Rational-slice knots via strongly negative-amphicheiral knots

Akio Kawauchi*

ABSTRACT

We show that certain satellite knots of every strongly negative-amphicheiral rational knot are rational-slice knots. This proof also shows that the 0-surgery manifold of a certain strongly negative amphicheiral knot such as the figure-eight knot bounds a compact oriented smooth 4-manifold homotopy equivalent to the 2-sphere such that a second homology class of the 4-manifold is represented by a smoothly embedded 2-sphere if and only if the modulo two reduction of it is zero.

1. Statement of Result

A knot K in the 3-sphere $\mathbf{S}^3$ is a *slice knot* if K bounds a smooth proper disk D in the 4-disk $\mathbf{B}^4$ bounded by $\mathbf{S}^3$. In this paper, we generalize the concept of a slice knot to a concept on a rational knot, i.e., a knot K in a rational-homology 3-sphere S (= a smooth oriented 3-manifold with the rational-homology of $\mathbf{S}^3$). A *rational* $(4,2)$-*disk pair* is a $(4,2)$-dimensional manifold pair (B,D) such that B is a *rational 4-disk*, namely a compact smooth oriented 4-manifold with the rational-homology of the 4-disk $\mathbf{B}^4$, and D is a smooth proper disk in B. The boundary pair $(S,K)=(\partial B,\partial D)$ is a rational knot, which we call a *weakly rational-slice knot.* We need a more detailed concept of a weakly rational-slice knot. To state it, we note that there is a natural isomorphism

$$H_2(S,S\backslash K)\to H_2(B,B\backslash D)$$

on infinite cyclic groups which can be seen by taking a relative tubular neighborhood of (D,K) in (B,S) and then considering excision isomorphisms. We denote by $\mathrm{b}H_*(\bullet)$ the quotient group of the integral homology group $H_*(\bullet)$ by the torsion subgroup $\mathrm{t}H_*(\bullet)$. Then we see that the natural homomorphism

$$\mathrm{b}H_1(S\backslash K)\to \mathrm{b}H_1(B\backslash D)$$

is a monomorphism on infinite cyclic groups. For an integer $d\geqq 1$, the knot (S,K) is a *d-rational-slice knot* if it bounds a rational $(4,2)$-disk pair (B,D) such

*Dedicating this paper to Professor Jose Maria Montesinos on the occasion of his 65th birthday.

that the cokernel of the natural monomorphism $\mathrm{b}H_1(S\backslash K) \to \mathrm{b}H_1(B\backslash D)$ is isomorphic to $Z_d(= Z/dZ)$. Let $o(K)$ denote the homological order of the element $[K] \in H_1(S)$, where the zero element is understood to have the order 1. A *rational-slice knot* is a 1-rational-slice knot (S, K) with $o(K) = 1$, meaning that the knot (S, K) bounds a rational $(4, 2)$-disk pair (B, D) which induces a meridian-preserving natural isomorphsim $\mathrm{b}H_1(S\backslash K) \to \mathrm{b}H_1(B\backslash D)$ on the infinite cyclic groups with meridian generators. We see that any rational-slice knot (S, K) is an algebraic-slice knot, that is, a knot with a null-cobordant Seifert matrix in the sense of J. Levine [10]. In fact, we can construct a Seifert surface F for K in S since $o(K) = 1$ and hence a compact smooth oriented 3-manifold A in B bounded by the closed surface $F \cup (-D)$ by applying the Pontrjagin-Thom construction to the natural isomorphism $H^1(B\backslash D) \cong H^1(S\backslash K) \cong Z$. The existence of this 3-manifold A means that K is an algebraic-slice knot (cf.[9, Theorem 12.2.3]). Let O be a link with components O_i $(i = 1, 2, \ldots, s)$ in the 3-sphere $\mathbf{S}^3$, We deform the link O into a link $\tilde{O} = \cup_{i=1}^{s} \tilde{O}_i$ in an unknotted solid torus $V \subset \mathbf{S}^3$. There are infinitely many ways of constructing links $\tilde{O} \subset V$ from O. The link $\tilde{O}$ in V is an m-satellite link and denoted by $\tilde{O}(m)$ if m is the greatest common divisor of the integers $m_i \geqq 0$ $(i = 1, 2, \ldots, s)$ such that the cokernel of the natural homomorphism $H_1(\tilde{O}_i) \to H_1(V)$ is isomorphic to Z_{m_i} for every i. Let $V(K)$ be a tubular neighborhood of a knot K in S. An *m-satellite link* of a link O in $\mathbf{S}^3$ along a knot K in S is a link in S which is the image $\tilde{O}(m; K) \subset V(K) \subset S$ of an m-satellite link $\tilde{O}(m) \subset V$ under a (meridian, longitude)-preserving and orientation-preserving homeomorphism (called a *faithful* homeomorphism) $V \to V(K)$. A knot K in S is *strongly negative-amphicheiral* if there is an orientation-reversing involution τ on S such that $\tau(K) = K$ and the fixed point set $\mathrm{Fix}(\tau) = \mathbf{S}^0 \subset K$. In this case, it turns out that there are two types of strongly negative-amphicheiral knots. To state it, let (S_τ, K_τ) be the orbit pair of the pair (S, K) under the action τ, and $\tau^* : H_1(S_\tau\backslash K_\tau) \to Z_2$ the monodromy map of the double covering $S\backslash K \to S_\tau\backslash K_\tau$. We say that K is of *type* I *or* II according to whether the restriction of τ^* to the torsion subgroup $\mathrm{t}H_1(S_\tau\backslash K_\tau)$ is non-trivial or trivial, respectively. If S is a Z_2-homology 3-sphere, then K is always of type II, as we shall show in Corollary 2.4. In Example 2.5, we shall give an example of a strongly negative-amphicheiral knot K with $o(K) = 2$ of type I in a rational-homology 3-sphere S with $H_1(S) = Z_2 \oplus Z_2$. The following theorem is our main theorem.

Theorem 1.1. Let K be a strongly negative-amphicheiral knot with $o(K) = r$ in a rational-homology 3-sphere S. Let O be a slice knot in $\mathbf{S}^3$. If K is of type I, then every mr-satellite knot $K' = \tilde{O}(mr; K)$ for every non-negative integer m is a rational-slice knot in S. If K is of type II, then every $2mr$-satellite knot $K' = \tilde{O}(2mr; K)$ for every non-negative integer m is a rational-slice knot in S.

The following generalization of Theorem 1.1 taking O to be a general knot in $\mathbf{S}^3$ is obtained immediately from Theorem 1.1 since the knot $O\#(-\bar{O})$ in $\mathbf{S}^3$ is a slice knot and we have

$$(\tilde{O}(m; K))\#(-\bar{O}) = (\widetilde{O\#(-\bar{O})})(m; K),$$

for the orientation-reversing mirror image $-\bar{O}$ of the knot O.

Corollary 1.2. Let K be a strongly negative-amphicheiral knot with $o(K) = r$ in a rational-homology 3-sphere S. Let $-\bar{O}$ be the orientation-reversing mirror image of any knot O in $\mathbf{S}^3$. If K is of type I, then the connected sum $K'\#(-\bar{O})$ for every mr-satellite knot $K' = \tilde{O}(mr; K)$ for every non-negative integer m is a rational-slice knot in S. If K is of type II, then the connected sum $K'\#(-\bar{O})$ for every $2mr$-satellite knot $K' = \tilde{O}(2mr; K)$ for every non-negative integer m is a rational-slice knot in S.

For a knot K in $\mathbf{S}^3$, let $K(m)$ be the untwisted double of K for $m = 0$ or the $(m, 1)$-cable knot along K for an integer $m \neq 0$, which is regarded as a $|m|$-satellite knot $\tilde{O}(|m|; K)$ of a trivial knot O in $\mathbf{S}^3$ along the knot K in $\mathbf{S}^3$. Hence the following corollary is direct from Theorem 1.1.

Corollary 1.3. Let K be a strongly negative-amphicheiral knot in $\mathbf{S}^3$. Then the knot $K(2m)$ in $\mathbf{S}^3$ is a rational-slice knot for every integer m.

Concerning this corollary, the author showed in 1980 that the knot $K(2m)$ with K the figure-eight knot, a famous strongly negative-amphicheiral knot is a rational-slice knot by a slightly different method in an unpublished handwritten manuscript [7], although by a result of K. Miyazaki [11] we see that $K(2m)$ is not any ribbon knot for every $m > 0$. It appears an unsettled problem to determine whether or not $K(2m)$ is a slice knot for any m (see Cha[1], Cha-Livingston-Ruberman [2]). As a final remark of the first section, we observe that a link version of our main theorem (Theorem 1.1) is directly obtained. A link L of the components K_i $(i = 1, 2, \ldots, s)$ in S is a *strongly rational-slice link* in S if the knots K_i $(i = 1, 2, \ldots, s)$ have $o(K_i) = 1$ and bound mutually disjoint smooth proper disks D_i $(i = 1, 2, \ldots, s)$ in a rational 4-disk B with $\partial B = S$ such that there is a meridian-preserving natural isomorphsim $\mathrm{b}H_1(S\backslash L) \to \mathrm{b}H_1(B\backslash \cup_{i=1}^{s} D_i)$ on the free abelian groups with meridian bases. In the case that $S = \mathbf{S}^3$ and $B = \mathbf{B}^4$, a strongly rational-slice link is nothing but a usual *strongly slice link* ([9]). If the components K_i $(i = 1, 2, \ldots, s)$ of a link L in S are rational-slice knots by mutually disjoint smooth proper disks D_i $(i = 1, 2, \ldots, s)$ in a rational 4-disk B with $\partial B = S$ and $\partial D_i = K_i$ $(i = 1, 2, \ldots, s)$, then the link L in S is a strongly rational-slice link in S. In fact, since there is a meridian-preserving isomorphism $\mathrm{b}H_1(S\backslash L) \to \oplus_{i=1}^{s}\mathrm{b}H_1(S\backslash K_i)$ on the free abelian groups with the meridian bases, we obtain a meridian-preserving isomorphism $\mathrm{b}H_1(S\backslash L) \to \mathrm{b}H_1(B\backslash \cup_{i=1}^{s} D_i)$ by composing the isomorphism $\oplus_{i=1}^{s}\mathrm{b}H_1(S\backslash K_i) \to \oplus_{i=1}^{s}\mathrm{b}H_1(B\backslash D_i)$ given obtained by the assumption of a rational-slice knot. Using this remark, we obtain the following corollary as a link version of Theorem 1.1.

Corollary 1.4. Let K be a strongly negative-amphicheiral knot with $o(K) = r$ in a rational-homology 3-sphere S. Let O be a strongly slice link in $\mathbf{S}^3$, and m a non-negative integer. If K is of type I, then every mr-satellite link $L = \tilde{O}(mr; K)$ is a strongly rational-slice link in S. If K is of type II, then every $2mr$-satellite link $L = \tilde{O}(2mr; K)$ is a strongly rational-slice link in S.

The proof is given after the proof of Theorem 1.1. For example, although the Bing double $BD_1(K)$ of the figure-eight knot K in $\mathbf{S}^3$ is NOT a strongly slice link by [1] and [2], we can see from Corollary 1.4 that it is a strongly rational-slice link. In §2, we show some properties of rational-slice knots and strongly amphicheiral knots. In §3, we study a composition of rational-homology cobordisms between rational-homology handles. In §4, the proofs of Main Theorem (Theorem 1.1) and Corollary 1.4 are given. In §5, we apply our result on a classical strongly negative-amphicheiral knot to the existence of a certain compact smooth 4-manifold.

The author would like to thank the referee for making corrections of errors and suggestions in an earlier version of this paper.

2. Some properties of rational-slice knots and strongly negative-amphicheiral knots

The *slope* $s(K)$ of a knot K in a rational-homology 3-sphere S is defined by the identity

$$s(K) = -\lambda_S([K],[K]) \in Q/Z$$

for the linking pairing $\lambda_S : H_1(S) \times H_1(S) \to Q/Z$ and the homology class $[K] \in H_1(S)$. A knot K in S with $s(K) = 0$ is called a *flat* knot in S (see [8]). We have the following lemma:

Lemma 2.1. If K is a weakly rational-slice knot or a strongly negative-amphicheiral knot in a rational-homology 3-sphere S, then K is flat in S.

Proof. Let (S,K) be a weakly rational-slice knot bounding a pair (B,D) such that D is a smooth proper disk in a rational 4-disk B. We take a rational 2-cycle $\hat{D}_Q = D - c_Q$ in B by taking a rational 2-chain c_Q in S with $\partial c_Q = K$. We take slight translations K', $\hat{D}'_Q$, D' and c'_Q of K, $\hat{D}_Q$, D and c_Q respectively such that

(1) the rational 2-cycle $\hat{D}'_Q = D' - c'_Q$ with $\partial D' = \partial c'_Q = K'$ intersects $\hat{D}_Q$ transversely,
(2) the rational 2-chain c'_Q and the knot K' are in a slight translation S' of S into the interior of B,
(3) the knot $K \subset S$ is identified with $K^* = D \cap S' \subset S'$ and $K^* \cap K' = \emptyset$.

Then we have the rational intersection number

$$\begin{aligned}\mathrm{Int}_B(\hat{D}_Q, \hat{D}'_Q) &= \mathrm{Int}_B(D - c_Q, D' - c'_Q) \\ &= \mathrm{Int}_B(D, D') - \mathrm{Int}_B(D, c'_Q) - \mathrm{Int}_B(c_Q, D') + \mathrm{Int}_B(c_Q, c'_Q) \\ &= \mathrm{Int}_B(D, D') - \mathrm{Int}_B(D, c'_Q).\end{aligned}$$

Since $H_2(B;Q) = 0$, we have $\mathrm{Int}_B(\hat{D}_Q, \hat{D}'_Q) = 0$ and the rational linking number

$$\mathrm{Link}_{S'}(K^*, K') = \mathrm{Int}_B(D, c'_Q) = \mathrm{Int}_B(D, D'),$$

so that

$$s(K) = -\mathrm{Link}_{S'}(K^*, K') \pmod 1 \equiv -\mathrm{Int}_B(D, D') \equiv 0 \pmod 1.$$

For a strongly negative-amphicheiral knot (S,K), let τ be an orientation-reversing involution on S with $\mathrm{Fix}(\tau) = \mathbf{S}^0 \subset K$. Then we have a simple loop ℓ on a τ-invariant tubular neighborhood $V(K)$ such that $\tau(\ell) \cap \ell = \emptyset$ and ℓ is isotopic to K in $V(K)$. Applying τ to the rational linking number $\mathrm{Link}_S(\ell, \tau(\ell))$, we have $\mathrm{Link}_S(\ell, \tau(\ell)) = -\mathrm{Link}_S(\tau(\ell), \ell)$ because τ reverses the orientation of S, and hence $\mathrm{Link}_S(\ell, \tau(\ell)) = 0$, showing that $s(K) = 0$. □

Let $E = \mathrm{cl}(S \backslash V(K))$ be the exterior of K in S. It is shown in [8] that if $s(K) = 0$ and $o(K) = r$, then there is a compact connected oriented proper surface F in E such that the boundary ∂F consists of r parallels of a longitude of $V(K)$ which are unique up to isotopies of E, so that we can specify a unique meridian-longitude system for every flat knot K in S. In our argument, the 0-surgery manifold M of a flat knot K in S which we can consider by a unique meridian-longitude system plays an important role. This manifold M is constructed as follows: Let $X = S \times [-1, 1] \cup \mathbf{B}^2 \times \mathbf{B}^2$ be a 4-manifold obtained by attaching the solid torus $(\partial \mathbf{B}^2) \times \mathbf{B}^2$ to $V(K) \times 1 \subset S \times 1$ with the 0-framing. Then the boundary ∂X consists of $S \times (-1)$ (regarded as $-S$) and M. A *rational-homology handle* is a closed oriented 3-manifold with the rational-homology of $\mathbf{S}^1 \times \mathbf{S}^2$. The following lemma shows that the 0-surgery manifold of a flat knot in a rational-homology 3-sphere is a rational-homology handle.

Lemma 2.2. Let E and M be the exterior and the 0-surgery manifold of a flat knot K in S with $o(K) = r$, respectively. Then we have the following natural short exact sequences

$$0 \to Z \to H_1(E) \to H_1(S) \to 0 \quad \text{and} \quad 0 \to Z_r \to H_1(E) \to H_1(M) \to 0,$$

where Z and Z_r are generated by the meridian and the longitude of K, respectively.

Proof. We note that $H_i(S, E) \cong H_i(M, E) \cong H_i(\mathbf{S}^1 \times \mathbf{B}^2, \mathbf{S}^1 \times \mathbf{S}^1)$ is isomorphic to 0 for $i = 1$ and Z for $i = 2$. Since $H_2(S) = 0$, we obtain the first short exact sequence from the homology sequence for the pair (S, E). We also obtain the exact sequence $Z \to H_1(E) \to H_1(M) \to 0$ from the homology sequence for the pair (M, E), where the map $Z \to H_1(E)$ sends a generator of Z to the longitude of K in E with homological order r in $H_1(E)$ since $o(K) = r$, which induces the exact sequence $0 \to Z_r \to H_1(E)$. □

We say that a rational-homology handle M is the boundary of a *rational-homology circle* Y *of degree* $r(\geqq 1)$ if Y is a compact oriented smooth 4-manifold with $\partial Y = M$ such that the pair (Y, M) has the rational-homology of $(\mathbf{S}^1 \times \mathbf{B}^3, \mathbf{S}^1 \times \mathbf{S}^2)$ such that the cokernel of the natural monomorphism $\mathrm{b}H_1(M) \to \mathrm{b}H_1(Y)$ is isomorphic to Z_r. The following lemma is fundamental to our construction.

Lemma 2.3. Let K be a strongly negative-amphicheiral knot in a rational-homology 3-sphere S. Then, according to whether K is of type I or II, the 0-surgery manifold M of K bounds a rational-homology circle Y of degree 1 or 2 such that $H_1(Y, M) \cong Z_2$, respectively.

Proof. Since K is a strongly negative-amphicheiral knot, there is an involution τ on S such that $\tau(K) = K$ and $\mathrm{Fix}(\tau) = \mathbf{S}^0 \subset K$. The involution τ induces a free involution τ_M on M. Let M_τ be the orbit manifold of M under the action of τ_M. Let $p : M \to M_\tau$ be the double covering projection. Since K is a flat knot in S by Lemma 2.1, we see from Lemma 2.2 that M is a rational-homology handle. Let

$V' = \mathbf{B}^2 \times (\partial \mathbf{B}^2) \subset M$ be the dual solid torus of $V(K)$, and V'_τ the orbit solid Klein bottle of V' under the action of τ_M. We note that the natural sequence

$$0 \to H_1(V') \to H_1(M) \to H_1(M, V') \to 0$$

is a short exact sequence by Lemma 2.2 and $H_1(M, V')$ is a torsion group. Since every element of $H_1(M_\tau, V'_\tau)$ is generated by simple proper arcs in $\mathrm{cl}(M_\tau \backslash V'_\tau)$, and the preimage $p^{-1}(a)$ of a simple proper arc a consists of two simple proper arcs a', a'' in $\mathrm{cl}(M \backslash V')$, we see that the covering homomorphism $p_* : H_1(M, V') \to H_1(M_\tau, V'_\tau)$ is onto (in fact, we have $p_*([a']) = [a]$). This means that the natural sequence

$$0 \to H_1(V'_\tau) \to H_1(M_\tau) \to H_1(M_\tau, V'_\tau) \to 0$$

is also a short exact sequence and $H_1(M_\tau, V'_\tau)$ is a torsion group, where the injectivity of the map $H_1(V'_\tau) \to H_1(M_\tau)$ follows from the fact that $H_1(M_\tau)$ must be an infinite group because M_τ is a closed *non-orientable* 3-manifold. Thus, $H_i(M_\tau; Q) \cong Q$ for $i = 0, 1$ and $H_3(M_\tau; Q) = 0$. Since the Euler characteristic $\chi(M_\tau) = 0$, we have that $H_*(M_\tau; Q) = H_*(\mathbf{S}^1; Q)$. The double covering $p : M \to M_\tau$ induces an exact sequence

$$H_1(M) \xrightarrow{p_*} H_1(M_\tau) \longrightarrow Z_2 \to 0.$$

Let Y be the twisted line-bundle of M_τ, in other words, the mapping cylinder of the double covering $p : M \to M_\tau$. We note that the boundary ∂Y of Y is the manifold M. Because M_τ is a strong deformation retract of Y, we have the following short exact sequence

$$0 \to H_1(M) \xrightarrow{i_*} H_1(Y) \longrightarrow Z_2 \to 0$$

for the inclusion $i : M \subset Y$. In particular, we have $H_1(Y, M) \cong Z_2$. Using that M_τ has the rational-homology of $\mathbf{S}^1$, we see that M is the boundary of a rational-homology circle Y. To determine the degree of Y, we consider the following exact sequence

$$H_1(E) \xrightarrow{(p|_E)_*} H_1(E_\tau) \xrightarrow{\tau^*} Z_2 \to 0,$$

where we note that the homomorphism $\tau^* : H_1(E_\tau) \to Z_2$ is identified with the map $\tau^* : H_1(S_\tau \backslash K_\tau) \to Z_2$. By Lemma 2.2, we note that $\mathrm{b}H_1(E) \cong Z$ and there is a natural isomorphism $\mathrm{b}H_1(E) \to \mathrm{b}H_1(M)$. The natural map $H_1(E_\tau) \to H_1(M_\tau)$ is onto because $H_1(M_\tau, E_\tau) = H_1(V_\tau, \partial V_\tau) = 0$, which implies that we have a natural isomorphism $\mathrm{b}H_1(E_\tau) \to \mathrm{b}H_1(M_\tau)$. Let K be of type I. Then $(p|_E)_*$ induces an isomorphism $\mathrm{b}H_1(E) \cong \mathrm{b}H_1(E_\tau)$, which induces an isomorphism $p_* : \mathrm{b}H_1(M) \cong \mathrm{b}H_1(M_\tau)$. This implies that Y is of degree 1. Let K be of type II. Then we have the following short exact sequence

$$0 \to \mathrm{b}H_1(E) \xrightarrow{(p|_E)_*} \mathrm{b}H_1(E_\tau) \to Z_2 \to 0,$$

which implies a short exact sequence

$$0 \to \mathrm{b}H_1(M) \xrightarrow{p_*} \mathrm{b}H_1(M_\tau) \to Z_2 \to 0.$$

This implies that Y is of degree 2. □

The following corollary which is direct from Lemma 2.3 is promised in the introduction.

Corollary 2.4. Let K be a strongly negative-amphicheiral knot in a Z_2-homology 3-sphere S. Then K is of type II.

Proof. In the proof of Lemma 2.3, we showed that the $p_* : H_1(M, V') \to H_1(M_\tau, V'_\tau)$ is onto. By the excision isomorphism, this implies that $(p|_E)_* : H_1(E, \partial E) \to H_1(E_\tau, \partial E_\tau)$ is onto. Since $H_1(E, \partial E; Z_2) = 0$, we have $H_1(E_\tau, \partial E_\tau; Z_2) = 0$. The image of the natural homomorphism $H_1(\partial E_\tau; Z_2) \to H_1(E_\tau; Z_2)$ is Z_2 by the Z_2-Poincaré duality, and thus we have $H_1(E_\tau; Z_2) \cong Z_2$. In the proof of Lemma 2.3, we have $\mathrm{b}H_1(M_\tau) \cong Z$, so that $\mathrm{b}H_1(E_\tau) \cong Z$ and $\mathrm{t}H_1(E_\tau)$ is an odd-torsion group. Hence the restriction of τ^* to the torsion subgroup $\mathrm{t}H_1(E_\tau) = \mathrm{t}H_1(S_\tau \backslash K_\tau)$ is a trivial homomorphism. □

Here is an example of a strongly negative-amphicheiral knot in a rational-homology 3-sphere of type I.

Example 2.5. For the projective plane $\mathbf{P}^2$, let $p : M = \mathbf{S}^1 \times \mathbf{S}^2 \to M_\tau = \mathbf{S}^1 \times \mathbf{P}^2$ be the double covering, which induces an isomorphism $p_* : H_1(M) \to \mathrm{b}H_1(M_\tau)$, showing that the twisted line bundle Y of M_τ is a rational-homology circle of degree 1 bounded by M. We look for a knot K'_τ in M_τ representing a generator of $\mathrm{b}H_1(M_\tau)$ such that K' admits a solid Klein bottle tubular neighborhood in M_τ. Then the preimage $K' = p^{-1}(K'_\tau)$ is a τ_M-invariant knot in M representing the 2 times of a generator of $H_1(M) \cong Z$. We replace a τ_M-invariant tubular neighborhood $V(K') = \mathbf{S}^1 \times \mathbf{B}^2$ in M with a solid torus $\mathbf{B}^2 \times \partial \mathbf{B}^2$ to obtain a rational-homology 3-sphere S with $H_1(S) \cong Z_2 \oplus Z_2$, where we note that any meridian of $V(K')$ represents an order 2 element of $H_1(\mathrm{cl}(M \backslash V(K')))$. The involution τ_M on M induces an orientation-reversing involution τ on S which makes the knot $K = 0 \times \partial \mathbf{B}^2$ invariant with $\mathrm{Fix}(\tau) = \mathbf{S}^0 \subset K$. Thus, K is a strongly negative-amphicheiral knot in S with $o(K) = 2$. Since Y is a rational-homology circle of degree 1 bounded by M, the knot K must be of type I by the proof of Lemma 2.3..

3. Composing rational-homology cobordisms between rational-homology handles

Two rational-homology handles M and M' are *rational-homology cobordant* of *degree* (r, r') for positive integers $r, r' > 0$ if there is a compact oriented 4-manifold C with boundary $\partial C = (-M) \cup M'$ such that the inclusions $M \subset C$ and $M' \subset C$ induce rational isomorphisms $H_*(M; Q) \cong H_*(C; Q)$ and $H_*(M'; Q) \cong H_*(C; Q)$, respectively and monomorphisms $\mathrm{b}H_1(M) \to \mathrm{b}H_1(C)$ and $\mathrm{b}H_1(M') \to \mathrm{b}H_1(C)$ with the cokernels isomorphic to Z_r and $Z_{r'}$, respectively. The triad $(C; M, M')$ is called a *rational-homology cobordism*. The following lemma shows how the indices of rational-homology cobordisms change by a composition of rational-homology cobordisms between rational-homology handles.

Lemma 3.1. Let $(C; M, M')$ and $(C'; M'.M'')$ be rational-homology cobordisms between rational-homology handles of indices (r, a) and (b, r''), respectively. Let d be the greatest common divisor of a and b, and $a = d\tilde{a}$, $b = d\tilde{b}$. Then the composite rational-homology cobordism $(C \cup C'; M, M'')$ sticked along M' is of degree $(r\tilde{b}, r''\tilde{a})$.

Proof. The Mayer-Vietoris sequence

$$H_1(M') \to H_1(C) \oplus H_1(C') \to H_1(C \cup C') \to 0$$

induces a short exact sequence

$$0 \to \mathrm{b}H_1(M') \to \mathrm{b}H_1(C) \oplus \mathrm{b}H_1(C') \to \hat{H}_1(C \cup C') \to 0,$$

where $\hat{H}_1(C \cup C')$ denoted the quotient group of $H_1(C \cup C')$ by a torsion subgroup. Let $e_{M'}$, e_C and $e_{C'}$ be generators of the infinite cyclic groups $\mathrm{b}H_1(M')$, $\mathrm{b}H_1(C)$ and $\mathrm{b}H_1(C')$, respectively such that the map $\mathrm{b}H_1(M') \to \mathrm{b}H_1(C) \oplus \mathrm{b}H_1(C')$ sends $e_{M'}$ to $ae_C - be_{C'} = d(\tilde{a}e_C - \tilde{b}e_{C'})$. Let $j : \mathrm{b}H_1(C) \oplus \mathrm{b}H_1(C') \to \mathrm{b}H_1(C \cup C') \cong Z$ be the natural epimorphism. Then we have $dj(\tilde{a}e_C - \tilde{b}e_{C'}) = 0$ and hence $j(\tilde{a}e_C) = j(\tilde{b}e_{C'})$. Let a^* and b^* be integers such that $\tilde{a}a^* + \tilde{b}b^* = 1$. Then we show that $j(b^*e_C + a^*e_{C'})$ is a generator of $\mathrm{b}H_1(C \cup C')$. In fact,

$$j(e_C) = j(a^*\tilde{a}e_C) + j(b^*\tilde{b}e_C) = j(a^*\tilde{b}e_{C'}) + j(b^*\tilde{b}e_C) = \tilde{b}j(b^*e_C + a^*e_{C'}),$$

and similarly, $j(e_{C'}) = \tilde{a}j(b^*e_C + a^*e_{C'})$. Since the natural monomorphisms $\mathrm{b}H_1(M) \to \mathrm{b}H_1(C \cup C')$ and $\mathrm{b}H_1(M'') \to \mathrm{b}H_1(C \cup C')$ send some generators e_M and $e_{M''}$ to $rj(e_C)$ and $r''j(e_{C'})$, respectively, we see that the degree of the rational-homology cobordism $(C \cup C'; M, M'')$ is $(r\tilde{b}, r''\tilde{a})$. □

In Lemma 3.1, let $M'' = \mathbf{S}^1 \times \mathbf{S}^2$. Then we can obtain a rational-homology circle Y' bounding M' by sticking $\mathbf{S}^1 \times \mathbf{B}^3$ and C' along M''. The following corollary is direct from Lemma 3.1.

Corollary 3.2. Let $(C; M, M')$ be a rational-homology cobordism between rational-homology handles M and M' of degree (r, a), and M' the boundary of a rational-homology circle Y' of degree b. Let d be the greatest common divisor of a and b, and $b = d\tilde{b}$. Then the rational-homology handle M is the boundary of the composite rational-homology circle $Y = C \cup Y'$ (sticked along M') of degree $r\tilde{b}$.

4. Proof of Main Theorem

Throughout the whole section, the proofs of the main theorem(Theorem 1.1) and Corollary 1.4 will be made. Let $X = S \times [0, 1] \cup \mathbf{B}^2 \times \mathbf{B}^2$ be the surgery trace from $S = S \times 0$ to the 0-surgery manifold M of (S, K) done by using the fact that it is a flat knot by Lemma 2.1. Let $D = K \times [0, 1] \cup \mathbf{B}^2 \times 0$ be a proper disk in X. For the rational-homology circle Y constructed in Lemma 2.3, we construct a 4-manifold $B = X \cup Y$ sticked along M. Then $\partial B = S$ and B is a rational 4-disk. By Lemma 2.3, according to whether K is of type I or II, the knot (S, K) is a 1-rational-slice or 2-rational-slice knot with $o(K) = r$ by the rational (4,2)-disk pair (B, D), respectively.

Let $N(D) = D \times \mathbf{B}^2 (\cong \mathbf{B}^4)$ be a tubular neighborhood of D in X. According to whether K is of type I or II, let $K' = \tilde{O}(mr; K)$ or $\tilde{O}(2mr; K)$ be any mr-satellite knot or any $2mr$-satellite knot of O along K in S, respectively. Then $o(K') = 1$. Since K' is equivalent to the slice knot O in $\mathbf{S}^3 = \partial N(D)$ by definition, the knot K' bounds a smooth proper disk D' in $N(D)$. We shall show that the knot (S, K') is a rational-slice knot (meaning a 1-rational-slice knot with $o(K') = 1$) by the rational (4,2)-disk pair (B, D'). Let $m = 0$. In this case, the knot K' bounds a Seifert surface F' in $V(K)$ and the union $-F' \cup D'$ bounds an oriented 3-manifold in $N(D)$. Then $\mathrm{b}H_1(S \backslash K') \cong Z$ and $\mathrm{b}H_1(B \backslash D') \cong Z$ are generated by meridians and we have a natural isomorphism $\mathrm{b}H_1(S \backslash K') \to \mathrm{b}H_1(B \backslash D')$. Thus, (B, D') is a desired smooth rational disk-pair bounding (S, K'). Let $m \neq 0$. Let $C = \mathrm{cl}(X \backslash N(D'))$. Since the knot (S, K') is a flat knot by Lemma 2.1, the manifold $M' = \partial C \backslash M$ is the 0-surgery manifold of (S, K'). We prove the following lemma later.

Lemma 4.1. According to whether K is of type I or II, the triad $(C; M', M)$ is a rational-homology cobordism of degree $(1, m)$ or $(1, 2m)$, respectively.

Since M is the boundary of a rational-homology circle Y of degree 1 or 2 respectively according to whether K is of type I or II, we see from Corollary 3.2 and Lemma 4.1 that M' bounds a rational-homology circle $C \cup Y$ of degree 1, meaning that (B, D') is a desired smooth rational disk-pair bounding (S, K'). This completes the proof of Theorem 1.1 assuming the proof of Lemma 4.1.

Proof of Lemma 4.1. Let $n = m$ or $2m$ according to whether K is of type I or II. The boundary ∂C consists of M and the 0-surgery manifold M' of K' in S. By excision, we have $H_2(X, S) \cong Z$ with $[D]$ a generator and $H_q(X, S) = 0$ for $q \neq 2$. Similarly, we have $H_2(D' \cup S, S) \cong Z$ with $[D']$ a generator and $H_q(D' \cup S, S) = 0$ for $q \neq 2$. Hence we have a natural exact sequence

$$0 \to H_2(D' \cup S, S) \to H_2(X, S) \to H_2(X, D' \cup S) \to 0.$$

The homology class $[D'] \in H_2(X, S)$ is equal to $nr[D]$ because K' is an nr-satellite knot of O along K, and hence by Poincaré duality and excision we have the following homology group:

$$H_{3-q}(C, M) \cong H_q(C, M') \cong H_q(X, D' \cup S) = \begin{cases} Z_{nr} & (q = 2) \\ 0 & (\text{otherwise}), \end{cases}$$

showing that the triad $(C; M', M)$ is a rational-homology cobordism. By Lemma 2.2,

$$\mathrm{b}H_1(M') \cong \mathrm{b}H_1(C) \cong \mathrm{b}H_1(M) \cong Z.$$

Since $H_1(C, M') = 0$, the natural homomorphism $H_1(M') \to H_1(C)$ is onto so that the induced homomorphism $\mathrm{b}H_1(M') \to \mathrm{b}H_1(C)$ is an isomorphism on infinite cyclic groups. Since $o(K') = 1$, the knot K' bounds a Seifert surface F' in S. We note that any loop in the exterior $E' = \mathrm{cl}(S \backslash N(K'))$ intersecting F' with the intersection number 1 represents a generator g' of $\mathrm{b}H_1(M') \cong \mathrm{b}H_1(C) \cong Z$ by non-singularity of the intersection pairing $\mathrm{Int} : \mathrm{b}H_1(M') \times \mathrm{b}H_2(M') \to Z$ since the surface $F' \cap E'$

with $F' \cap \partial V(K')$ a longitude of K' extends to a closed connected oriented surface in M'. Similarly, since K is flat and $o(K) = r$, we see that r-parallel copies of the longitude of K on $\partial V(K)$ bound a connected oriented proper surface F in the exterior $E = \mathrm{cl}(S \backslash N(K))$ which represents a generator of $H_2(E, \partial E) \cong Z$ and extends to a closed connected oriented surface in M (cf.[8]). Let ℓ be a loop in E intersecting F with the intersection number 1, which represents a generator g of $\mathrm{b}H_1(M) \cong Z$. Taking $V(K')$ in the interior of $V(K)$, we shall show that there is a Seifert surface F'' of K' in S constructed from n-paralell copies of the surface $F \subset E$ by adding a compact surface in $V(K)$. To see this, by the definition of an nr-satellite knot, we note that the meridian of $V(K)$ meets any Seifert surface F' of K' with the intersection number nr, which is the intersection number of the meridian of $V(K)$ and the closed 1-manifold $F' \cap \partial V(K)$. We modify F' so that the closed 1-manifold $F' \cap \partial V(K)$ consists of parallel simple loops with the same orientation in $\partial V(K)$. Then the closed 1-manifold $F' \cap \partial V(K)$ is isotopic to nr-parallel copies of the longitude of K on $\partial V(K)$ which is the boundary of n-parallel copies of $F \subset E$ by uniqueness of a characteristic surface for E in [8]. Then a desired Seifert surface F'' of K' is obtained from F' by replacing the surface $F' \cap E$ with n-parallel copies of $F \subset E$ after an isotopic deformation of $F' \cap V(K)$ in $V(K)$ keeping K' fixed. Since the intersection numer $\mathrm{Int}_{E'}(\ell, F'') = n$, the homomorphism $\mathrm{b}H_1(M) \to \mathrm{b}H_1(C)$ sends a generator g of $\mathrm{b}H_1(M)$ to the element ng' for a generator g' of $\mathrm{b}H_1(C) \cong \mathrm{b}H_1(M')$. This shows that $(C; M', M)$ is a rational-homology cobordism of degree $(1, n)$. □

This completes the proof of Theorem 1.1. □

The proof of Corollary 1.4 is given here.

Proof of Crollary 1.4. In the proof of Theorem 1.1, let $L = \cup_{i=1}^{s} L_i = \tilde{O}(mr; K)$ or $\tilde{O}(2mr; K)$ be any mr-satellite link or any $2mr$-satellite link of an s-component strongly slice link $O = \cup_{i=1}^{s} O_i$ along K in S according to whether K is of type I or II. Then $o(L_i) = 1$ for every i. Since L is nothing but the strongly slice link O in $\mathbf{S}^3 = \partial N(D)$, the knot components L_i bounds mutually disjoint smooth proper disks D_i $(i = 1, 2, \ldots, s)$ in $N(D)$. By the proof of Theorem 1.1, the existence of (B, D_i) means a rational-slice knot (S, L_i) for every i, and hence the remark following Theorem 1.1 shows that L is a strongly rational-slice link in S. □

5. Applying our result to the existence of a certain compact smooth 4-manifold

For a $(2k-1)$-knot K in $\mathbf{S}^{2k+1}$ with $k \geqq 1$, a compact smooth $(2k+2)$-manifold W homotopy equivalent to $\mathbf{S}^{2k}$ is constructed as the union $\mathbf{B}^{2k+2} \cup \mathbf{B}^{2k} \times \mathbf{B}^2$ attaching the submanifold $(\partial \mathbf{B}^{2k}) \times \mathbf{B}^2$ to a tubular neighborhood $N(K)(= K \times \mathbf{B}^2)$ of K in $\mathbf{S}^{2k+1} = \partial \mathbf{B}^{2k+2}$, where we take the 0-framing on $N(K)$ for $k = 1$. The boundary $M = \partial W$ has the same homology as $S^{2k} \times S^1$. It is well-known that if K is a slice knot, then every homology class of $H_{2k}(W)(\cong Z)$ is represented by a $2k$-sphere smoothly embedded in W. On the other hand, for some non-slice knots K, every non-zero homology class of $H_{2k}(W)$ cannot be represented by any $2k$-sphere smoothly embedded in W. In [6], we showed not only this result for every $k \geqq 1$, but also gave, for every $k \geqq 2$, a compact smooth $(2k+2)$-manifold W homotopy equivalent to $\mathbf{S}^{2k}$ such that a homology class $w \in H_{2k}(W)$ is represented by a $2k$-sphere smoothly embedded in W if and only if the modulo two reduction $w_2 \in H_{2k}(W; Z_2)$ of w is 0. Using our construction in Lemma 2.3, we have a result, filling up the absence of $k = 1$. To describe it, we say that a knot polynomial $A(t)$ is *of m-slice type* (for an integer m) if $A(t^m) = \pm t^i F(t) F(t^{-1})$ for an integer i and an integral polynomial $F(t)$. The polynomial $A(t)$ of every strongly negative-amphicheiral knot in $\mathbf{S}^3$ is of $2m$-slice type for every integer m, because it is shown in [4] that $A(t)$ has the identity $A(t^2) = \pm t^i F(t) F(t^{-1})$ for an integer i and an integral polynomial $F(t)$ with $F(t^{-1}) = \pm t^j F(-t)$ for an integer j. Incidentally, we mention that this identity holds for every negative-amphicheiral knot in $\mathbf{S}^3$, conjectured by the author in [5] and proved by R. Hartley in [3](cf.[9]) (although we do not use this fact). If a knot polynomial $A(t)$ is of $(2m+1)$-slice type for an integer m, then $|A(-1)|$ is a square. Thus, the polynomial $A(t) = t^2 - 3t + 1$ of the figure-eight knot which is a strongly negative-amphicheiral knot is not of $(2m+1)$-slice type for any integer m. More generally, it is suggested by the referee that *if the polynomial $A(t)$ of a strongly negative-amphicheiral knot in $\mathbf{S}^3$ is of degree 2, then $A(t)$ is not of $(2m+1)$-slice type for any integer m.* In fact, in this case, $A(t)$ has the form $A(t) = \pm(a^2t^2 - (2a^2+1)t + a^2)$ for a non-zero integer a, so that $|A(-1)| = 4a^2 + 1$ is not a square. Using this notion, we have the following theorem:

Theorem 5.1. Let K be a strongly negative-amphicheiral knot in $\mathbf{S}^3$ whose polynomial $A(t)$ is not of $(2m+1)$-slice type for any interger m. Then the 0-surgery manifold M of K bounds a compact smooth 4-manifold W homotopy equivalent to $\mathbf{S}^2$ such that a homology class $w \in H_2(W)$ is represented by a 2-sphere smoothly embedded in W if and only if the modulo two reduction $w_2 \in H_2(W; Z_2)$ of w is 0.

Proof. Let Y be the 4-manifold in Lemma 2.3 with $\partial Y = M$ and $H_1(Y, M) \cong Z_2$. Since K is of type II and $H_1(M) \cong Z$, we have $H_1(Y) \cong Z$. Let ℓ be a loop in Y representing a generator of $H_1(Y)$. Let $N(\ell) \cong \mathbf{S}^1 \times \mathbf{B}^3$ be a regular neighborhood of ℓ in Y. We do a surgery on Y replacing $N(\ell)$ with $\mathbf{B}^2 \times \mathbf{S}^2$ to obtain a 4-manifold W with $\partial W = M$ and $H_1(W) = 0$. Since the Euler characteristic $\chi(W) = 2$, we see that $H_*(W) \cong H_*(\mathbf{S}^2)$. By a careful choice of ℓ, we show that W is simply connected, which is sufficient to see from J. H. C. Whitehead's theorem that W is homotopy equivalent to $\mathbf{S}^2$. Since the orbit manifold M_τ of M under the action of a free involution τ_M on M induced from τ is a strong deformation retract of Y, it is sufficient to specify a loop ℓ in the manifold M_τ which is a union of E_τ and a solid Klein bottle with the boundaries pasted. Let ℓ be a loop $\mathbf{P}^1$ in the boundary $\mathbf{P}^2$ of a neighborhood of the image of a fixed point of $\mathrm{Fix}(\tau)$ in $\mathbf{S}^3_\tau$, which is regarded as a loop in the Klein bottle ∂E_τ. We note that $H_1(E_\tau) \cong Z$ and ℓ represents a generator. The fundamental group $\pi_1(W, x)$ is isomorphic to the group $\pi = \pi_1(M_\tau, x)/\langle [\ell] = 1 \rangle$ which obtained from the fundamental group $\pi_1(M_\tau, x)$ by adding the relation $[\ell] = 1$. Since the element $[\ell]^2$ is represented by the image of a meridian of E in the group $\pi_1(E_\tau, x)$ and the group $\pi_1(E, y)/\langle [m] = 1 \rangle$ for a meridian m is the trivial group, we see that $\pi_1(E_\tau, x)/\langle [\ell]^2 = 1 \rangle$ is isomorphic to Z_2, so that $\pi = \{1\}$. This implies that

W is homotopy equivalent to $\mathbf{S}^2$. By the excision isomorphism, we have

$$H_2(W, \mathbf{B}^2 \times \mathbf{S}^2) \cong H_2(Y, N(\ell)) \cong H_2(Y, \ell) \cong H_2(Y) \cong Z_2,$$

which shows that $\pm 2e \in H_2(W)$ for a generator $e \in H_2(W)$ is represented by the 2-sphere $0 \times \mathbf{S}^2$, which is embedded smoothly in W. By tubing some parallels of $0 \times \mathbf{S}^2$ in $\mathbf{B}^2 \times \mathbf{S}^2$, we see that the element $2me \in H_2(W)$ for every integer m is represented by a 2-sphere smoothly embedded in W. If $(2m+1)e \in H_2(W)$ for an integer m is represented by a smoothly embedded 2-sphere in W, then we obtain a rational-homology cobordism $(C; \mathbf{S}^1 \times \mathbf{S}^2, M)$ of degree $(1, 2m+1)$ by removing an open tubular neighborhood of the 2-sphere from W, and it is shown in [6] that the polynomial $A(t)$ is of $(2m+1)$-slice type, which contradicts our assumption. Thus, the element $(2m+1)e \in H_2(W)$ for any integer m cannot be represented by any 2-sphere smoothly embedded in W. □

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Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
kawauchi@sci.osaka-cu.ac.jp