From KAWAUCHI 1

# THE (1,2)- CABLE OF ~~A~~ THE FIGURE EIGHT KNOT IS RATIONALLY SLICE

A knot $k \subset S^3$ is <u>rationally slice</u> if $\exists\, D \subset W$, $W =$ a compact ~~compa~~ oriented smooth 4-mfd with $\tilde{H}_*(W;\mathbb{Q}) = 0$, $D =$ a smooth ~~p~~ props disk in $W$ s.t. $\partial(D \subset W) = (k \subset S^3)$ and $H_1(S^3 - k; \mathbb{Z}) \xrightarrow{\cong} H_1(W - D; \mathbb{Z})/\text{torsion}\ (\cong \mathbb{Z})$.

Clearly, rationally slice $\Rightarrow$ algebraically slice.

Let $k \subset S^3$ be a knot which admits an orientation-reversi involution $\alpha$ with $\mathrm{Fix}(\alpha, S^3) = S^0 \subset k$. That is, let $k$ be a strongly-amphicheiral knot. For example, we can tak as $k$, ~~a~~ the figure eight knot.

$\boxed{\text{Theorem}}$ The (1,2)-cable of $k$ is rationally slice.

$\boxed{\text{Problem}}$ Show that the (1,2)-cable of ~~a~~ the figure eight ~~k~~ kn is <u>not</u> a slice knot.

Proof of Theorem. Let $N$ be a small $\alpha$-invariant regular nbd of $\mathrm{Fix}(\alpha) = S^0$ in $S^3$. Let $M^4$ be the mapping cylinder of the projection

$$\begin{array}{ccc} S^3 - \mathrm{Int}\,N & \longrightarrow & S^3 - \mathrm{Int}\,N/\alpha \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ S^2 \times [0,1] & & P^2 \times [0,1] \end{array}$$

. $M^4 \cong P_0^3 \times [0,1]$, $P_0^3 = P^3 - \mathrm{Int}\,\Delta^3$. The boundary $\partial M^4 \cong S^3 \# P^3 \# -P^3$ contains a knot sum $k \# A \# -A$ bounding a smooth disk in $M^4$, where $A$ is a knot in $P^3$ representing a generator of $H_1(P^3; Z) \cong Z_2$. But, $A \# -A$ ($\subset P^3 \# -P^3$) bounds a smooth disk in $P_0^3 \times [0,1]$. Let $W_I = M^4 \cup_{P^3 \# -P^3 - \mathrm{Int}\,B^3} P_0^3 \times [0,1]$ so that $\tilde{H}_*(W_I; Q) = 0$ and $\partial W_I = S^3$. By construction, the knot $k$ bounds a smooth disk $D_I$ in $W_I$. (The construction of $D_I \subset W_I$ has been suggested by Galewski–Stern's work (cf. [Math. Proc. Camb. Phil. Soc. 85, 449–451]).)

Note that
$$0 \to \underset{\overset{\|}{Z}}{H_1(S^3 - k; Z)} \longrightarrow \underset{\overset{\|}{Z}}{H_1(W_I - D_I; Z)/\mathrm{torsion}} \to Z_2 \to 0$$
is exact. Let $E = W_I - \mathrm{Int}\,N(D_I)$. Then the boundary $\partial E$ is

the 0-surgery of $k \subset S^3$, say $M(k)$. It follows that

$$0 \to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_1(M(k);\mathbb{Z})}} \to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_1(E;\mathbb{Z})}}/\text{torsion} \to \mathbb{Z}_2 \to 0$$

is exact. Let

$W_{\mathrm{II}} = D^4 \cup D^2\times D^2 / \underset{\substack{\uparrow \\ \text{a tubular nbd of } k \text{ in } S^3}}{T(k)} \equiv (\partial D^2)\times D^2$ s.t. $\partial W_{\mathrm{II}} = M(k)$. Then $2\times(\text{generator}) \in H_2(W_{\mathrm{II}};\mathbb{Z})\,(\cong \mathbb{Z})$ is represented by a 2-sphere $\Sigma$ with just one non-locally flat point of the knot type of the (1,2)-cable, $\frac{1}{2}k$, of $k$ (cf. Kawauchi [Topology] 19, 321–334). Let $H = W_{\mathrm{II}} - \mathrm{Int}\, N(\Sigma)$, $N(\Sigma)$ = a regular nbd of $\Sigma$ in $W_{\mathrm{II}}$. Note that $\partial H = M(k) \cup M^*$ where $M^*$ = the 0-surgery of $\frac{1}{2}k \subset S^3$. Let $W_{\mathrm{III}} = H \cup_{M(k)} E$.

$$H_q(H, M^*;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & (q=2) \\ 0 & (q\neq 2) \end{cases}, \quad H_q(H, M(k);\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & (q=1) \\ 0 & (q\neq 1) \end{cases}.$$

The inclusion $M^* \subset W_{\mathrm{III}}$ induces an iso. $H_1(M^*;\mathbb{Z}) \xrightarrow{\cong} H_1(W_{\mathrm{III}};\mathbb{Z})/\text{torsion}\ (\cong \mathbb{Z})$.

Let $W_{\mathrm{IV}} = S^3\times[0,1] \cup D^2\times D^2 / T(\frac{1}{2}k)\times 1 \equiv (\partial D^2)\times D^2$ be the trace of the 0-surgery from $\frac{1}{2}k \subset S^3$ to $M^*$. The knot $\frac{1}{2}k$ bounds a smooth disk $D = (\frac{1}{2}k)\times[0,1] \cup D^2\times 0$ in $W_{\mathrm{IV}} \cup_{M^*} W_{\mathrm{III}} = W$. $\tilde{H}_*(W;\mathbb{Q}) = 0$ and $H_1(S^3 - \frac{1}{2}k;\mathbb{Z}) \xrightarrow{\cong} H_1(W - D;\mathbb{Z})/\text{torsion}$, that is, $\frac{1}{2}k$ is rationally slice.