k *and* k' *are even,* $n=n'\geq 3$, $b=b'$ *and* $(p'_1,\, p'_2,\dots, p'_n)\cong(p_1,\, p_2,\dots, p_n)$ *or* $(p_n,\dots, p_2,\, p_1)$.

THEOREM II. *Every pretzel knot is simple.*

THEOREM III. *A pretzel knot is equivalent to a torus knot if and only if it is equivalent to* $k(-p;\, -)$ *for some odd* p, $k(0;\, 3\varepsilon,\, 3\varepsilon,\, -2\varepsilon)$ *or* $k(0;\, 3\varepsilon,\, 5\varepsilon,\, -2\varepsilon)$, $\varepsilon=\pm 1$.

It is directly checked that $k(p\varepsilon;\, -)(p>0)$, $k(0;\, 3\varepsilon,\, 3\varepsilon,\, -2\varepsilon)$ or $k(0;\, 3\varepsilon,\, 5\varepsilon,\, -2\varepsilon)$ are equivalent to the torus knots of type $(p,\, 2\varepsilon)$, $(3,\, 4\varepsilon)$ and $(3,\, 5\varepsilon)$, respectively, $\epsilon=\pm 1$(cf. Fig. 4).

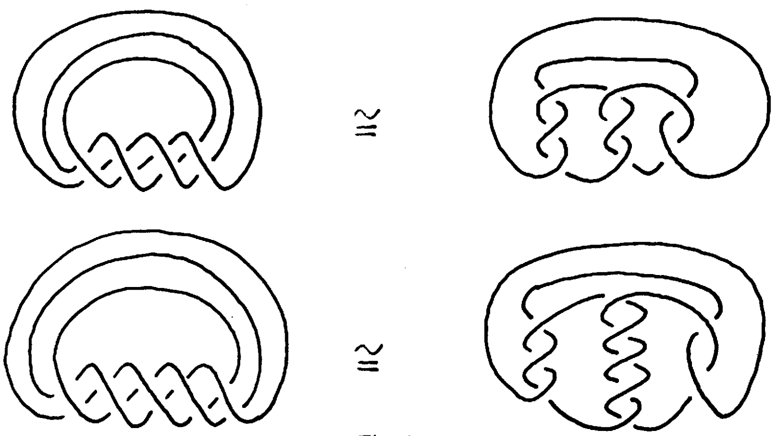

Fig. 4

To obtain THEOREM III, we shall also determine the pretzel knots whose branched double covering spaces are homeomorphic to those of torus knots. Note that a knot is a torus knot iff the exterior is a Seifert manifold (cf. Burde/Murasugi [B/M]). Then according to Thurston [TH], the pretzel knot exterior is a hyperbolic manifold except the torus knots of THEOREM III. THEOREM I is obtained by adding several remarks to Parris's arguments in [P], but for the sakes of convenience and clarity, we shall give here a full proof. After having done this work, the author learned from Boileau [B] that Bonahon/Boileau/Siebenmann have obtained similar results*) for the Montesions knots (and links) containing the pretzel knots, by using different methods. Some results of this note will be used in [K/K/S]. Spaces and maps will be considered in the piecewise-linear category.

1. Proof of Theorem I. Let $k=k(-b;\, p_1,\, p_2,\dots,\, p_n)$ and $G=\pi_1(S^3-k)$. Following Reidemeister [R], Trotter [TR] and [P], we consider the quotient $G_*=G/\langle m^2=1\rangle$, where m is a meridian element of k. Let $x_1, x_2,\dots, x_r$, $r=n+|b|$, be the meridian elements of k about the maximal points in Fig. 2, in the direction from the bottom to the top. We have

$$G_* = (x_1,\, x_2,\dots,\, x_r | x_1^2 = x_2^2 = \cdots = x_r^2 = 1,$$
$$(x_1x_2)^{p_1} = \cdots = (x_nx_{n+1})^{p_n} = (x_{n+1}x_{n+2})^{\varepsilon} = \cdots = (x_rx_1)^{\varepsilon}),$$

where $\varepsilon=\varepsilon(b)$ (if $b\neq 0$). (When $b=0$, we understand that $x_{n+1}=x_1$ and the relation $(x_{n+1}x_{n+2})^{\varepsilon}=\cdots=(x_rx_1)^{\varepsilon}$ does not appear.) Let C be the cyclic subgroup of G_* generated by $(x_1x_2)^{p_1}=\cdots=(x_rx_1)^{\varepsilon}$. Since C is normal in G_*, we can consider the quotient $G_{**}=G_*/C$. We have

*) H. Zieschang has also obtained them.

$$G_{**} = (x_1, x_2, \ldots, x_n | x_1^2 = x_2^2 = \cdots = x_n^2 = (x_1 x_2)^{p_1} = \cdots = (x_n x_1)^{p_n} = 1).$$

Note that $H_1(G_*; Z) = H_1(G_{**}; Z) = Z_2$. Let QG_*, QG_{**} be the commutator (index 2) subgroups of G_*, G_{**}, respectively. Writing $a_i = x_i x_{i+1}$, $a_r = x_r x_1$, we have

$$QG_* = (a_1, a_2, \ldots, a_r | a_1^{p_1} = a_2^{p_2} = \cdots = a_n^{p_n} = a_{n+1}^{\varepsilon} = \cdots = a_r^{\varepsilon}, a_1 a_2 \cdots a_r = 1),$$

$$QG_{**} = (a_1, a_2, \ldots, a_n | a_1^{p_1} = a_2^{p_2} = \cdots = a_n^{p_n} = a_1 a_2 \cdots a_n = 1).$$

Clearly, G_* and QG_* are invariants of k. Similarly, G'_*, G'_{**}, QG'_* and QG'_{**} are defined for $G' = \pi_1(S^3 - k')$ with $k' = k(-b'; p'_1, p'_2, \ldots, p'_{n'})$.

LEMMA 1.1. *If $k \cong k'$, then n and n' are ≤ 2 or ≥ 3 at the same time.*

PROOF. For $n \leq 2$, QG_* is abelian (cyclic). We show that QG_* is non-abelian for $n \geq 3$. It suffices to show that QG_{**} is non-abelian for $n \geq 3$. According to if $\sum_{i=1}^n 1/|p_i|$ is $> n-2$ (then, $n=3$), $= n-2$ or $< n-2$, we can construct an n-sided convex polygon $P = (v_1 v_2 \cdots v_n)$ in the spherical, Euclidean or hyperbolic plane (S^2, E^2 or H^2) so that the interior angle at the vertex v_i is $\pi/|p_i|$ and for the geodesics $\ell_1, \ell_2, \ldots, \ell_n$ determined by the edges $v_n v_1, v_1 v_2, \ldots, v_{n-1} v_n$, $\ell_i \cap \ell_j \neq \phi$ iff $j \equiv i \pm 1 \pmod n$. Then G_{**} is a discrete group of isometries of S^2, E^2 or H^2 such that the generators $x_1, x_2, \ldots, x_n$ correspond to the reflections in $\ell_1, \ell_2, \ldots, \ell_n$ (see Coxeter/Moser [C/M], Magnus [M]). Suppose that $\sum_{i=1}^n 1/|p_i| \leq n-2$. Then G_{**} and hence QG_{**} are infinite groups. Since $H_1(QG_{**}; Z)$ is finite, QG_{**} is non-abelian. Suppose that $\sum_{i=1}^n 1/|p_i| > n-2$. Then $n=3$ and $\{|p_1|, |p_2|, |p_3|\} = \{2, 3, 3\}$ or $\{2, 3, 5\}$. For example, by Fox [F1] the natural map $QG_{**} \to H_1(QG_{**}; Z)$ has a non-trivial kernel, implying that QG_{**} is non-abelian. Similarly, QG'_* is abelian or non-abelian according to if $n' \leq 2$ or ≥ 3. The result follows.

For $n \leq 2$, k is a 2-bridge knot. Let (α, β) be a normal form of k due to Suhubert [SC2]. Then $c(k) \cong \beta/\alpha$. In fact, Fig. 5 shows that for $n=2$ and

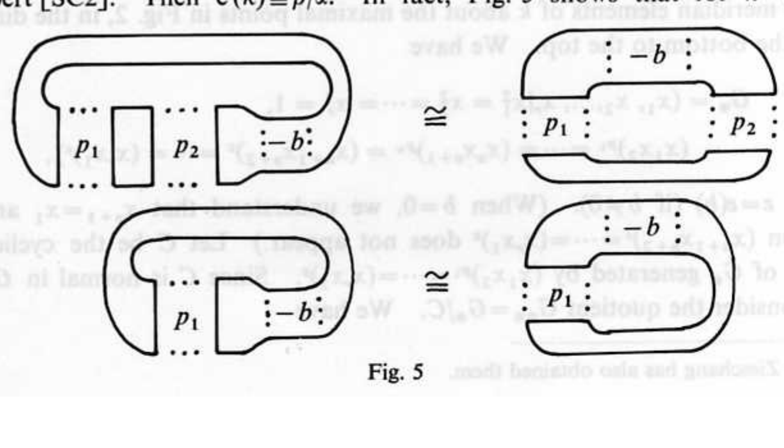

Fig. 5

$b \neq 0$, $\beta/\alpha \cong 1/(p_1+1/(b+1/p_2)) = (p_1 b+1)/p_1 p_2 \mathrm{e}(k)$ and for $n=1$ and $b \neq 0$, $\beta/\alpha \cong 1/(p_1+1/b) = b/(p_1 b+1) \cong -p_1/(p_1 b+1) = -1/\mathrm{e}(k)$. The other case ($n=0$ or $b=0$) is easier checked.

PROOF of THEOREM I for $n \leq 2$. By Lemma 1.1, $n' \leq 2$. Schubert's classification of 2-bridge knots [SC2] and the above remark imply that $k \cong k'$ iff $\mathrm{c}(k) \cong \mathrm{c}(k')$, completing the proof.

To conclude the proof of THEOREM I, it suffices to show the "only if" part, assuming that n, $n' \geq 3$, since the "if" part was observed in the Introduction.

LEMMA 1.2. *If $k \cong k'$ and n, $n' \geq 3$, then $n=n'$, $b=b'$ and $\{p_1, p_2, \ldots, p_n\} = \{p'_1, p'_2, \ldots, p'_n\}$. In particular, $\mathrm{e}(k) = \mathrm{e}(k')$ and k, k' are odd or even at the same time.*

PROOF. The double covering spaces $S^3(k)_2$, $S^3(k')_2$ of S^3 branched along k, k' are Seifert manifolds over S^2 with invariants $(b; (p_1, 1), (p_2, 1), \ldots, (p_n, 1))$, $(b'; (p'_1, 1), (p'_2, 1), \ldots, (p'_{n'}, 1))$, respectively (cf. Montesions [MO]). Note that there is an orientation-preserving homeomorphism $h: S^3(k)_2 \cong S^3(k')_2$. If $\pi_1(S^3(k)_2)$ is infinite, then by Orlik/Voget/Zieschang [O/V/Z] or Conner/Raymond [C/R] h is homotopic to a fiber-preserving homeomorphism. By Neumann/Raymond [N/R, Theorem 1.1] (where we understand $(p_i, 1)$ as $(|p_i|, \varepsilon(p_i))$ and b as $(1, b)$), we have $n=n'$, $p_i = p'_i$ and $b=b'$, noting our assumption on minimal presentations and changing the indices of $p_1, p_2, \ldots, p_n$, if necessary. Assume that $\pi_1(S^3(k)_2)$ is finite. Since $\pi_1(S^3(k)_2) = QG_*$, QG_{**} is finite. So, $n=3$ and $\{|p_1|, |p_2|, |p_3|\} = \{2, 3, 3\}$ or $\{2, 3, 5\}$ (cf. the proof of LEMMA 1.1). Similarly, $n'=3$ and $\{|p'_1|, |p'_2|, |p'_3|\} = \{2, 3, 3\}$ or $\{2, 3, 5\}$. By Seifert [SE], the orders of $H_1(S^3(k)_2; Z)$, $H_1(S^3(k')_2; Z)$ are $|p_1 p_2 p_3 \mathrm{e}(k)|$, $|p'_1 p'_2 p'_3 \mathrm{e}(k')|$, respectively. Since they are equal, it follows that $(b', p'_1, p'_2, p'_3) = (b\varepsilon, p_1\varepsilon, p_2\varepsilon, p_3\varepsilon)$ for $\varepsilon = \pm 1$, noting our assumption on minimal presentations and changing the indices of p_1, p_2, p_3, if necessary. If $\varepsilon = -1$ occurs, then $S^3(k)_2$ admits an orientation-reversing auto-homeomorphism. But, [N/R, Theorem 8.2] shows that $S^3(k)_2$ never does, for $H_1(S^3(k)_2; Q) = 0$. Thus, $\varepsilon = 1$. This completes the proof.

PROOF of the "only if" part of THEOREM I for $n \geq 3$. When $n=3$ and k is odd with $(p_1, p_2, p_3) \cong (p_3, p_2, p_1)$ or even, then the result follows from LEMMA 1.2, since $\{p_1, p_2, p_3\} = \{p'_1, p'_2, p'_3\}$ implies $(p'_1, p'_2, p'_3) \cong (p_1, p_2, p_3)$ or (p_3, p_2, p_1). So, assume that $n=3$ and k is odd with $(p_1, p_2, p_3) \not\cong (p_3, p_2, p_1)$ or $n \geq 4$. Then $\sum_{i=1}^{n} 1/|p_i| < n-2$ and G_{**} is a discrete group of isometries of H^2 as in the proof of LEMMA 1.1. Note that $\ell_i \cap \ell_j = \phi$ iff $x_i x_j$ is of infinite order in G_{**}. QG_{**} is a discrete group of orientation-preserving isometries of H^2 and it is well-known (easily proved) that the center of QG_{**} is trivial. Since $C \subseteq$ (the center of QG_*) and $QG_*/C = G_{**}$, we see that C is equal to the center of QG_*. It follows that

$G_{**}=G_*/C$ is an invariant of k. Assume that $k'\cong k$. Then by LEMMA 1.2, $n'=n$ and $b'=b$. We have an isomorphism $G'_{**}=(x'_1, x'_2,\dots, x'_n | x'^2_1=x'^2_2=\cdots=x'^2_n=(x'_1x'_2)^{p'_1}=\cdots=(x'_nx'_1)^{p'_n}=1)\cong G_{**}$. Identify x'_i with the isomorphic image of it. Since $x'_1, x'_2,\dots, x'_n$ are mutually conjugates in $G'_{**}\cong G_{**}$, we see that $x'_1, x'_2,\dots, x'_n$ act on H^2 orientation-reversingly, so that $x'_1, x'_2,\dots, x'_n$ are reflections in some geodesics $\ell'_1, \ell'_2,\dots, \ell'_n$. Noting that $x'_1, x'_2,\dots, x'_n$ are mutually distinct and $x'_ix'_j$ is of infinite order unless $j\equiv i\pm 1 \pmod n$, we see that $\ell'_1, \ell'_2,\dots, \ell'_n$ are mutually distinct and $\ell'_i\cap\ell'_j=\phi$ unless $j\equiv i\pm 1 \pmod n$. Since $(x'_1x'_2)^{p'_1}=\cdots=(x'_nx'_1)^{p'_n}=1$, we see that ℓ'_i and ℓ'_{i+1} meet at a point v'_i, $i=1, 2,\dots, n(\ell'_{n+1}=\ell'_1)$. The ℓ'_i's and the v'_i's determine an n-sided convex polygon P'. Since the interior angle of P' at v'_i is a multiple of $\pi/|p'_i|$ and by LEMMA 1.2 $\{p'_1, p'_2,\dots, p'_n\}=\{p_1, p_2,\dots, p_n\}$, it follows that (the total curvature of P')$\leq \sum_{i=1}^n (\pi-\pi/|p_i|)=$(the total curvature of P). Let D, D' be the finite regions in H^2 bounded by P, P', respectively. By the Gauss/Bonnet theorem, (the area of D')$\leq$ (the area of D). Since D is a fundamental region of G_{**} and D' is a union of isometric copies of D, it follows that D' is an isometric copy of D, that is, $tD'=D$ for some $t\in G_{**}$. Write $tx'_it^{-1}=x_{j_i}$, $i=1, 2,\dots, n$. We have $(j_1, j_2,\dots, j_n)\cong(1, 2,\dots, n)$ or $(n,\dots, 2, 1)$. By composing the isomorphism $G'_{**}\cong G_{**}$ to the inner automorphism induced by t^{-1}, we consider that $x'_i=x_{j_i}$, $i=1, 2,\dots, n$. If k is even, we can assume by using an equivalence $k(-b; p_1, p_2,\dots, p_n)\cong k(-b; p_n,\dots, p_2, p_1)$ that $(j_1, j_2,\dots, j_n)\cong(1, 2,\dots, n)$. The following two lemmas will complete the proof :

LEMMA 1.3. *If k is odd, then we have necessarily $(j_1, j_2,\dots, j_n)\cong(1, 2,\dots, n)$.*

LEMMA 1.4. *If $(j_1, j_2,\dots, j_n)\cong(1, 2,\dots, n)$, then $(p'_1, p'_2,\dots, p'_n)\cong(p_1, p_2,\dots, p_n)$.*

PROOF of LEMMA 1.3. Suppose $(j_1, j_2,\dots, j_n)\not\cong(1, 2,\dots, n)$. Then $(j_1, j_2,\dots, j_n)\cong(n,\dots, 2, 1)$. By changing the indices of p_{j_i}, x_{j_i} cyclically, we can assume that $x'_i=x_{n+2-i}$ and $|p'_i|=|p_{n+1-i}|$, $i=1, 2,\dots, n$ $(x_{n+1}=x_1)$. Let L, L' be the longitude elements of k, k' in G_{**}, G'_{**}, respectively. The equivalence $k'\cong k$ means that $uL'u^{-1}=L$ for some $u\in G_{**}$. We can write L, L' as follows ([TR], [P]):

$$L = [(x_1x_2)^{-d_1}(x_2x_3)^{-d_2}\cdots(x_nx_1)^{-d_n}]^2,$$
$$L' = [(x'_1x'_2)^{-d'_1}(x'_2x'_3)^{-d'_2}\cdots(x'_nx'_1)^{-d'_n}]^2,$$

where $d_i=(|p_i|-1)/2=(|p'_{n+1-i}|-1)/2=d'_{n+1-i}$, $i=1, 2,\dots, n$. Then we find $w\in G_{**}$ such that $wLw^{-1}=L^{-1}$. We show that there are no such elements in G_{**}. This is due to [TR, p. 279], but we give the proof. Note that L is a translation algong ℓ_1 through a distance equal to twice the perimeter of P in the direction from v_1 to v_n (cf. [TR], [P]). Regard L as a real Möbius transformation acting on the Riemann sphere $C \cup \{\infty\}$ and H^2 as the upper half plane. Since L is of infinite order and fixes the geodesic ℓ_1 setwise, L must be a hyperbolic element (see Lehner [LE, p. 8]). By applying a real Möbius transformation, we can assume that the fixed points of L are 0 and ∞, so that there is a constant $r>0$ with $L(z)=rz$ for all $z \in H^2$ and ℓ_1 is the y-axis within H^2. First assume $w^2 \neq 1$. Using that $w^2Lw^{-2}=L$, we see that the fixed points of w^2 are 0 and ∞ (cf. [LE, p.9]) and hence there is a constant $r'>0$ such that $w^2(z)=r'z$ for all $z \in H^2$. Then $w(z)=\sqrt{r'}z$ or $-\sqrt{r'}\bar{z}$($\bar{z}$=the complex conjugation of z) for all $z \in H^2$, according to if w is orientation-preserving or -reversing. [To see this, note that $w(z)$ can be written as $(az+b)/(cz+d)$ or $-(a\bar{z}+b)/(c\bar{z}+d)$ for real a, b, c, d with $ad-bc=1$, according to if w is orientation-preserving or -reversing.] We have $wLw^{-1}=L \neq L^{-1}$, a contradiction. Thus, $w^2=1$. Since $p_1, p_2, \dots, p_n$ are odd, w must be orientation-reversing. We can write $w(z)=-(a\bar{z}+b)/(c\bar{z}+a)$ for real a, b, c with $a^2-bc=1$. Using that $(wL)^2=1$, $L(z)=rz$ and $r\neq 1$, we have $a=0$ and $w(z)=b^2/\bar{z}$. This implies that w is a reflection in the geodesic $S^+=\{z \in \boldsymbol{C} \mid |z|=|b|, \operatorname{Im} z>0\}$. Since S^+ meets ℓ_1 orthogonally, at most one of the p_i's must be even, which is a contradiction. This completes the proof.

PROOF of LEMMA 1.4. The proof is essentially due to [P]. By changing the indices of p_{j_i} and x_{j_i} cyclically, we can assume that $x_i'=x_i$ in G_{**} and $|p_i'|=|p_i|$, $i=1, 2, \dots, n$. For a generator g of C, we have

(1) $x_i'=x_ig^{m_i}=g^{-m_i}x_i$, $i=1, 2, \dots, n$,

(2) $(x_1'x_2')^{p_1'}=(x_2'x_3')^{p_2'}=\cdots=(x_n'x_1'g^{-\varepsilon b})^{p_n'}=g^{\varepsilon}$, $\varepsilon=\pm 1$,

(3) $(x_1x_2)^{p_1}=(x_{23})^{p_2}=\cdots=(x_nx_1g^{-b})^{p_n}=g$

in G_*. Note that QG_* is torsion-free [PROOF. $QG_*=\pi_1(S^3(k)_2)$ and $S^3(k)_2$ is a Seifert Z_2-homology 3-sphere (cf. the proof of LEMMA 1.2). By [SE], $S^3(k)_2$ is irreducible. By the sphere theorem $S^3(k)_2$ is aspherical, for QG_* is infinite. So, QG_* is torsion-free (cf. Hempel [H])]. Thus, C is infinite cyclic, because C is non-trivial in QG_*. We assume that $|p_i'|$, $i=1, 2, \dots, n-1$, are odd (≥ 3) and $|p_n'|\geq 2$. Using that C is the center of QG_*, we see that

$$g^{\varepsilon} = (x_1'x_2')^{p_1'} = (g^{-m_1}x_1x_2g^{m_2})^{p_1'}=(x_1x_2)^{p_1'}\, g^{p_1'(m_2-m_1)}$$

If $p_1'=\varepsilon_1 p_1$, $\varepsilon_1=\pm 1$, then $\varepsilon=\varepsilon_1+p_1'(m_2-m_1)$. For $|p_1'|\geq 3$, we have $\varepsilon=\varepsilon_1$ and $m_1=m_2$. Similarly, we have $p_i'=\varepsilon p_i$ for $i=1, 2, \dots, n-1$, and $m_1=m_2=\cdots=m_n$. Note that

$$(x_n'x_1')^{p_n'} = g^{\varepsilon b p_n'+\varepsilon} \quad\text{and}\quad (x_nx_1)^{p_n} = g^{bp_n+1}.$$

If $p_n'=\varepsilon_n p_n$, $\varepsilon_n=\pm 1$, then $\varepsilon\varepsilon_n p_n b+\varepsilon=\varepsilon_n p_n b+\varepsilon_n$. For $|p_n|\geq 2$, $\varepsilon_n=\varepsilon$. In conclusion, we have $p_i'=\varepsilon p_i$, $i=1, 2, \dots, n$, and $b=\varepsilon b$. Suppose $\varepsilon=-1$. Then $b=0$ and $p_i'=-p_i$, $i=1, 2, \dots, n$. If k is odd, then $e(k)=-e(k')\neq 0$. By LEMMA 1.2, $e(k)=e(k')$, a contradiction. If k is even, then p_n and p_n' are the unique non-zero even numbers in the p_i's and the p_i''s, respectively. By LEMMA 1.2, $p_n = p_n'$, a contradiction. Therefore, $\varepsilon = 1$. This completes the proof.

2. Proof of Theorem II.

PROOF of THEOREM II for $n \leq 3$. When $n \leq 3$, the bridge index of k is ≤ 3. If k is not simple (i.e., k has a non-trivial companion), then by Schubert [SC, 1] k must be the sum of two non-trivial 2-bridge knots, so that $S^3(k)_2$ is not irreducible. But, it is a Seifert Z_2-homology 3-sphere and by [SE] irreducible, which is a contradiction. This completes the proof.

For $n \geq 4$ we shall use a concept of simple tangles by Soma [SO]. Let a_1, a_2 be disjoint arcs properly imbedded in a 3-ball B. The union $t = a_1 \cup a_2$ is called a *tangle* in B. Two tangles t_1, t_2 are *equivalent* and denoted by $t_1 \cong t_2$, if there is an orientation-preserving auto-homeomorphism of B sending t_1 to t_2 setwise. A tangle t is *simple*, if t is prime and $B - t$ has no incompressible imbedded torus. Note that t is prime iff the doulbe covering space $B(t)_2$ of B branched along t is irreducible and not homeomorphic to a solid torus (cf. Lickorish [LI]). We use the following three lemmas:

LEMMA 2.1. *Let $t = a_1 \cup a_2$ be a prime tangle in B. Assume that there is a disk D in B with $a_1 \subset \partial D$ and $\mathrm{cl}(\partial D - a_1) \subset \partial B$ such that a_2 and $\mathrm{Int}\, D$ intersect transversally in a single point and $\pi_1(B - D \cup a_2)$ is free. Then t is simple.*

LEMMA 2.2. *Let a tangle $t \subset B$ be a sum of a trivial tangle $t_0 \subset B_0$ and a prime tangle $t_1 \subset B_1$ along a disk $D^* = (\partial B_0) \cap (\partial B_1)$ such that $D^* - t_0 \cap D^*$ is incompressible in $B_0 - t_0$. Then t is simple if and only if t_1 is simple.*

LEMMA 2.3. *A knot is simple if it is a sum of two simple tangles.*

PROOF of LEMMA 2.1. The proof is implicitly contained in [SO]. Suppose that there is an incompressible torus T in $B - t$. T splits B into two parts E_1, E_2 with $\partial E_1 = T$, $\partial E_2 = T \cup \partial B$. Note that $t \subset E_2$. T intersects D, since otherwise, we would have a monomorphims $\pi_1(T) = Z \times Z \to \pi_1(B - D \cup a_2) =$ *a free group*, which is impossible. Let $D_0 = D - D \cap a_2$. Since D_0 is incompressible in $B - t$, we can assume that $D \cap T$ consists of essential loops in both T and D_0. Let ℓ be a loop in $D \cap T$, innermost in D. Let D' be the disk in D bounded by ℓ. Note that $a_2 \cap D \subset D' \subset E_2$. Let $N(D')$ be a collar of D' in E_2 such that $a_2' = a_2 \cap N(D')$ is a proper unknotted arc in $N(D')$. Then a_2' is unknotted in the 3-ball $E_1 \cup N(D')$, for otherwise a_2 and hence $t = a_1 \cup a_2$ has a local knot, contradicting the primeness of t. So, E_1 is a solid torus which contradicts the incompressibility of T in $B - t$. The proof is completed.

PROOF of LEMMA 2.2. The "if part" is proved in [SO]. To see the "only if'' part, we take a torus T in B_1-t_1. Since $B-t$ is simple, T is compressible in $B-t$. Let D be a compressible disk. Using that $D^*-t_0\cap D^*=D^*-t_1\cap D^*$ is incompressible in both B_0-t_0 and B_1-t_1, we can deform D (by an isotopy of B keeping $T\cup t$ fixed) so that $D\subset B_1-t_1$. Thus, T is compressible in B_1-t_1 and t_1 is simple, completing the proof.

LEMMA 2.3 is proved in [SO].

PROOF of THEOREM II for $n\geq 4$. Denote by $t(-b;\, p_1,\ldots,\, p_m)$ the tangle illustrated in Fig. 6(a), where the p_i's are non-zero, non-unit integers and odd except for some one. We shall show that $t(0;\, p_1,\ldots,\, p_m)$ is simple for $m\geq 2$. The proof will be then completed by LEMMA 2.3, because for $n\geq 4$ $k(-b;\, p_1,\, p_2,\ldots,\, p_n)$ is a sum of the tangles $t(0;\, p_1,\ldots,\, p_{n-2})$ and $t(-b;\, p_{n-1},\, p_n)$, and $t(-b;\, p_{n-1},\, p_n)\cong t(0;\, p_{n-1},\, p_n)$. The tangle $t=t(0;\, p_1,\ldots,\, p_m)$ $(m\geq 2)$ is prime, since $B(t)_2$ is a bounded Seifert-manifold with non-abelian fundamental group that is irreducible and not homeomorphic to $S^1\times B^2$. Note that t is a sum of the trivial tangles $t(0;\, p_1),\ldots,\, t(0;\, p_m)$. By LEMMA 2.1, $t(0;\, 2,\, p)$ with p odd is simple (cf. Fig. 6(b)). Now we apply LEMMA 2.2 to each arrow of the following sequence: $t(0;\, 2,\, p)$ with p odd$\rightarrow t(0;\, 2,\, p,\, p')$ with $p,\, p'$ odd$\rightarrow t(0;\, p,\, p')$ with $p,\, p'$ odd$\rightarrow t(0;\, p,\, p',\, p'')$ with $p,\, p'$ odd and p'' even$\rightarrow t(0;\, p',\, p'')$ with p' odd and p'' even. Since $t(0;\, p_1,\, p_2)\cong t(0;\, p_2,\, p_1)$, it follows that $t(0;\, p_1,\, p_2)$ is always simple. For $m\geq 3$ we further apply LEMMA 2.2 to each arrow of the following sequence: $t(0;\, p_1,\, p_2)\rightarrow t(0;\, p_1,\, p_2,\, p_3)\rightarrow\cdots\rightarrow t(0;\, p_1,\, p_2,\ldots,\, p_m)$. The proof is completed.

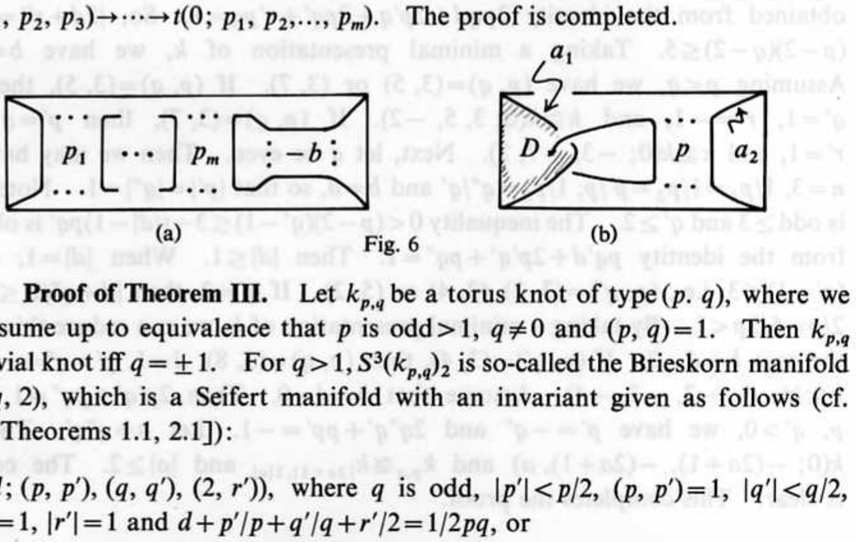

Fig. 6

3. Proof of Theorem III. Let $k_{p,q}$ be a torus knot of type $(p.\, q)$, where we can assume up to equivalence that p is odd>1, $q\neq 0$ and $(p,\, q)=1$. Then $k_{p,q}$ is a trivial knot iff $q=\pm 1$. For $q>1$, $S^3(k_{p,q})_2$ is so-called the Brieskorn manifold $M(p,\, q,\, 2)$, which is a Seifert manifold with an invariant given as follows (cf. [N/R, Theorems 1.1, 2.1]):

(1) $(d;\, (p,\, p'),\, (q,\, q'),\, (2,\, r'))$, where q is odd, $|p'|<p/2$, $(p,\, p')=1$, $|q'|<q/2$, $(q,\, q')=1$, $|r'|=1$ and $d+p'/p+q'/q+r'/2=1/2pq$, or

(2) $(d;\, (p,\, p'),\, (p,\, p'),\, (q',\, q''))$, where q is even, $q=2q'$, $|p'|<p/2$, $(p,\, p')=1$, $|q''|\leq q'/2$, $(q',\, q'')=1$ and $d+2p'/p+q''/q'=1/pq'$.

Note that for $q>1$, $\pi_1(S^3(k_{p,q})_2)$ is abelian iff $q=2$. Since $k_{p,2}\cong k(p;\, -)$, we see from Schubert's classification of 2-bridge knots the following lemma:

LEMMA 3.1. *For* $k=k(-b;\, p_1,\dots, p_n)$ *with* $n\leq 2$ *and* $k_{p,q}$ *with* p *odd*>1 *and* $q\neq 0$, *the following are equivalent*:

(1) $k\cong k_{p,q}$,

(2) $S^3(k)_2\cong S^3(k_{p,q})_2$ *by an orientation-preserving homeomorphim*,

(3) $k\cong k_{p,q}$ *where* $q=\pm 1,\ \pm 2$,

(4) $k\cong k(p^*;\, -)$ *for some odd* p^*.

LEMMA 3.2. *For* $k=k(-b;\, p_1,\dots, p_n)$ *with* $n\geq 3$ *and* $k_{p,q}$ *with* p *odd*>1 *and* $q\neq 0$, *there is an orientation-preserving homeomorphism* $S^3(k)_2\cong S^3(k_{p,q})_2$ *if and only if one of the following cases occurs* $(\varepsilon=\varepsilon(q))$:

(1) $k\cong k(0;\, 3\varepsilon,\, 5\varepsilon,\, -2\varepsilon)$, $k_{p,q}\cong k_{3,5\varepsilon}$,

(2) $k\cong k(0;\, -3\varepsilon,\, -7\varepsilon,\, 2\varepsilon)$, $k\cong k_{3,7\varepsilon}$,

(3) $k\cong k(-\varepsilon;\, -3\varepsilon,\, -3\varepsilon,\, -4\varepsilon)$, $k_{p,q}\cong k_{3,8\varepsilon}$,

(4) $k\cong k(0;\, -(2a+1)\varepsilon,\, -(2a+1)\varepsilon,\, a\varepsilon)$, $k_{p,q}\cong k_{|2a+1|,2|a|\varepsilon}$ *for an integer* a *with* $|a|\geq 2$.

PROOF. It suffices to give the proof for $q>0$ (i.e., $\varepsilon=1$) by reversing, if necessary, the orientation of S^3. Using that $\pi_1(S^3(k)_2)$ is non-abelian, we see that $q\geq 3$. If $S^3(k)_2\cong S^3(k_{p,q})_2$, then there is a fiber-preserving homeomorphism $S^3(k)_2\cong S^3(k_{p,q})_2$ as Seifert manifolds (cf. the proof of LEMMA 1.2). First, let q be odd. Then we may have that $n=3$, $1/p_1=p'/p$, $1/p_2=q'/q$, $1/p_3=r'/2$ and $b=d$, so that $|p'|=|q'|=|r'|=1$. The inequality $|2d+r'|pq\leq 2p+2q+1$ is obtained from the identity $2pqd+2p'q+2pq'+r'pq=1$. So, $|2d+r'|=1$ and $(p-2)(q-2)\leq 5$. Taking a minimal presentation of k, we have $b=d=0$. Assuming $p<q$, we have $(p, q)=(3, 5)$ or $(3, 7)$. If $(p, q)=(3, 5)$, then $p'=q'=1$, $r'=-1$, and $k\cong k(0;\, 3, 5, -2)$. If $(p, q)=(3, 7)$, then $p'=q'=-1$, $r'=1$, and $k\cong k(0;\, -3, -7, 2)$. Next, let q be even. Then we may have that $n=3$, $1/p_1=1/p_2=p'/p$, $1/p_3=q''/q'$ and $b=d$, so that $|p'|=|q''|=1$. Note that p is odd≥ 3 and $q'\geq 2$. The inequality $0<(p-2)(q'-1)\leq 3-(|d|-1)pq'$ is obtained from the identity $pq'd+2p'q'+pq''=1$. Then $|d|\leq 1$. When $|d|=1$, $(p-2)\cdot(q'-1)\leq 3$, i.e., $(p, q')=(3, 2)$, $(3, 4)$ or $(5, 2)$. If $q'=2$, then $|d+q''/2|\leq 1/2p+2/p=5/2p<1$. By taking a minimal presentation of k, we can reduce this case to the case $b=d=0$. If $(p, q')=(3, 4)$, then $(p, q)=(3, 8)$, $d=1$, $p'=q''=-1$, and $k\cong k(-1;\, -3, -3, -4)$. Assume that $b=d=0$. Then $2p'q'+pq''=1$. Since $p,\ q'>0$, we have $p'=-q''$ and $2q''q'+pp'=-1$. Let $a=q''q'$. Then $k\cong k(0;\, -(2a+1), -(2a+1), a)$ and $k_{p,q}\cong k_{|2a+1|,2|a|}$ and $|a|\geq 2$. The converse is clear. This completes the proof.

PROOF of THEOREM III. For $n\leq 2$, it is due to LEMMA 3.1. Let $n\geq 3$. From the Introduction and LEMMA 3.2, it suffices to prove that $k(0;\, -3\varepsilon, -7\varepsilon, 2\varepsilon)\not\cong k_{3,7\varepsilon}$, $k(-\varepsilon;\, -3\varepsilon, -3\varepsilon, -4\varepsilon)\not\cong k_{3,8\varepsilon}$, $k(0;\, -(2a+1)\varepsilon, -(2a+1)\varepsilon, a\varepsilon)\not\cong k_{|2a+1|,2|a|\varepsilon}$ for $a=2$ or $|a|\geq 3$. To do this, we use the following classical lemma

(See Fox [F2, pp. 140–141] for a proof):

LEMMA 3.3 *Let γ be the crossing number of a knot diagram of a non-trivial knot, and δ, the degree of the Alexander polynomial. Then $\gamma > \delta$.*

Note that $\delta(k_{p,q}) = (|p|-1)(|q|-1)$ and $\gamma(k(-b;\, p_1,\ldots,\, p_n)) = |b| + \sum_{i=1}^{n} |p_i|$. Then the above non-equivalences are easily proved. This completes the proof.

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