Representing 3-manifolds in the complex number plane

AKIO KAWAUCHI and Ikuo TAYAMA

Osaka City University Advanced Mathematical Institute

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@sci.osaka-cu.ac.jp, qqyh6es9k@adagio.ocn.ne.jp

ABSTRACT

A complete invariant defined for (closed, connected, orientable) 3-manifolds is an invariant defined for the 3-manifolds such that any two 3-manifolds with the same invariant are homeomorphic. Further, if the 3-manifold itself is reconstructed from the data of the complete invariant, then it is called a characteristic invariant defined for the 3-manifolds. In previous papers by the first author, a characteristic lattice point invariant and a characteristic rational invariant defined for the 3-manifolds were constructed which also produced a smooth real function with the definition interval $(-1, 1)$ as a characteristic invariant defined for the 3-manifolds. In this paper, a complex number-valued characteristic invariant for the 3-manifolds whose norm is smaller than or equal to one half is introduced by using an embedding of a set of lattice points called the ADelta set, distinct from the PDelta set, into the set of complex numbers. The distributive situation for the invariants of the 3-manifolds of lengths up to 10 is plotted in the complex number plane with radius smaller than or equal to one half. By using this complex number-valued characteristic invariant, a holomorphic function with the unit open disk as the definition domain which is called the characteristic quantity function is constructed as a characteristic invariant defined for the 3-manifolds.

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1. Introduction

An idea of the classification of the set of prime links and the set of 3-manifolds was explained in the paper [5] (see [3] for general terminologies). Let $\mathbb{M}$ be the set of (unoriented types of) closed connected orientable 3-manifolds. Let $\mathbb{L}$ be the set of (unoriented types of) links in S^3 (including the knots as one-component links). An element $\mathbf{x}$ of $\mathbb{Z}^n$ for a positive integer n is called a *lattice point* of length n, where $\mathbb{Z}$ denotes the set of integers. We also consider the *empty lattice point* $\emptyset$ of length 0. Let $\mathbb{X}$ be the set of all lattice points. We have a canonical map

$$\mathrm{cl}\beta : \mathbb{X} \to \mathbb{L}$$

sending a lattice point $\mathbf{x}$ to a closed braid diagram $\mathrm{cl}\beta(\mathbf{x})$, which is surjective by the Alexander theorem (cf. J. S. Birman [1]), where we take the image $\mathrm{cl}\beta(\emptyset) = \phi$ the *empty knot*. It was shown in [5] that every well-order of the set $\mathbb{X}$ induces an injection

$$\sigma : \mathbb{L} \to \mathbb{X}$$

which is a right inverse of the map $\mathrm{cl}\beta$. In particular, by taking the canonical well-order which is explained in [5, 6, 7, 9, 10, 12, 13, 14], we consider the subset $\mathbb{L}^p \subset \mathbb{L}$ consisting of prime links as a well-ordered set with the order inherited from $\mathbb{X}$ by σ, where the two-component trivial link is excluded from $\mathbb{L}^p$ by definition. The *length* $\ell(L)$ of a prime link $L \in \mathbb{L}^p$ is the length $\ell(\sigma(L))$ of the lattice point $\sigma(L)$. Let $\mathbb{G}$ be the set of (isomorphism types of) the link groups $\pi_1(S^3 \setminus L)$ for all links L in S^3. Let $\pi : \mathbb{L} \to \mathbb{G}$ be the map sending a link L to the link group $\pi_1(S^3 \setminus L)$. Let $\mathbb{L}^\pi$ be the subset of $\mathbb{L}^p$ consisting of a π-*minimal link*, that is, a prime link L such that L is the initial element of the subset

$$\{L' \in \mathbb{L}^p | \, \pi_1(S^3 \setminus L') = \pi_1(S^3 \setminus L)\} \subset \mathbb{L}^p.$$

We are interested in this subset $\mathbb{L}^\pi$ because it has a crucial property that the restriction of π to $\mathbb{L}^\pi$ is injective. Since the restriction of σ to $\mathbb{L}^\pi$ is also injective, we can consider $\mathbb{L}^\pi$ as a well-ordered set by the order induced from the order of $\mathbb{X}$. In [4], we showed that the set

$$\mathbb{L}^\pi(M) = \{L \in \mathbb{L}^\pi | \chi(L, 0) = M\}$$

is not empty for every 3-manifold $M \in \mathbb{M}$, where $\chi(L, 0)$ denotes the 0-surgery manifold of S^3 along L and we define $\chi(L, 0) = S^3$ when L is the empty knot ϕ. By Kirby's theorem [15] on the Dehn surgeries of framed links, we note that the set $\mathbb{L}^\pi(M)$ is defined in terms of only links so that any two π-minimal links in $\mathbb{L}^\pi(M)$ are related by two kinds of Kirby moves and choices of orientations of S^3. Sending every 3-manifold M to the initial element of $\mathbb{L}^\pi(M)$ induces an embedding

$$\alpha : \mathbb{M} \to \mathbb{L}$$

with $\chi(\alpha(M), 0) = M$ for every 3-manifold $M \in \mathbb{M}$, which induces two embeddings

$$\begin{aligned} \sigma_\alpha &= \sigma\alpha : \mathbb{M} \to \mathbb{X}, \\ \pi_\alpha &= \pi\alpha : \mathbb{M} \to \mathbb{G}. \end{aligned}$$

In this construction, we can reconstruct the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself from the lattice point $\sigma_\alpha(M) \in \mathbb{X}$. The *length* $\ell(M)$ of a 3-manifold $M \in \mathbb{M}$ is the length $\ell(\sigma_\alpha(M))$ of the lattice point $\sigma_\alpha(M)$. To calculate $\sigma_\alpha(M)$, we proposed a program on the classification problem of $\mathbb{M}$ in [5] and classified the 3-manifolds of lengths $\leqq 10$ (See [10, 12, 13, 14]). In this process, the prime links of lengths $\leqq 10$ and their exteriors are classified (See [7, 8, 9, 11]).

In general, we say that an invariant Inv for a set of topological objects is *complete* if any two members A and A' with $\mathrm{Inv}(A) = \mathrm{Inv}(A')$ are homeomorphic. The complete invariant $\mathrm{Inv}(A)$ is a *characteristic invariant* if the topological object A can be reconstructed from the data of $\mathrm{Inv}(A)$. For example, the group $\pi_\alpha(M)$ is a complete invariant for $M \in \mathbb{M}$ taking the value in groups and the lattice point $\sigma_\alpha(M)$ is a characteristic invariant for $M \in \mathbb{M}$ taking the value in lattice points. For a real interval I, we put $I_{\mathbb{Q}} = I \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. In [6], we constructed a characteristic invariant $g(M)$, called the characteristic genus, with value in $[0, +\infty)_{\mathbb{Q}}$ for all $M \in \mathbb{M}$. For the construction, we considered a subset $P\Delta \subset \mathbb{X}$, called the PDelta set which contains the the image $\sigma(\mathbb{L}^p)$ and established an embedding

$$g : P\Delta \to [0, +\infty)_{\mathbb{Q}}$$

such that the subset $g(P\Delta)$ recovers the set $P\Delta$. By using the embedding g, a smooth function $G_{\mathbb{S}}(t)$ with the definition interval $(-1, 1)$ was also constructed in [6] for every subset $\mathbb{S} \subset P\Delta$ containing the empty lattice point $\emptyset$ of length 0 and the lattice point $\mathbf{0}$ of length 1. This smooth function $G_{\mathbb{S}}(t)$ is a characteristic invariant defined for the set $\mathbb{S}$. More concretely, there is a power series function

$$G_{\mathbb{S}}(t) = \sum_{n=0}^{n=+\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \dots \ (a_n \in \mathbb{Q},\ a_0 = a_1 = 1)$$

with the convergence interval $(-1, 1)$ such that the coefficients $a_n\,(n = 0, 1, 2, \dots)$ construct the set S uniquely. By restricting $P\Delta$ to the subsets $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p)$, the characteristic prime link function $G_{\mathbb{L}^p}(t)$ and the characteristic genus function $G_{\mathbb{M}}(t)$ were obtained.

In this paper, a subset of the lattice point set $\mathbb{X}$ called the ADelta set $A\Delta$ which is distinct from $P\Delta$ but contains the set $\sigma(\mathbb{L}^p)$ is introduced to define an embedding

$$q : A\Delta \to \mathbb{C},$$

where $\mathbb{C}$ denotes the complex number plane, such that the norm $|q(\mathbf{x})|$ has $|q(\mathbf{x})| \leqq \frac{1}{2}$ and the lattice point $\mathbf{x}$ is reconstructed from the value $q(\mathbf{x})$. Thus, the embedding q gives a characteristic invariant called the *characteristic quantity* defined for the ADelta set $A\Delta$, and by restricting $A\Delta$ to the subsets $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p)$, the *characteristic quantity* defined for the prime link set $\mathbb{L}^p$ and the *characteristic quantity* defined for the 3-manifold set $\mathbb{M}$. The ADelta set $A\Delta$ and the embedding q are explained in § 2.

In § 3, the table of the characteristic quantities of the prime links and the 3-manifolds of lengths up to 10 is given by using the list of [14]. Further, the distributive situation of the characteristic quantities in the table is drawn in the complex number plane with radius smaller than or equal to one half.

In § 4, from the characteristic quantity q, a complex number-valued holomorphic function (called the *characteristic quantity function*) $Q_{\mathbb{S}}(z)$ with the definition domain $D = \{z \in \mathbb{C} |\, |z| < 1\}$ is constructed so that the function $Q_{\mathbb{S}}(z)$ is a characteristic invariant of the lattice point set $\mathbb{S}$ for every ADelta subset $\mathbb{S} \subset A\Delta$ containing $\emptyset$ and $\mathbf{0}$. More concretely, there is a power series function

$$Q_{\mathbb{S}}(z) = \sum_{n=0}^{n=+\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \dots \ (c_n \in \mathbb{C},\ c_0 = c_1 = 1)$$

with the convergence domain D such that the coefficients c_n $(n = 0, 1, 2, \dots)$ construct the set S uniquely.

By taking $\mathbb{S} = \sigma(\mathbb{L}^p)$ and $\sigma_\alpha(\mathbb{M})$, the characteristic quantity functions $Q_{\mathbb{L}^p}(z)$ and $Q_{\mathbb{M}}(z)$ which are characteristic invariants of $\mathbb{L}^p$ and $\mathbb{M}$ are respectively obtained.

2. Embedding the ADelta set into the complex number plane

For a lattice point $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{X}$, let

$$\begin{aligned} |\mathbf{x}| &= (|x_1|, |x_2|, \dots, |x_n|), \\ \min |\mathbf{x}| &= \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max |\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|. \end{aligned}$$

The *Delta set*[1] is the subset Δ of $\mathbb{X}$ consisting of the empty lattice point $\emptyset$ of length 0 and the lattice point $\mathbf{0}$ of length 1 and all the lattice points $\mathbf{x}$ of lengths $n \geqq 2$ satisfying $x_1 = 1$ and

$$1 \leqq \min |\mathbf{x}| \leqq \max |\mathbf{x}| \leqq \frac{n}{2}.$$

The *ADelta set* $A\Delta$ is the subset of Δ consisting of

$$\emptyset, \quad \mathbf{0}, \quad 1^2$$

and all the lattice points $\mathbf{x}$ of lengths $n \geqq 3$ such that

$$x_1 = 1,\ |x_2| \leqq 2 \text{ and } |x_i| < \frac{n}{2} \text{ for some } i \geqq 2.$$

The following lemma is needed to our argument:

Lemma 2.1. $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p) \subset A\Delta$.

[1]The present definition of the Delta set coincides with the definition of [6], but slightly wider than the definitions of [5, 7, 9, 10, 12, 13, 14].

This lemma means that the collections of the links $\mathrm{cl}\beta(\mathbf{x})$ and the 3-manifolds $\chi(\mathrm{cl}\beta(\mathbf{x}), 0)$ for all lattice points $\mathbf{x} \in A\Delta$ contain all the prime links and all the 3-manifolds, respectively.

Proof of Lemma 2.1. It is implicitly proved in [5, Lemma (3.4)] that every lattice point $\mathbf{x} \in \sigma(\mathbb{L}^p)$ is in the ADelta set $A\Delta$. An outline of the proof is given as follows. A *sublattice point* of a lattice point $\mathbf{x}$ is a lattice point $\mathbf{x}'$ such that $\mathbf{x} = (\mathbf{u}, \mathbf{x}', \mathbf{v})$ for some lattice points $\mathbf{u}, \mathbf{v}$ (which may be the empty lattice point). For every lattice point $\mathbf{x} \in \sigma(\mathbb{L}^p)$ with $\ell(\mathbf{x}) \geqq 2$, it is noted that the absolute value $|\mathbf{x}|$ contains a sublattice $(1, 1)$, $(1, 2)$ or $(2, 1)$ by [5, the property (4) of Definition (3.2) and Lemma (3.4)]. By [5, the property (8) of Definition (3.2) and Lemma (3.4)], it is shown that the lattice point $\mathbf{x} \in \sigma(\mathbb{L}^p)$ must be $(1, 1, \dots)$ or $(1, \pm 2, \dots)$. The other properties of $A\Delta$ are given in [5, Lemma (3.4)]. □

The main purpose of this paper is to discuss some representations of the ADelta set $A\Delta$. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in A\Delta$ be a lattice point with $n \geqq 2$. The *signature sequence* of $\mathbf{x}$ is the lattice point

$$\varepsilon(\mathbf{x}) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n),$$

where

$$\varepsilon_i = \begin{cases} 0 & (x_i > 0) \\ 1 & (x_i < 0) \end{cases}$$

Then $\varepsilon_1 = 0$. We define the rational number

$$\theta(\mathbf{x}) = \frac{1}{2^{n-1}}(\varepsilon_2 + \varepsilon_3 2 + \cdots + \varepsilon_n 2^{n-2}),$$

which we call the *angle* of $\mathbf{x}$. The rational number

$$\tau(\mathbf{x}) = \frac{1}{n^{n-1}}(x_2 + x_3 n + \cdots + x_n n^{n-2}),$$

called the *decimal torsion* in [6]. For $\mathbf{x} \in A\Delta$ of length $n \geqq 3$, let $|\mathbf{x}| = (|x_1|, |x_2|, \dots, |x_n|)$, and $|\mathbf{x}|_0$ the lattice point in $\mathbb{X}$ obtained from $|\mathbf{x}|$ by replacing every $|x_i| = \frac{n}{2}$ with 0. The *real quantity* of $\mathbf{x}$ is defined to be the decimal torsion

$$r(\mathbf{x}) = \tau(|\mathbf{x}|_0).$$

The angle and the real quantity for $\emptyset$, $\mathbf{0}$, 1^2 are defined as follows:

$$\theta(\emptyset) = r(\emptyset) = 0,$$
$$\theta(\mathbf{0}) = r(\mathbf{0}) = \frac{1}{2},$$
$$\theta(\mathbf{1}^2) = 0,\ r(\mathbf{1}^2) = \frac{1}{2}.$$

Here are some remarks on the angle and the real quantity for $\emptyset$, $\mathbf{0}$, 1^2.

Remark 2.2. The angle and the real quantity for $\emptyset$, $\mathbf{0}$ and 1^2 are not definite values. At present, the angle and the real quantity of $\mathbf{0}$ are determined by regarding it as the lattice point $(1,-1)$ to compute that $\theta(1,-1) = \tau(|(1,-1)|) = \frac{1}{2}$. The angle and the real quantity of 1^2 are at present determined by computing that $\theta(1^2) = 0$ and $\tau(1^2) = \frac{1}{2}$. On the other hand, if we grant the assumption of the case of the length $n \geqq 3$ to the case $n = 2$, then we have $|(1,-1)|_0 = (1^2)_0 = 0^2$.

We put the following definition.

Definition 2.3. The *characteristic quantity* of a lattice point $\mathbf{x} \in A\Delta$ is the complex number

$$q(\mathbf{x}) = r(\mathbf{x}) \exp(2\pi \mathbf{i} \theta(\mathbf{x})).$$

For example, we have

$$q(\emptyset) = 0, \quad q(\mathbf{0}) = -\frac{1}{2}, \quad q(1^2) = \frac{1}{2}$$

We show the following theorem:

Theorem 2.4. The characteristic quantity $q(\mathbf{x})$ of a lattice point $\mathbf{x} \in A\Delta$ except $\emptyset$, $\mathbf{0}$ and 1^2 has the following properties (1) and (2):

(1) $0 < |q(\mathbf{x})| = r(\mathbf{x}) < \frac{1}{2}$.
(2) The lattice point $\mathbf{x} \in A\Delta$ is uniquely reconstructed from the value of $q(\mathbf{x})$.

Proof. By definition of the ADelta set $A\Delta$ and the assumption of exceptional lattice points, the lattice point $\mathbf{x}$ has $\ell(\mathbf{x}) = n \geqq 3$ and $0 < \tau(|\mathbf{x}|_0)$. Let $\mathbf{x} = (x_1, x_2, \dots)$. Since $\max |\mathbf{x}|_0 < \frac{n}{2}$, we see from [6, Lemma 3.1] that $\tau(|\mathbf{x}|_0) < \frac{1}{2}$. Hence we have

$$0 < |q(\mathbf{x})| = r(\mathbf{x}) = \tau(|\mathbf{x}|_0) < \frac{1}{2},$$

showing (1). To show (2), suppose $q(\mathbf{x}) = q(\mathbf{x}')$ for any lattice point $\mathbf{x}' \in A\Delta$. First, we show that the length $\ell(\mathbf{x}) = \ell(\mathbf{x}') = n$ is determined. Denote the same real quantity $\tau(|\mathbf{x}|_0) = \tau(|\mathbf{x}'|_0)$ by $\frac{k'}{k}$ for coprime positive integers k', k. For $|\mathbf{x}|_0$, if $|x_2| = 1$ or $|x_2| = 2$ and n is odd, then k must be n^{n-1}. If $|x_2| = 2$ and n is even, then let $n = 2n'$. If $n' = 2$, then k is 4^i $(1 \leqq i \leqq 3)$. If n' is even $\geqq 4$, then k must be $n'n^{n-2}$. If n' is odd, then the odd factor of k must be $(n')^{2n'-1}$ and $n = 2n'$. Applying the same argument to $|\mathbf{x}'|_0$, we see that the length n is uniquely determined. Then, by [6, Lemma 3.1], the identity $\tau(|\mathbf{x}|_0) = \tau(|\mathbf{x}'|_0)$ implies $|\mathbf{x}|_0 = |\mathbf{x}'|_0$, so that $|\mathbf{x}| = |\mathbf{x}'|$ by definition. The angle $\theta(\mathbf{x})$ is in the interval $[0, 1)$. In fact, by definition,

$$0 \leqq \theta(\mathbf{x}) \leqq \frac{1}{2^{n-1}} (\sum_{i=0}^{n-2} 2^i) = \frac{1}{2^{n-1}} (2^{n-1} - 1) < 1.$$

Thus, $\exp(2\pi\mathbf{i}\theta(\mathbf{x})) = \exp(2\pi\mathbf{i}\theta(\mathbf{x}'))$ if and only if $\theta(\mathbf{x}) = \theta(\mathbf{x}')$. Let $\varepsilon(\mathbf{x}') = (\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_n)$ be the signature sequence of $\mathbf{x}'$, where we have $\varepsilon'_1 = 0$. The same angle $\theta(\mathbf{x}) = \theta(\mathbf{x}')$ implies

$$|\varepsilon_i - \varepsilon'_i| \leqq 1 \quad \text{and} \quad \varepsilon_i - \varepsilon'_i \equiv 0 \pmod 2 \quad (i = 1, 2, \ldots, n).$$

Hence we have the same signature sequence $\varepsilon(\mathbf{x}) = \varepsilon(\mathbf{x}')$. Combining this with the identity $|\mathbf{x}| = |\mathbf{x}'|$, we have the identity $\mathbf{x} = \mathbf{x}'$, showing (2). □

3. The characteristic quantities of the 3-manifolds of lengths up to 10 and the distributive situation

In the following table, the numerical data of the characteristic quantities are given for the prime links of lengths up to 10 and for the 3-manifolds with lengths up to 10 by using the data of [14]. In the column M of the table the ordering of the 3-manifolds are shown, where $\times$ means a lattice point of a prime link which does not belong to the lattice points of 3-manifolds.

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
ϕ	1	0	0	0	0
0	2	$\frac{1}{2}$	0.5	$\frac{1}{2}$	0.5
1^2	$\times$	$\frac{1}{2}$	0.5	0	0
1^3	3	$\frac{4}{9}$	0.44444444444444	0	0
1^4	4	$\frac{21}{64}$	0.328125	0	0
$(1,-2,1,-2)$	5	$\frac{1}{16}$	0.0625	$\frac{5}{8}$	0.625
1^5	6	$\frac{156}{625}$	0.2496	0	0
$(1^2,-2,1,-2)$	7	$\frac{286}{625}$	0.4576	$\frac{5}{8}$	0.625
1^6	8	$\frac{1555}{7776}$	0.19997427983539	0	0
$(1^3,2,-1,2)$	9	$\frac{2887}{7776}$	0.37127057613169	$\frac{1}{4}$	0.25
$(1^3,-2,1,-2)$	10	$\frac{2887}{7776}$	0.37127057613169	$\frac{5}{8}$	0.625
$(1^2,2,1^2,2)$	11	$\frac{2857}{7776}$	0.36741255144033	0	0
$(1^2,-2,1^2,-2)$	12	$\frac{2857}{7776}$	0.36741255144033	$\frac{9}{16}$	0.5625
$(1^2,-2,1,-2^2)$	13	$\frac{3073}{7776}$	0.39519032921811	$\frac{13}{16}$	0.8125
$(1,-2,1,-2,1,-2)$	14	$\frac{361}{972}$	0.37139917695473	$\frac{21}{32}$	0.65625
$(1,-2,1,3,-2,3)$	15	$\frac{55}{972}$	0.056584362139918	$\frac{9}{32}$	0.28125
1^7	16	$\frac{19608}{117649}$	0.16666525002337	0	0
$(1^4,2,-1,2)$	17	$\frac{36758}{117649}$	0.31243784477556	$\frac{1}{4}$	0.25
$(1^4,-2,1,-2)$	18	$\frac{36758}{117649}$	0.31243784477556	$\frac{5}{8}$	0.625
$(1^3,2,1^2,2)$	$\times$	$\frac{36464}{117649}$	0.30993888600838	0	0
$(1^3,2,-1^2,2)$	$\times$	$\frac{36464}{117649}$	0.30993888600838	$\frac{3}{8}$	0.375
$(1^3,-2,1^2,-2)$	19	$\frac{36464}{117649}$	0.30993888600838	$\frac{9}{16}$	0.5625
$(1^3,-2,1,-2^2)$	20	$\frac{38865}{117649}$	0.33034704927369	$\frac{13}{16}$	0.8125
$(1^2,-2,1^2,-2^2)$	21	$\frac{38823}{117649}$	0.32999005516409	$\frac{25}{32}$	0.78125
$(1^2,-2,1,-2,1,-2)$	22	$\frac{36765}{117649}$	0.31249734379383	$\frac{21}{32}$	0.65625
$(1^2,2,-1,-3,2,-3)$	23	$\frac{56316}{117649}$	0.47867810181132	$\frac{11}{16}$	0.6875
$(1^2,-2,1,3,-2,3)$	24	$\frac{56316}{117649}$	0.47867810181132	$\frac{9}{32}$	0.28125
$(1,-2,1,-2,3,-2,3)$	25	$\frac{56359}{117649}$	0.47904359578067	$\frac{21}{64}$	0.328125
$(1,-2,1,3,-2^2,3)$	26	$\frac{56065}{117649}$	0.47654463701349	$\frac{25}{64}$	0.390625
1^8	27	$\frac{299593}{2097152}$	0.14285707473755	0	0
$(1^5,2,-1,2)$	28	$\frac{565833}{2097152}$	0.26981019973755	$\frac{1}{4}$	0.25
$(1^5,-2,1,-2)$	29	$\frac{565833}{2097152}$	0.26981019973755	$\frac{5}{8}$	0.625
$(1^4,2,1^2,2)$	$\times$	$\frac{562249}{2097152}$	0.26810121536255	0	0
$(1^4,2,-1^2,2)$	$\times$	$\frac{562249}{2097152}$	0.26810121536255	$\frac{3}{8}$	0.375
$(1^4,-2,1^2,-2)$	30	$\frac{562249}{2097152}$	0.26810121536255	$\frac{9}{16}$	0.5625
$(1^3,2,1^3,2)$	31	$\frac{561801}{2097152}$	0.26788759231567	0	0
$(1^3,2,-1^3,2)$	32	$\frac{561801}{2097152}$	0.26788759231567	$\frac{7}{16}$	0.4375
$(1^3,-2,1^3,-2)$	33	$\frac{561801}{2097152}$	0.26788759231567	$\frac{17}{32}$	0.53125
$(1^4,2,-1,2^2)$	34	$\frac{595017}{2097152}$	0.28372621536255	$\frac{1}{8}$	0.125
$(1^4,-2,1,-2^2)$	35	$\frac{595017}{2097152}$	0.28372621536255	$\frac{13}{16}$	0.8125
$(1^3,2,-1^2,2^2)$	36	$\frac{594569}{2097152}$	0.28351259231567	$\frac{3}{16}$	0.1875
$(1^3,-2,1^2,-2^2)$	37	$\frac{594569}{2097152}$	0.28351259231567	$\frac{25}{32}$	0.78125
$(1^3,2,-1,2,-1,2)$	38	$\frac{565897}{2097152}$	0.26984071731567	$\frac{5}{16}$	0.3125
$(1^3,-2,1,-2,1,-2)$	39	$\frac{565897}{2097152}$	0.26984071731567	$\frac{21}{32}$	0.65625
$(1^2,-2,1^2,-2,1,-2)$	40	$\frac{565841}{2097152}$	0.26981401443481	$\frac{41}{64}$	0.640625
$(1^3,-2,1,-2^3)$	41	$\frac{598665}{2097152}$	0.28546571731567	$\frac{29}{32}$	0.90625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^3, -2^2, 1, -2^2)$	42	$\frac{595081}{2097152}$	0.28375673294067	$\frac{27}{32}$	0.84375
$(1^2, -2, 1, -2, 1, -2^2)$	43	$\frac{595025}{2097152}$	0.28373003005981	$\frac{53}{64}$	0.828125
$(1^2, -2, 1, -2^2, 1, -2)$	44	$\frac{566353}{2097152}$	0.27005815505981	$\frac{45}{64}$	0.703125
$(1^2, 2^2, 1^2, 2^2)$	45	$\frac{594577}{2097152}$	0.28351640701294	0	0
$(1^2, -2^2, 1^2, -2^2)$	46	$\frac{594577}{2097152}$	0.28351640701294	$\frac{51}{64}$	0.796875
$(1, -2, 1, -2, 1, -2, 1, -2)$	47	$\frac{282949}{1048576}$	0.26984119415283	$\frac{85}{128}$	0.6640625
$(1^3, 2, -1, -3, 2, -3)$	48	$\frac{864905}{2097152}$	0.41241884231567	$\frac{11}{16}$	0.6875
$(1^3, -2, 1, 3, -2, 3)$	49	$\frac{864905}{2097152}$	0.41241884231567	$\frac{9}{32}$	0.28125
$(1^2, 2, 1^2, -3, 2, -3)$	$\times$	$\frac{864849}{2097152}$	0.41239213943481	$\frac{5}{8}$	0.625
$(1^2, 2, -1^2, -3, 2, -3)$	$\times$	$\frac{864849}{2097152}$	0.41239213943481	$\frac{23}{32}$	0.71875
$(1^2, -2, 1^2, 3, -2, 3)$	50	$\frac{864849}{2097152}$	0.41239213943481	$\frac{17}{64}$	0.265625
$(1^2, -2, 1, -2, 3, -2, 3)$	51	$\frac{865361}{2097152}$	0.41263628005981	$\frac{21}{64}$	0.328125
$(1^2, -2, 1, 3, -2^2, 3)$	52	$\frac{861777}{2097152}$	0.41092729568481	$\frac{25}{64}$	0.390625
$(1, -2, 1, -2, 1, 3, -2, 3)$	53	$\frac{432453}{1048576}$	0.41241931915283	$\frac{37}{128}$	0.2890625
$(1^2, -2, 1, 3, -2, 3^2)$	54	$\frac{894545}{2097152}$	0.42655229568481	$\frac{9}{64}$	0.140625
$(1, -2, 1, -2, 3, -2^2, 3)$	$\times$	$\frac{430917}{1048576}$	0.41095447540283	$\frac{53}{128}$	0.4140625
$(1, -2, 1, 3, -2^3, 3)$	55	$\frac{430693}{1048576}$	0.41074085235596	$\frac{57}{128}$	0.4453125
$(1, 2^2, 1, 3, 2^2, 3)$	56	$\frac{430889}{1048576}$	0.41092777252197	0	0
$(1, 2^2, 1, 3, -2^2, 3)$	$\times$	$\frac{430889}{1048576}$	0.41092777252197	$\frac{3}{8}$	0.375
$(1, -2^2, 1, 3, -2^2, 3)$	$\times$	$\frac{430889}{1048576}$	0.41092777252197	$\frac{51}{128}$	0.3984375
$(1, -2, 3, -2, 1, -2, 3, -2)$	57	$\frac{315725}{1048576}$	0.30109882354736	$\frac{85}{128}$	0.6640625
$(1, -2, 1, 3, -2, -4, 3, -4)$	58	$\frac{49765}{1048576}$	0.047459602355957	$\frac{89}{128}$	0.6953125
1^9	59	$\frac{5380840}{43046721}$	0.12499999709618	0	0
$(1^6, 2, -1, 2)$	60	$\frac{10222858}{43046721}$	0.23748285031977	$\frac{1}{4}$	0.25
$(1^6, -2, 1, -2)$	61	$\frac{10222858}{43046721}$	0.23748285031977	$\frac{5}{8}$	0.625
$(1^5, 2, 1^2, 2)$	$\times$	$\frac{10170370}{43046721}$	0.23626352399757	0	0
$(1^5, 2, -1^2, 2)$	$\times$	$\frac{10170370}{43046721}$	0.23626352399757	$\frac{3}{8}$	0.375
$(1^5, -2, 1^2, -2)$	62	$\frac{10170370}{43046721}$	0.23626352399757	$\frac{9}{16}$	0.5625
$(1^4, 2, 1^3, 2)$	$\times$	$\frac{10164538}{43046721}$	0.2361280432951	0	0
$(1^4, 2, -1^3, 2)$	$\times$	$\frac{10164538}{43046721}$	0.2361280432951	$\frac{7}{16}$	0.4375
$(1^4, -2, 1^3, -2)$	63	$\frac{10164538}{43046721}$	0.2361280432951	$\frac{17}{32}$	0.53125
$(1^4, -2, -1^3, -2)$	64	$\frac{10164538}{43046721}$	0.2361280432951	$\frac{31}{32}$	0.96875
$(1^5, 2, -1, 2^2)$	65	$\frac{10701811}{43046721}$	0.24860920300991	$\frac{1}{8}$	0.125
$(1^5, -2, 1, -2^2)$	66	$\frac{10701811}{43046721}$	0.24860920300991	$\frac{13}{16}$	0.8125
$(1^4, 2, -1^2, 2^2)$	67	$\frac{10695979}{43046721}$	0.24847372230744	$\frac{3}{16}$	0.1875
$(1^4, -2, 1^2, -2^2)$	68	$\frac{10695979}{43046721}$	0.24847372230744	$\frac{25}{32}$	0.78125
$(1^4, 2, -1, 2, -1, 2)$	69	$\frac{10223587}{43046721}$	0.23749978540758	$\frac{5}{16}$	0.3125
$(1^4, -2, 1, -2, 1, -2)$	70	$\frac{10223587}{43046721}$	0.23749978540758	$\frac{21}{32}$	0.65625
$(1^3, 2, 1^3, 2^2)$	71	$\frac{10695331}{43046721}$	0.24845866889606	0	0
$(1^3, 2, -1^3, 2^2)$	72	$\frac{10695331}{43046721}$	0.24845866889606	$\frac{7}{32}$	0.21875
$(1^3, -2, 1^3, 2^2)$	73	$\frac{10695331}{43046721}$	0.24845866889606	$\frac{1}{64}$	0.015625
$(1^3, -2, 1^3, -2^2)$	74	$\frac{10695331}{43046721}$	0.24845866889606	$\frac{49}{64}$	0.765625
$(1^3, 2, -1^2, 2, -1, 2)$	75	$\frac{10222939}{43046721}$	0.2374847319962	$\frac{11}{32}$	0.34375
$(1^3, -2, 1^2, -2, 1, -2)$	76	$\frac{10222939}{43046721}$	0.2374847319962	$\frac{41}{64}$	0.640625
$(1^2, -2, 1^2, -2, 1^2, -2)$	77	$\frac{10170379}{43046721}$	0.23626373307272	$\frac{73}{128}$	0.5703125

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^4,-2,1,-2^3)$	78	$\frac{10755028}{43046721}$	0.24984546441993	$\frac{29}{32}$	0.90625
$(1^4,-2^2,1,-2^2)$	79	$\frac{10702540}{43046721}$	0.24862613809772	$\frac{27}{32}$	0.84375
$(1^3,-2,1^2,-2^3)$	80	$\frac{10754380}{43046721}$	0.24983041100854	$\frac{57}{64}$	0.890625
$(1^3,-2,1,-2,1,-2^2)$	81	$\frac{10701892}{43046721}$	0.24861108468633	$\frac{53}{64}$	0.828125
$(1^3,-2,1,-2^2,1,-2)$	82	$\frac{10229500}{43046721}$	0.23763714778647	$\frac{45}{64}$	0.703125
$(1^3,2^2,1^2,2^2)$	83	$\frac{10696060}{43046721}$	0.24847560398387	0	0
$(1^3,-2^2,1^2,-2^2)$	84	$\frac{10696060}{43046721}$	0.24847560398387	$\frac{51}{64}$	0.796875
$(1^2,-2,1^2,-2,1,-2^2)$	85	$\frac{10701820}{43046721}$	0.24860941208507	$\frac{105}{128}$	0.8203125
$(1^2,2,-1,2,1^2,2^2)$	86	$\frac{10695988}{43046721}$	0.2484739313826	$\frac{1}{64}$	0.015625
$(1^2,-2,1,-2,1^2,-2^2)$	87	$\frac{10695988}{43046721}$	0.2484739313826	$\frac{101}{128}$	0.7890625
$(1^2,-2,1,-2,1,-2,1,-2)$	88	$\frac{10223596}{43046721}$	0.23749999448274	$\frac{85}{128}$	0.6640625
$(1^4,2,-1,-3,2,-3)$	89	$\frac{15597046}{43046721}$	0.36232831764352	$\frac{11}{16}$	0.6875
$(1^4,-2,1,3,-2,3)$	90	$\frac{15597046}{43046721}$	0.36232831764352	$\frac{9}{32}$	0.28125
$(1^3,2,1^2,-3,2,-3)$	91	$\frac{15596398}{43046721}$	0.36231326423214	$\frac{5}{8}$	0.625
$(1^3,2,-1^2,-3,2,-3)$	92	$\frac{15596398}{43046721}$	0.36231326423214	$\frac{23}{32}$	0.71875
$(1^3,-2,1^2,3,-2,3)$	93	$\frac{15596398}{43046721}$	0.36231326423214	$\frac{17}{64}$	0.265625
$(1^3,-2,-1^2,3,-2,3)$	94	$\frac{15596398}{43046721}$	0.36231326423214	$\frac{23}{64}$	0.359375
$(1^3,2,-1,2,3,-2,3)$	95	$\frac{15602959}{43046721}$	0.36246568002241	$\frac{9}{32}$	0.28125
$(1^3,2,-1,2,-3,2,-3)$	96	$\frac{15602959}{43046721}$	0.36246568002241	$\frac{21}{32}$	0.65625
$(1^3,-2,1,-2,3,-2,3)$	97	$\frac{15602959}{43046721}$	0.36246568002241	$\frac{21}{64}$	0.328125
$(1^3,-2,1,-2,-3,2,-3)$	98	$\frac{15602959}{43046721}$	0.36246568002241	$\frac{45}{64}$	0.703125
$(1^3,2,-1,-3,2^2,-3)$	99	$\frac{15550471}{43046721}$	0.3612463537002	$\frac{19}{32}$	0.59375
$(1^3,-2,1,3,-2^2,3)$	100	$\frac{15550471}{43046721}$	0.3612463537002	$\frac{25}{64}$	0.390625
$(1^2,2,1^2,2,-3,2,-3)$	$\times$	$\frac{15602887}{43046721}$	0.36246400742115	$\frac{5}{8}$	0.625
$(1^2,2,-1^2,2,-3,2,-3)$	$\times$	$\frac{15602887}{43046721}$	0.36246400742115	$\frac{43}{64}$	0.671875
$(1^2,-2,1^2,-2,3,-2,3)$	$\times$	$\frac{15602887}{43046721}$	0.36246400742115	$\frac{41}{128}$	0.3203125
$(1^2,2,1^2,-3,2^2,-3)$	$\times$	$\frac{15550399}{43046721}$	0.36124468109894	$\frac{9}{16}$	0.5625
$(1^2,2,-1^2,-3,2^2,-3)$	$\times$	$\frac{15550399}{43046721}$	0.36124468109894	$\frac{39}{64}$	0.609375
$(1^2,-2,1^2,3,-2^2,3)$	101	$\frac{15550399}{43046721}$	0.36124468109894	$\frac{49}{128}$	0.3828125
$(1^2,2,-1,2,1,3,-2,3)$	102	$\frac{15597055}{43046721}$	0.36232852671868	$\frac{17}{64}$	0.265625
$(1^2,-2,1,-2,1,3,-2,3)$	103	$\frac{15597055}{43046721}$	0.36232852671868	$\frac{37}{128}$	0.2890625
$(1^2,-2,1,3,-2,1,3,-2)$	104	$\frac{11234719}{43046721}$	0.26098896127303	$\frac{73}{128}$	0.5703125
$(1^3,2,-1,-3,2,-3^2)$	105	$\frac{16081912}{43046721}$	0.37359203271255	$\frac{27}{32}$	0.84375
$(1^3,-2,1,3,-2,3^2)$	106	$\frac{16081912}{43046721}$	0.37359203271255	$\frac{9}{64}$	0.140625
$(1^2,-2,1^2,3,-2,3^2)$	$\times$	$\frac{16081840}{43046721}$	0.37359036011128	$\frac{17}{128}$	0.1328125
$(1^2,2,-1,2^2,3,-2,3)$	107	$\frac{15603616}{43046721}$	0.36248094250895	$\frac{17}{64}$	0.265625
$(1^2,2,-1,2^2,-3,2,-3)$	108	$\frac{15603616}{43046721}$	0.36248094250895	$\frac{41}{64}$	0.640625
$(1^2,-2,1,-2^2,3,-2,3)$	109	$\frac{15603616}{43046721}$	0.36248094250895	$\frac{45}{128}$	0.3515625
$(1^2,-2,1,-2^2,-3,2,-3)$	110	$\frac{15603616}{43046721}$	0.36248094250895	$\frac{93}{128}$	0.7265625
$(1^2,-2,1,3,2^3,3)$	111	$\frac{15545296}{43046721}$	0.36112613548428	$\frac{1}{128}$	0.0078125
$(1^2,-2,1,3,-2^3,3)$	112	$\frac{15545296}{43046721}$	0.36112613548428	$\frac{57}{128}$	0.4453125
$(1^2,-2^2,1,-2,3,-2,3)$	113	$\frac{15602968}{43046721}$	0.36246588909757	$\frac{43}{128}$	0.3359375
$(1^2,2^2,1,3,2^2,3)$	$\times$	$\frac{15550480}{43046721}$	0.36124656277536	0	0
$(1^2,2^2,1,3,-2^2,3)$	$\times$	$\frac{15550480}{43046721}$	0.36124656277536	$\frac{3}{8}$	0.375
$(1^2,2^2,1,-3,2^2,-3)$	114	$\frac{15550480}{43046721}$	0.36124656277536	$\frac{9}{16}$	0.5625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^2, -2^2, 1, 3, -2^2, 3)$	115	$\frac{15550480}{43046721}$	0.36124656277536	$\frac{51}{128}$	0.3984375
$(1, -2, 1, -2, 1, -2, 3, -2, 3)$	116	$\frac{15602960}{43046721}$	0.36246570325298	$\frac{85}{256}$	0.33203125
$(1, -2, 1, -2, 1, 3, 2^2, 3)$	$\times$	$\frac{15550472}{43046721}$	0.36124637693078	$\frac{5}{256}$	0.01953125
$(1, -2, 1, -2, 1, 3, -2^2, 3)$	117	$\frac{15550472}{43046721}$	0.36124637693078	$\frac{101}{256}$	0.39453125
$(1, -2, 1, -2, 1, -3, 2^2, -3)$	118	$\frac{15550472}{43046721}$	0.36124637693078	$\frac{149}{256}$	0.58203125
$(1, -2, 1, -2^2, 1, 3, -2, 3)$	119	$\frac{15597128}{43046721}$	0.36233022255052	$\frac{77}{256}$	0.30078125
$(1, 2, -1, 2, 3, -2, 1, -2, 3)$	$\times$	$\frac{15486320}{43046721}$	0.35975608920364	$\frac{41}{128}$	0.3203125
$(1, -2, 1, -2, 3, -2, 1, -2, 3)$	120	$\frac{15486320}{43046721}$	0.35975608920364	$\frac{85}{256}$	0.33203125
$(1, -2, 1, -2, -3, -2, 1, -2, -3)$	121	$\frac{15486320}{43046721}$	0.35975608920364	$\frac{221}{256}$	0.86328125
$(1^2, -2, 1, -2, 3, -2, 3^2)$	122	$\frac{16082569}{43046721}$	0.37360729519909	$\frac{21}{128}$	0.1640625
$(1^2, -2, 1, 3, -2^2, 3^2)$	123	$\frac{16076737}{43046721}$	0.37347181449663	$\frac{25}{128}$	0.1953125
$(1, -2, 1, 3, -2, 1, 3, -2, 3)$	124	$\frac{15597209}{43046721}$	0.36233210422694	$\frac{73}{256}$	0.28515625
$(1^2, -2, 1, 3, -2, -4, 3, -4)$	125	$\frac{20977804}{43046721}$	0.4873264098327	$\frac{89}{128}$	0.6953125
$(1, -2, 1, -2^3, 3, -2, 3)$	126	$\frac{15603689}{43046721}$	0.36248263834079	$\frac{93}{256}$	0.36328125
$(1, -2, 1, -2, 3, -2^3, 3)$	127	$\frac{15545369}{43046721}$	0.36112783131612	$\frac{117}{256}$	0.45703125
$(1, -2, 1, 3, -2^4, 3)$	128	$\frac{15544721}{43046721}$	0.36111277790473	$\frac{121}{256}$	0.47265625
$(1, -2^2, 1, -2, 3, -2^2, 3)$	129	$\frac{15551129}{43046721}$	0.36126163941732	$\frac{107}{256}$	0.41796875
$(1, -2^2, 3, -2, 1, -2, 3, -2)$	130	$\frac{11286641}{43046721}$	0.26219513909085	$\frac{171}{256}$	0.66796875
$(1, -2, 1, -2, 3, -2, -4, 3, -4)$	131	$\frac{20977877}{43046721}$	0.48732810566454	$\frac{181}{256}$	0.70703125
$(1, -2, 1, -2, -3, 2, 4, -3, 4)$	132	$\frac{20977877}{43046721}$	0.48732810566454	$\frac{77}{256}$	0.30078125
$(1, -2, 1, 3, -2^2, -4, 3, -4)$	133	$\frac{20977229}{43046721}$	0.48731305225316	$\frac{185}{256}$	0.72265625
1^{10}	134	$\frac{111111111}{1000000000}$	0.111111111	0	0
$(1^7, 2, -1, 2)$	135	$\frac{212111111}{1000000000}$	0.212111111	$\frac{1}{4}$	0.25
$(1^7, -2, 1, -2)$	136	$\frac{212111111}{1000000000}$	0.212111111	$\frac{5}{8}$	0.625
$(1^6, 2, 1^2, 2)$	$\times$	$\frac{211211111}{1000000000}$	0.211211111	0	0
$(1^6, 2, -1^2, 2)$	$\times$	$\frac{211211111}{1000000000}$	0.211211111	$\frac{3}{8}$	0.375
$(1^6, -2, 1^2, -2)$	137	$\frac{211211111}{1000000000}$	0.211211111	$\frac{9}{16}$	0.5625
$(1^5, 2, 1^3, 2)$	138	$\frac{211121111}{1000000000}$	0.211121111	0	0
$(1^5, 2, -1^3, 2)$	139	$\frac{211121111}{1000000000}$	0.211121111	$\frac{7}{16}$	0.4375
$(1^5, -2, 1^3, -2)$	140	$\frac{211121111}{1000000000}$	0.211121111	$\frac{17}{32}$	0.53125
$(1^5, -2, -1^3, -2)$	141	$\frac{211121111}{1000000000}$	0.211121111	$\frac{31}{32}$	0.96875
$(1^4, 2, 1^4, 2)$	142	$\frac{211112111}{1000000000}$	0.211112111	0	0
$(1^4, 2, -1^4, 2)$	143	$\frac{211112111}{1000000000}$	0.211112111	$\frac{15}{32}$	0.46875
$(1^4, -2, 1^4, -2)$	144	$\frac{211112111}{1000000000}$	0.211112111	$\frac{33}{64}$	0.515625
$(1^6, 2, -1, 2^2)$	145	$\frac{221211111}{1000000000}$	0.221211111	$\frac{1}{8}$	0.125
$(1^6, -2, 1, -2^2)$	146	$\frac{221211111}{1000000000}$	0.221211111	$\frac{13}{16}$	0.8125
$(1^5, 2, -1^2, 2^2)$	147	$\frac{221121111}{1000000000}$	0.221121111	$\frac{3}{16}$	0.1875
$(1^5, -2, 1^2, -2^2)$	148	$\frac{221121111}{1000000000}$	0.221121111	$\frac{25}{32}$	0.78125
$(1^5, 2, -1, 2, -1, 2)$	149	$\frac{212121111}{1000000000}$	0.212121111	$\frac{5}{16}$	0.3125
$(1^5, -2, 1, -2, 1, -2)$	150	$\frac{212121111}{1000000000}$	0.212121111	$\frac{21}{32}$	0.65625
$(1^4, 2, 1^3, 2^2)$	151	$\frac{221112111}{1000000000}$	0.221112111	0	0
$(1^4, 2, -1^3, 2^2)$	152	$\frac{221112111}{1000000000}$	0.221112111	$\frac{7}{32}$	0.21875
$(1^4, -2, 1^3, 2^2)$	153	$\frac{221112111}{1000000000}$	0.221112111	$\frac{1}{64}$	0.015625
$(1^4, -2, 1^3, -2^2)$	154	$\frac{221112111}{1000000000}$	0.221112111	$\frac{49}{64}$	0.765625
$(1^4, -2, -1^3, -2^2)$	155	$\frac{221112111}{1000000000}$	0.221112111	$\frac{63}{64}$	0.984375

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^4, 2, -1^2, 2, -1, 2)$	156	$\frac{212112111}{1000000000}$	0.212112111	$\frac{11}{32}$	0.34375
$(1^4, -2, 1^2, -2, 1, -2)$	157	$\frac{212112111}{1000000000}$	0.212112111	$\frac{41}{64}$	0.640625
$(1^3, 2, -1^3, 2, -1, 2)$	$\times$	$\frac{212111211}{1000000000}$	0.212111211	$\frac{23}{64}$	0.359375
$(1^3, -2, 1^3, 2, -1, 2)$	158	$\frac{212111211}{1000000000}$	0.212111211	$\frac{33}{128}$	0.2578125
$(1^3, -2, 1^3, -2, 1, -2)$	159	$\frac{212111211}{1000000000}$	0.212111211	$\frac{81}{128}$	0.6328125
$(1^3, 2, -1^2, 2, -1^2, 2)$	160	$\frac{211211211}{1000000000}$	0.211211211	$\frac{27}{64}$	0.421875
$(1^3, -2, 1^2, -2, 1^2, -2)$	161	$\frac{211211211}{1000000000}$	0.211211211	$\frac{73}{128}$	0.5703125
$(1^5, -2, 1, -2^3)$	162	$\frac{222121111}{1000000000}$	0.222121111	$\frac{29}{32}$	0.90625
$(1^5, -2^2, 1, -2^2)$	163	$\frac{221221111}{1000000000}$	0.221221111	$\frac{27}{32}$	0.84375
$(1^4, -2, 1^2, -2^3)$	164	$\frac{222112111}{1000000000}$	0.222112111	$\frac{57}{64}$	0.890625
$(1^4, 2, -1, 2, -1, 2^2)$	165	$\frac{221212111}{1000000000}$	0.221212111	$\frac{5}{32}$	0.15625
$(1^4, -2, 1, -2, 1, -2^2)$	166	$\frac{221212111}{1000000000}$	0.221212111	$\frac{53}{64}$	0.828125
$(1^4, 2, -1, 2^2, -1, 2)$	167	$\frac{212212111}{1000000000}$	0.212212111	$\frac{9}{32}$	0.28125
$(1^4, -2, 1, -2^2, 1, -2)$	168	$\frac{212212111}{1000000000}$	0.212212111	$\frac{45}{64}$	0.703125
$(1^4, 2^2, 1^2, 2^2)$	$\times$	$\frac{221122111}{1000000000}$	0.221122111	0	0
$(1^4, 2^2, -1^2, 2^2)$	169	$\frac{221122111}{1000000000}$	0.221122111	$\frac{3}{16}$	0.1875
$(1^4, -2^2, 1^2, -2^2)$	170	$\frac{221122111}{1000000000}$	0.221122111	$\frac{51}{64}$	0.796875
$(1^3, -2, 1^3, 2^3)$	171	$\frac{222111211}{1000000000}$	0.222111211	$\frac{1}{128}$	0.0078125
$(1^3, -2, 1^3, -2^3)$	172	$\frac{222111211}{1000000000}$	0.222111211	$\frac{113}{128}$	0.8828125
$(1^3, 2, -1^2, 2, -1, 2^2)$	$\times$	$\frac{221211211}{1000000000}$	0.221211211	$\frac{11}{64}$	0.171875
$(1^3, -2, 1^2, -2, 1, -2^2)$	173	$\frac{221211211}{1000000000}$	0.221211211	$\frac{105}{128}$	0.8203125
$(1^3, -2, 1^2, -2^2, 1, -2)$	174	$\frac{212211211}{1000000000}$	0.212211211	$\frac{89}{128}$	0.6953125
$(1^3, 2, -1, 2, 1^2, 2^2)$	175	$\frac{221121211}{1000000000}$	0.221121211	$\frac{1}{64}$	0.015625
$(1^3, 2, -1, 2, -1^2, 2^2)$	176	$\frac{221121211}{1000000000}$	0.221121211	$\frac{13}{64}$	0.203125
$(1^3, -2, 1, -2, 1^2, -2^2)$	177	$\frac{221121211}{1000000000}$	0.221121211	$\frac{101}{128}$	0.7890625
$(1^3, -2, 1, -2, 1, -2, 1, -2)$	178	$\frac{212121211}{1000000000}$	0.212121211	$\frac{85}{128}$	0.6640625
$(1^3, 2^2, 1^3, 2^2)$	179	$\frac{221112211}{1000000000}$	0.221112211	0	0
$(1^3, -2^2, 1^3, -2^2)$	180	$\frac{221112211}{1000000000}$	0.221112211	$\frac{99}{128}$	0.7734375
$(1^2, 2, -1^2, 2, 1^2, 2^2)$	$\times$	$\frac{221121121}{1000000000}$	0.221121121	$\frac{3}{128}$	0.0234375
$(1^2, -2, 1^2, -2, 1^2, -2^2)$	181	$\frac{221121121}{1000000000}$	0.221121121	$\frac{201}{256}$	0.78515625
$(1^2, -2, 1^2, -2, 1, -2, 1, -2)$	182	$\frac{212121121}{1000000000}$	0.212121121	$\frac{169}{256}$	0.66015625
$(1^2, -2, 1, -2, 1^2, -2, 1, -2)$	183	$\frac{212112121}{1000000000}$	0.212112121	$\frac{165}{256}$	0.64453125
$(1^5, 2, -1, -3, 2, -3)$	184	$\frac{323121111}{1000000000}$	0.323121111	$\frac{11}{16}$	0.6875
$(1^5, -2, 1, 3, -2, 3)$	185	$\frac{323121111}{1000000000}$	0.323121111	$\frac{9}{32}$	0.28125
$(1^4, 2, 1^2, -3, 2, -3)$	186	$\frac{323112111}{1000000000}$	0.323112111	$\frac{5}{8}$	0.625
$(1^4, 2, -1^2, -3, 2, -3)$	$\times$	$\frac{323112111}{1000000000}$	0.323112111	$\frac{23}{32}$	0.71875
$(1^4, -2, 1^2, 3, -2, 3)$	187	$\frac{323112111}{1000000000}$	0.323112111	$\frac{17}{64}$	0.265625
$(1^4, -2, -1^2, 3, -2, 3)$	$\times$	$\frac{323112111}{1000000000}$	0.323112111	$\frac{23}{64}$	0.359375
$(1^3, 2, 1^3, -3, 2, -3)$	188	$\frac{323111211}{1000000000}$	0.323111211	$\frac{5}{8}$	0.625
$(1^3, 2, -1^3, -3, 2, -3)$	$\times$	$\frac{323111211}{1000000000}$	0.323111211	$\frac{47}{64}$	0.734375
$(1^3, -2, 1^3, 3, -2, 3)$	189	$\frac{323111211}{1000000000}$	0.323111211	$\frac{33}{128}$	0.2578125
$(1^4, -2, 1, -2^4)$	190	$\frac{222212111}{1000000000}$	0.222212111	$\frac{61}{64}$	0.953125
$(1^4, -2^2, 1, -2^3)$	191	$\frac{222122111}{1000000000}$	0.222122111	$\frac{59}{64}$	0.921875
$(1^3, -2, 1, -2, 1, -2^3)$	192	$\frac{222121211}{1000000000}$	0.222121211	$\frac{117}{128}$	0.9140625
$(1^3, -2, 1, -2^2, 1, -2^2)$	193	$\frac{221221211}{1000000000}$	0.221221211	$\frac{109}{128}$	0.8515625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^3, -2, 1, -2^3, 1, -2)$	194	$\frac{212221211}{1000000000}$	0.212221211	$\frac{93}{128}$	0.7265625
$(1^3, 2^2, 1^2, 2^3)$	195	$\frac{222112211}{1000000000}$	0.222112211	0	0
$(1^3, -2^2, 1^2, -2^3)$	196	$\frac{222112211}{1000000000}$	0.222112211	$\frac{115}{128}$	0.8984375
$(1^3, 2^2, -1, 2, -1, 2^2)$	197	$\frac{221212211}{1000000000}$	0.221212211	$\frac{5}{32}$	0.15625
$(1^3, -2^2, 1, -2, 1, -2^2)$	198	$\frac{221212211}{1000000000}$	0.221212211	$\frac{107}{128}$	0.8359375
$(1^2, -2, 1^2, -2^2, 1, -2^2)$	199	$\frac{221221121}{1000000000}$	0.221221121	$\frac{217}{256}$	0.84765625
$(1^2, -2, 1, -2, 1, -2, 1, -2^2)$	200	$\frac{221212121}{1000000000}$	0.221212121	$\frac{213}{256}$	0.83203125
$(1^2, -2, 1, -2, 1, -2^2, 1, -2)$	201	$\frac{212212121}{1000000000}$	0.212212121	$\frac{181}{256}$	0.70703125
$(1^2, -2, 1, -2^2, 1^2, -2^2)$	202	$\frac{221122121}{1000000000}$	0.221122121	$\frac{205}{256}$	0.80078125
$(1, -2, 1, -2, 1, -2, 1, -2, 1, -2)$	203	$\frac{53030303}{250000000}$	0.212121212	$\frac{341}{512}$	0.666015625
$(1^4, 2, -1, 2, -3, 2, -3)$	204	$\frac{323212111}{1000000000}$	0.323212111	$\frac{21}{32}$	0.65625
$(1^4, -2, 1, -2, 3, -2, 3)$	205	$\frac{323212111}{1000000000}$	0.323212111	$\frac{21}{64}$	0.328125
$(1^4, -2, 1, -2, -3, 2, -3)$	206	$\frac{323212111}{1000000000}$	0.323212111	$\frac{45}{64}$	0.703125
$(1^4, 2, -1, -3, 2^2, -3)$	207	$\frac{322312111}{1000000000}$	0.322312111	$\frac{19}{32}$	0.59375
$(1^4, -2, 1, 3, -2^2, 3)$	208	$\frac{322312111}{1000000000}$	0.322312111	$\frac{25}{64}$	0.390625
$(1^3, 2, 1^2, 2, -3, 2, -3)$	209	$\frac{323211211}{1000000000}$	0.323211211	$\frac{5}{8}$	0.625
$(1^3, 2, -1^2, 2, -3, 2, -3)$	210	$\frac{323211211}{1000000000}$	0.323211211	$\frac{43}{64}$	0.671875
$(1^3, -2, 1^2, -2, 3, -2, 3)$	211	$\frac{323211211}{1000000000}$	0.323211211	$\frac{41}{128}$	0.3203125
$(1^3, -2, -1^2, -2, 3, -2, 3)$	212	$\frac{323211211}{1000000000}$	0.323211211	$\frac{47}{128}$	0.3671875
$(1^3, 2, 1^2, -3, 2^2, -3)$	213	$\frac{322311211}{1000000000}$	0.322311211	$\frac{9}{16}$	0.5625
$(1^3, 2, -1^2, -3, 2^2, -3)$	214	$\frac{322311211}{1000000000}$	0.322311211	$\frac{39}{64}$	0.609375
$(1^3, -2, 1^2, 3, -2^2, 3)$	215	$\frac{322311211}{1000000000}$	0.322311211	$\frac{49}{128}$	0.3828125
$(1^3, -2, -1^2, 3, -2^2, 3)$	216	$\frac{322311211}{1000000000}$	0.322311211	$\frac{55}{128}$	0.4296875
$(1^3, 2, -1, 2, 1, 3, -2, 3)$	$\times$	$\frac{323121211}{1000000000}$	0.323121211	$\frac{17}{64}$	0.265625
$(1^3, 2, -1, 2, -1, -3, 2, -3)$	217	$\frac{323121211}{1000000000}$	0.323121211	$\frac{45}{64}$	0.703125
$(1^3, -2, 1, -2, 1, 3, -2, 3)$	218	$\frac{323121211}{1000000000}$	0.323121211	$\frac{37}{128}$	0.2890625
$(1^3, 2, 1, 3, -2, 1, 3, 2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{1}{16}$	0.0625
$(1^3, 2, 1, -3, 2, 1, -3, 2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{9}{32}$	0.28125
$(1^3, 2, -1, 3, -2, 1, 3, -2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{37}{64}$	0.578125
$(1^3, 2, -1, 3, -2, -1, 3, 2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{13}{64}$	0.203125
$(1^3, 2, -1, -3, 2, 1, -3, -2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{51}{64}$	0.796875
$(1^3, 2, -1, -3, 2, -1, -3, 2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{27}{64}$	0.421875
$(1^3, -2, 1, 3, -2, 1, 3, -2)$	219	$\frac{231231211}{1000000000}$	0.231231211	$\frac{73}{128}$	0.5703125
$(1^3, -2, 1, -3, 2, 1, -3, -2)$	$\times$	$\frac{231231211}{1000000000}$	0.231231211	$\frac{101}{128}$	0.7890625
$(1^2, 2, -1^2, 2, 1, 3, -2, 3)$	220	$\frac{323121121}{1000000000}$	0.323121121	$\frac{35}{128}$	0.2734375
$(1^2, -2, 1^2, -2, 1, 3, -2, 3)$	221	$\frac{323121121}{1000000000}$	0.323121121	$\frac{73}{256}$	0.28515625
$(1^2, 2, 1^2, -3, -2, 1, -2, -3)$	222	$\frac{321231121}{1000000000}$	0.321231121	$\frac{27}{32}$	0.84375
$(1^2, 2, -1^2, -3, -2, 1, -2, -3)$	223	$\frac{321231121}{1000000000}$	0.321231121	$\frac{111}{128}$	0.8671875
$(1^2, -2, 1^2, 3, 2, -1, 2, 3)$	224	$\frac{321231121}{1000000000}$	0.321231121	$\frac{33}{256}$	0.12890625
$(1^2, -2, 1^2, 3, -2, 1, -2, 3)$	225	$\frac{321231121}{1000000000}$	0.321231121	$\frac{81}{256}$	0.31640625
$(1^4, 2, -1, -3, 2, -3^2)$	226	$\frac{332312111}{1000000000}$	0.332312111	$\frac{27}{32}$	0.84375
$(1^4, -2, 1, 3, -2, 3^2)$	227	$\frac{332312111}{1000000000}$	0.332312111	$\frac{9}{64}$	0.140625
$(1^3, -2, 1^2, 3, -2, 3^2)$	228	$\frac{332311211}{1000000000}$	0.332311211	$\frac{17}{128}$	0.1328125
$(1^3, -2, 1, -2^2, 3, -2, 3)$	229	$\frac{323221211}{1000000000}$	0.323221211	$\frac{45}{128}$	0.3515625
$(1^3, -2, 1, -2^2, -3, 2, -3)$	230	$\frac{323221211}{1000000000}$	0.323221211	$\frac{93}{128}$	0.7265625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^3, 2, -1, 2, 3, 2^2, 3)$	$\times$	$\frac{322321211}{1000000000}$	0.322321211	$\frac{1}{64}$	0.015625
$(1^3, 2, -1, 2, -3, 2^2, -3)$	231	$\frac{322321211}{1000000000}$	0.322321211	$\frac{37}{64}$	0.578125
$(1^3, -2, 1, -2, 3, 2^2, 3)$	$\times$	$\frac{322321211}{1000000000}$	0.322321211	$\frac{5}{128}$	0.0390625
$(1^3, -2, 1, -2, 3, -2^2, 3)$	232	$\frac{322321211}{1000000000}$	0.322321211	$\frac{53}{128}$	0.4140625
$(1^3, -2, 1, 3, -2^3, 3)$	233	$\frac{322231211}{1000000000}$	0.322231211	$\frac{57}{128}$	0.4453125
$(1^3, 2^2, -1, 2, -3, 2, -3)$	234	$\frac{323212211}{1000000000}$	0.323212211	$\frac{21}{32}$	0.65625
$(1^3, -2^2, 1, -2, 3, -2, 3)$	235	$\frac{323212211}{1000000000}$	0.323212211	$\frac{43}{128}$	0.3359375
$(1^3, -2^2, 1, -2, -3, 2, -3)$	236	$\frac{323212211}{1000000000}$	0.323212211	$\frac{91}{128}$	0.7109375
$(1^3, 2^2, 1, 3, 2^2, 3)$	$\times$	$\frac{322312211}{1000000000}$	0.322312211	0	0
$(1^3, 2^2, 1, 3, -2^2, 3)$	$\times$	$\frac{322312211}{1000000000}$	0.322312211	$\frac{3}{8}$	0.375
$(1^3, 2^2, 1, -3, 2^2, -3)$	$\times$	$\frac{322312211}{1000000000}$	0.322312211	$\frac{9}{16}$	0.5625
$(1^3, 2^2, -1, -3, 2^2, -3)$	$\times$	$\frac{322312211}{1000000000}$	0.322312211	$\frac{19}{32}$	0.59375
$(1^3, -2^2, 1, 3, -2^2, 3)$	$\times$	$\frac{322312211}{1000000000}$	0.322312211	$\frac{51}{128}$	0.3984375
$(1^3, 2^3, 1, 3, -2, 3)$	237	$\frac{323122211}{1000000000}$	0.323122211	$\frac{1}{4}$	0.25
$(1^3, 2, -3, 2, -1, 2, -3, 2)$	238	$\frac{232123211}{1000000000}$	0.232123211	$\frac{21}{64}$	0.328125
$(1^3, -2, 3, -2, 1, -2, 3, -2)$	239	$\frac{232123211}{1000000000}$	0.232123211	$\frac{85}{128}$	0.6640625
$(1^2, 2, -1^2, 2^2, 3, -2, 3)$	$\times$	$\frac{323221121}{1000000000}$	0.323221121	$\frac{35}{128}$	0.2734375
$(1^2, 2, -1^2, 2^2, -3, 2, -3)$	$\times$	$\frac{323221121}{1000000000}$	0.323221121	$\frac{83}{128}$	0.6484375
$(1^2, -2, 1^2, -2^2, 3, -2, 3)$	240	$\frac{323221121}{1000000000}$	0.323221121	$\frac{89}{256}$	0.34765625
$(1^2, -2, 1^2, -2^2, -3, 2, -3)$	241	$\frac{323221121}{1000000000}$	0.323221121	$\frac{185}{256}$	0.72265625
$(1^2, 2, 1^2, 2, 3, 2^2, 3)$	$\times$	$\frac{322321121}{1000000000}$	0.322321121	0	0
$(1^2, 2, 1^2, 2, 3, -2^2, 3)$	$\times$	$\frac{322321121}{1000000000}$	0.322321121	$\frac{3}{8}$	0.375
$(1^2, 2, 1^2, 2, -3, 2^2, -3)$	$\times$	$\frac{322321121}{1000000000}$	0.322321121	$\frac{9}{16}$	0.5625
$(1^2, 2, -1^2, 2, -3, 2^2, -3)$	$\times$	$\frac{322321121}{1000000000}$	0.322321121	$\frac{75}{128}$	0.5859375
$(1^2, -2, 1^2, -2, 3, -2^2, 3)$	242	$\frac{322321121}{1000000000}$	0.322321121	$\frac{105}{256}$	0.41015625
$(1^2, 2, 1^2, -3, 2^3, -3)$	$\times$	$\frac{322231121}{1000000000}$	0.322231121	$\frac{17}{32}$	0.53125
$(1^2, 2, 1^2, -3, -2^3, -3)$	$\times$	$\frac{322231121}{1000000000}$	0.322231121	$\frac{31}{32}$	0.96875
$(1^2, -2, 1^2, 3, 2^3, 3)$	243	$\frac{322231121}{1000000000}$	0.322231121	$\frac{1}{256}$	0.00390625
$(1^2, -2, 1^2, 3, -2^3, 3)$	244	$\frac{322231121}{1000000000}$	0.322231121	$\frac{113}{256}$	0.44140625
$(1^2, -2, 1, -2, 1, -2, 3, -2, 3)$	245	$\frac{323212121}{1000000000}$	0.323212121	$\frac{85}{256}$	0.33203125
$(1^2, 2, -1, 2, 1, 3, -2^2, 3)$	$\times$	$\frac{322312121}{1000000000}$	0.322312121	$\frac{49}{128}$	0.3828125
$(1^2, -2, 1, -2, 1, 3, 2^2, 3)$	$\times$	$\frac{322312121}{1000000000}$	0.322312121	$\frac{5}{256}$	0.01953125
$(1^2, -2, 1, -2, 1, 3, -2^2, 3)$	246	$\frac{322312121}{1000000000}$	0.322312121	$\frac{101}{256}$	0.39453125
$(1^2, -2, 1, -2, 1, -3, 2^2, -3)$	$\times$	$\frac{322312121}{1000000000}$	0.322312121	$\frac{149}{256}$	0.58203125
$(1^2, 2, -1, 2^2, 1, 3, -2, 3)$	$\times$	$\frac{323122121}{1000000000}$	0.323122121	$\frac{33}{128}$	0.2578125
$(1^2, -2, 1, -2^2, 1, 3, -2, 3)$	247	$\frac{323122121}{1000000000}$	0.323122121	$\frac{77}{256}$	0.30078125
$(1^2, 2, -1, 2, 3, 2, -1, 2, 3)$	$\times$	$\frac{321232121}{1000000000}$	0.321232121	$\frac{17}{128}$	0.1328125
$(1^2, 2, -1, 2, -3, 2, -1, 2, -3)$	$\times$	$\frac{321232121}{1000000000}$	0.321232121	$\frac{85}{128}$	0.6640625
$(1^2, -2, 1, -2, 3, -2, 1, -2, 3)$	248	$\frac{321232121}{1000000000}$	0.321232121	$\frac{85}{256}$	0.33203125
$(1^2, -2, 1, -2, -3, -2, 1, -2, -3)$	249	$\frac{321232121}{1000000000}$	0.321232121	$\frac{221}{256}$	0.86328125
$(1^2, -2, 1, -2, 3, -2, 1, 3, -2)$	250	$\frac{231232121}{1000000000}$	0.231232121	$\frac{149}{256}$	0.58203125
$(1^2, -2, 1, -2, -3, 2, 1, -3, -2)$	251	$\frac{231232121}{1000000000}$	0.231232121	$\frac{205}{256}$	0.80078125
$(1^2, -2, 1, 3, 2, -1, 2^2, 3)$	252	$\frac{322123121}{1000000000}$	0.322123121	$\frac{17}{256}$	0.06640625
$(1^2, -2, 1, 3, -2, 1, -2^2, 3)$	253	$\frac{322123121}{1000000000}$	0.322123121	$\frac{105}{256}$	0.41015625
$(1^2, -2, 1, 3, -2, 1, -2, 3, -2)$	254	$\frac{232123121}{1000000000}$	0.232123121	$\frac{169}{256}$	0.66015625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^2,-2,1,3,-2,1,3,-2^2)$	255	$\frac{223123121}{1000000000}$	0.223123121	$\frac{201}{256}$	0.78515625
$(1^2,2,-1,3,2^2,1,3,-2)$	256	$\frac{231223121}{1000000000}$	0.231223121	$\frac{65}{128}$	0.5078125
$(1^2,2,-1,-3,2^2,1,-3,-2)$	$\times$	$\frac{231223121}{1000000000}$	0.231223121	$\frac{99}{128}$	0.7734375
$(1^2,-2,1,3,2^2,1,3,-2)$	257	$\frac{231223121}{1000000000}$	0.231223121	$\frac{129}{256}$	0.50390625
$(1^2,-2,1,3,-2^2,1,3,-2)$	258	$\frac{231223121}{1000000000}$	0.231223121	$\frac{153}{256}$	0.59765625
$(1^2,-2,1,-3,2^2,1,-3,-2)$	259	$\frac{231223121}{1000000000}$	0.231223121	$\frac{197}{256}$	0.76953125
$(1^2,2^2,1^2,2,-3,2,-3)$	$\times$	$\frac{323211221}{1000000000}$	0.323211221	$\frac{5}{8}$	0.625
$(1^2,-2^2,1^2,-2,3,-2,3)$	260	$\frac{323211221}{1000000000}$	0.323211221	$\frac{83}{256}$	0.32421875
$(1^2,-2^2,1^2,-2,-3,2,-3)$	261	$\frac{323211221}{1000000000}$	0.323211221	$\frac{179}{256}$	0.69921875
$(1^2,2^2,1^2,3,2^2,3)$	262	$\frac{322311221}{1000000000}$	0.322311221	0	0
$(1^2,2^2,1^2,3,-2^2,3)$	$\times$	$\frac{322311221}{1000000000}$	0.322311221	$\frac{3}{8}$	0.375
$(1^2,2^2,1^2,-3,2^2,-3)$	$\times$	$\frac{322311221}{1000000000}$	0.322311221	$\frac{9}{16}$	0.5625
$(1^2,-2^2,1^2,3,-2^2,3)$	$\times$	$\frac{322311221}{1000000000}$	0.322311221	$\frac{99}{256}$	0.38671875
$(1,-2,1,-2,1,-2,1,3,-2,3)$	263	$\frac{80780303}{250000000}$	0.323121212	$\frac{149}{512}$	0.291015625
$(1,-2,1,-2,1,-2,1,-3,2,-3)$	264	$\frac{80780303}{250000000}$	0.323121212	$\frac{341}{512}$	0.666015625
$(1,2,-1,2,1,3,-2,1,-2,3)$	265	$\frac{80307803}{250000000}$	0.321231212	$\frac{81}{256}$	0.31640625
$(1,2,-1,2,1,-3,-2,1,-2,-3)$	266	$\frac{80307803}{250000000}$	0.321231212	$\frac{217}{256}$	0.84765625
$(1,-2,1,-2,1,3,-2,1,-2,3)$	267	$\frac{80307803}{250000000}$	0.321231212	$\frac{165}{512}$	0.322265625
$(1,-2,1,-2,1,-3,-2,1,-2,-3)$	268	$\frac{80307803}{250000000}$	0.321231212	$\frac{437}{512}$	0.853515625
$(1^3,-2,1,-2,3,-2,3^2)$	269	$\frac{332321211}{1000000000}$	0.332321211	$\frac{21}{128}$	0.1640625
$(1^3,2,-1,-3,2^2,-3^2)$	270	$\frac{332231211}{1000000000}$	0.332231211	$\frac{51}{64}$	0.796875
$(1^3,-2,1,3,-2^2,3^2)$	271	$\frac{332231211}{1000000000}$	0.332231211	$\frac{25}{128}$	0.1953125
$(1^3,2,-1,-3,2,-3,2,-3)$	272	$\frac{323231211}{1000000000}$	0.323231211	$\frac{43}{64}$	0.671875
$(1^3,-2,1,3,-2,3,-2,3)$	273	$\frac{323231211}{1000000000}$	0.323231211	$\frac{41}{128}$	0.3203125
$(1^2,2,1^2,2,-3,2,-3^2)$	$\times$	$\frac{332321121}{1000000000}$	0.332321121	$\frac{13}{16}$	0.8125
$(1^2,2,-1^2,2,-3,2,-3^2)$	$\times$	$\frac{332321121}{1000000000}$	0.332321121	$\frac{107}{128}$	0.8359375
$(1^2,-2,1^2,-2,3,-2,3^2)$	$\times$	$\frac{332321121}{1000000000}$	0.332321121	$\frac{41}{256}$	0.16015625
$(1^2,2,1^2,-3,2^2,-3^2)$	$\times$	$\frac{332231121}{1000000000}$	0.332231121	$\frac{25}{32}$	0.78125
$(1^2,2,-1^2,-3,2^2,-3^2)$	$\times$	$\frac{332231121}{1000000000}$	0.332231121	$\frac{103}{128}$	0.8046875
$(1^2,-2,1^2,3,-2^2,3^2)$	$\times$	$\frac{332231121}{1000000000}$	0.332231121	$\frac{49}{256}$	0.19140625
$(1^2,2,1^2,-3,2,-3,2,-3)$	$\times$	$\frac{323231121}{1000000000}$	0.323231121	$\frac{21}{32}$	0.65625
$(1^2,-2,1^2,3,-2,3,-2,3)$	$\times$	$\frac{323231121}{1000000000}$	0.323231121	$\frac{81}{256}$	0.31640625
$(1^2,-2,1,-2,1,3,-2,3^2)$	274	$\frac{332312121}{1000000000}$	0.332312121	$\frac{37}{256}$	0.14453125
$(1^2,-2,1,3,-2,1,3,-2,3)$	275	$\frac{323123121}{1000000000}$	0.323123121	$\frac{73}{256}$	0.28515625
$(1^2,-2,1,3,-2,1,3^2,-2)$	276	$\frac{233123121}{1000000000}$	0.233123121	$\frac{137}{256}$	0.53515625
$(1^3,-2,1,3,-2,3^3)$	277	$\frac{333231211}{1000000000}$	0.333231211	$\frac{9}{128}$	0.0703125
$(1^3,-2,1,3^2,-2,3^2)$	278	$\frac{332331211}{1000000000}$	0.332331211	$\frac{17}{128}$	0.1328125
$(1^2,-2,1^2,3^2,-2,3^2)$	279	$\frac{332331121}{1000000000}$	0.332331121	$\frac{33}{256}$	0.12890625
$(1^3,2,-1,-3,2,4,-3,4)$	280	$\frac{434231211}{1000000000}$	0.434231211	$\frac{19}{64}$	0.296875
$(1^3,-2,1,3,-2,-4,3,-4)$	281	$\frac{434231211}{1000000000}$	0.434231211	$\frac{89}{128}$	0.6953125
$(1^2,2,1^2,-3,2,4,-3,4)$	$\times$	$\frac{434231121}{1000000000}$	0.434231121	$\frac{9}{32}$	0.28125
$(1^2,2,-1^2,-3,2,4,-3,4)$	$\times$	$\frac{434231121}{1000000000}$	0.434231121	$\frac{39}{128}$	0.3046875
$(1^2,-2,1^2,3,-2,-4,3,-4)$	282	$\frac{434231121}{1000000000}$	0.434231121	$\frac{177}{256}$	0.69140625
$(1^2,-2,1,-2^3,3,-2,3)$	283	$\frac{323222121}{1000000000}$	0.323222121	$\frac{93}{256}$	0.36328125
$(1^2,2,-1,2^2,-3,2^2,-3)$	$\times$	$\frac{322322121}{1000000000}$	0.322322121	$\frac{73}{128}$	0.5703125

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1^2, -2, 1, -2^2, 3, -2^2, 3)$	$\times$	$\frac{322322121}{1000000000}$	0.322322121	$\frac{109}{256}$	0.42578125
$(1^2, -2, 1, 3, -2^4, 3)$	284	$\frac{322223121}{1000000000}$	0.322223121	$\frac{121}{256}$	0.47265625
$(1^2, -2^2, 1, -2^2, 3, -2, 3)$	285	$\frac{323221221}{1000000000}$	0.323221221	$\frac{91}{256}$	0.35546875
$(1^2, -2^2, 1, -2, 3, -2^2, 3)$	286	$\frac{322321221}{1000000000}$	0.322321221	$\frac{107}{256}$	0.41796875
$(1^2, 2^2, 1, 3, 2^3, 3)$	287	$\frac{322231221}{1000000000}$	0.322231221	0	0
$(1^2, 2^2, 1, 3, -2^3, 3)$	$\times$	$\frac{322231221}{1000000000}$	0.322231221	$\frac{7}{16}$	0.4375
$(1^2, 2^2, 1, -3, 2^3, -3)$	$\times$	$\frac{322231221}{1000000000}$	0.322231221	$\frac{17}{32}$	0.53125
$(1^2, 2^2, 1, -3, -2^3, -3)$	$\times$	$\frac{322231221}{1000000000}$	0.322231221	$\frac{31}{32}$	0.96875
$(1^2, -2^2, 1, 3, 2^3, 3)$	$\times$	$\frac{322231221}{1000000000}$	0.322231221	$\frac{3}{256}$	0.01171875
$(1^2, -2^2, 1, 3, -2^3, 3)$	$\times$	$\frac{322231221}{1000000000}$	0.322231221	$\frac{115}{256}$	0.44921875
$(1^2, -2^3, 1, -2, 3, -2, 3)$	288	$\frac{323212221}{1000000000}$	0.323212221	$\frac{87}{256}$	0.33984375
$(1, -2, 1, -2, 1, -2^2, 3, -2, 3)$	289	$\frac{80805303}{250000000}$	0.323221212	$\frac{181}{512}$	0.353515625
$(1, -2, 1, -2, 1, -2^2, -3, 2, -3)$	$\times$	$\frac{80805303}{250000000}$	0.323221212	$\frac{373}{512}$	0.728515625
$(1, -2, 1, -2, 1, 3, -2^3, 3)$	$\times$	$\frac{80557803}{250000000}$	0.322231212	$\frac{229}{512}$	0.447265625
$(1, -2, 1, -2, 1, -3, -2^3, -3)$	$\times$	$\frac{80557803}{250000000}$	0.322231212	$\frac{501}{512}$	0.978515625
$(1, -2, 1, -2^2, 1, 3, 2^2, 3)$	$\times$	$\frac{80578053}{250000000}$	0.322312212	$\frac{13}{512}$	0.025390625
$(1, -2, 1, -2^2, 1, 3, -2^2, 3)$	$\times$	$\frac{80578053}{250000000}$	0.322312212	$\frac{205}{512}$	0.400390625
$(1, -2, 1, -2^2, 1, -3, 2^2, -3)$	$\times$	$\frac{80578053}{250000000}$	0.322312212	$\frac{301}{512}$	0.587890625
$(1, -2, 1, -2^3, 1, 3, -2, 3)$	290	$\frac{80780553}{250000000}$	0.323122212	$\frac{157}{512}$	0.306640625
$(1, 2, -1, 2^2, 3, -2, 1, -2, 3)$	291	$\frac{80308053}{250000000}$	0.321232212	$\frac{81}{256}$	0.31640625
$(1, -2, 1, -2^2, 3, -2, 1, -2, 3)$	292	$\frac{80308053}{250000000}$	0.321232212	$\frac{173}{512}$	0.337890625
$(1, -2, 1, -2, 3, -2, 1, -2^2, 3)$	293	$\frac{80530803}{250000000}$	0.322123212	$\frac{213}{512}$	0.416015625
$(1, -2, 1, 3, -2^2, 1, -2^2, 3)$	294	$\frac{40265289}{125000000}$	0.322122312	$\frac{217}{512}$	0.423828125
$(1^2, -2, 1, -2^2, 3, -2, 3^2)$	295	$\frac{332322121}{1000000000}$	0.332322121	$\frac{45}{256}$	0.17578125
$(1^2, -2, 1, -2, 3, -2^2, 3^2)$	296	$\frac{332232121}{1000000000}$	0.332232121	$\frac{53}{256}$	0.20703125
$(1^2, -2, 1, -2, 3, -2, 3, -2, 3)$	297	$\frac{323232121}{1000000000}$	0.323232121	$\frac{85}{256}$	0.33203125
$(1^2, -2, 1, 3, 2^3, 3^2)$	298	$\frac{332223121}{1000000000}$	0.332223121	$\frac{1}{256}$	0.00390625
$(1^2, -2, 1, 3, -2^3, 3^2)$	299	$\frac{332223121}{1000000000}$	0.332223121	$\frac{57}{256}$	0.22265625
$(1^2, 2^2, 1, 3, 2^2, 3^2)$	300	$\frac{332231221}{1000000000}$	0.332231221	0	0
$(1^2, 2^2, 1, 3, -2^2, 3^2)$	301	$\frac{332231221}{1000000000}$	0.332231221	$\frac{3}{16}$	0.1875
$(1^2, 2^2, 1, -3, 2^2, -3^2)$	302	$\frac{332231221}{1000000000}$	0.332231221	$\frac{25}{32}$	0.78125
$(1^2, 2^2, 1, -3, -2^2, -3^2)$	303	$\frac{332231221}{1000000000}$	0.332231221	$\frac{31}{32}$	0.96875
$(1^2, -2^2, 1, 3, -2^2, 3^2)$	304	$\frac{332231221}{1000000000}$	0.332231221	$\frac{51}{256}$	0.19921875
$(1^2, -2^2, 1, 3, -2, 3, -2, 3)$	$\times$	$\frac{323231221}{1000000000}$	0.323231221	$\frac{83}{256}$	0.32421875
$(1^2, -2, 3, -2, 1, -2, 3^2, -2)$	305	$\frac{233212321}{1000000000}$	0.233212321	$\frac{149}{256}$	0.58203125
$(1, -2, 1, -2, 1, 3, -2, 3, -2, 3)$	$\times$	$\frac{80807803}{250000000}$	0.323231212	$\frac{165}{512}$	0.322265625
$(1, -2, 1, -2, 1, -3, 2, -3, 2, -3)$	306	$\frac{80807803}{250000000}$	0.323231212	$\frac{341}{512}$	0.666015625
$(1, -2, 1, -2, 3, -2, 1, 3, -2, 3)$	307	$\frac{80780803}{250000000}$	0.323123212	$\frac{149}{512}$	0.291015625
$(1^2, 2, -1, 2, 3, -2, -4, 3, -4)$	308	$\frac{434232121}{1000000000}$	0.434232121	$\frac{89}{128}$	0.6953125
$(1^2, 2, -1, 2, -3, 2, 4, -3, 4)$	309	$\frac{434232121}{1000000000}$	0.434232121	$\frac{37}{128}$	0.2890625
$(1^2, -2, 1, -2, 3, -2, -4, 3, -4)$	310	$\frac{434232121}{1000000000}$	0.434232121	$\frac{181}{256}$	0.70703125
$(1^2, -2, 1, -2, -3, 2, 4, -3, 4)$	311	$\frac{434232121}{1000000000}$	0.434232121	$\frac{77}{256}$	0.30078125
$(1^2, -2, 1, 3, 2^2, -4, 3, -4)$	312	$\frac{434223121}{1000000000}$	0.434223121	$\frac{161}{256}$	0.62890625
$(1^2, -2, 1, 3, -2^2, -4, 3, -4)$	313	$\frac{434223121}{1000000000}$	0.434223121	$\frac{185}{256}$	0.72265625
$(1, -2, 1, -2, 1, 3, -2, -4, 3, -4)$	314	$\frac{108557803}{250000000}$	0.434231212	$\frac{357}{512}$	0.697265625

$\mathbf{x}$	M	r	$float(r)$	θ	$float(\theta)$
$(1,-2,1,-2,1,-3,2,4,-3,4)$	$\times$	$\frac{108557803}{250000000}$	0.434231212	$\frac{149}{512}$	0.291015625
$(1^2,2,-1,-3,2,-3,4,-3,4)$	315	$\frac{434323121}{1000000000}$	0.434323121	$\frac{43}{128}$	0.3359375
$(1^2,2,-1,-3,2,-3,-4,3,-4)$	316	$\frac{434323121}{1000000000}$	0.434323121	$\frac{91}{128}$	0.7109375
$(1^2,-2,1,3,-2,3,4,-3,4)$	317	$\frac{434323121}{1000000000}$	0.434323121	$\frac{73}{256}$	0.28515625
$(1^2,-2,1,3,-2,3,-4,3,-4)$	318	$\frac{434323121}{1000000000}$	0.434323121	$\frac{169}{256}$	0.66015625
$(1^2,-2,1,3,-2,-4,3^2,-4)$	$\times$	$\frac{433423121}{1000000000}$	0.433423121	$\frac{153}{256}$	0.59765625
$(1^2,-2,1,3,-2,-4,3,-4^2)$	319	$\frac{443423121}{1000000000}$	0.443423121	$\frac{217}{256}$	0.84765625
$(1,-2,1,-2^4,3,-2,3)$	320	$\frac{80805553}{250000000}$	0.323222212	$\frac{189}{512}$	0.369140625
$(1,-2,1,-2^3,3,-2^2,3)$	$\times$	$\frac{80580553}{250000000}$	0.322322212	$\frac{221}{512}$	0.431640625
$(1,-2,1,-2,3,-2^4,3)$	321	$\frac{80555803}{250000000}$	0.322223212	$\frac{245}{512}$	0.478515625
$(1,-2,1,3,-2^5,3)$	322	$\frac{40277789}{125000000}$	0.322222312	$\frac{249}{512}$	0.486328125
$(1,-2^2,1,-2,3,-2^3,3)$	$\times$	$\frac{161116061}{500000000}$	0.322232122	$\frac{235}{512}$	0.458984375
$(1,-2^2,1,3,-2^4,3)$	$\times$	$\frac{161111561}{500000000}$	0.322223122	$\frac{243}{512}$	0.474609375
$(1,2^3,1,-3,2^3,-3)$	323	$\frac{161115611}{500000000}$	0.322231222	$\frac{17}{32}$	0.53125
$(1,-2^3,1,3,-2^3,3)$	324	$\frac{161115611}{500000000}$	0.322231222	$\frac{231}{512}$	0.451171875
$(1,-2^3,3,-2,1,-2,3,-2)$	325	$\frac{116061611}{500000000}$	0.232123222	$\frac{343}{512}$	0.669921875
$(1,-2^2,3,-2,1,-2^2,3,-2)$	326	$\frac{116106161}{500000000}$	0.232212322	$\frac{363}{512}$	0.708984375
$(1,-2^2,3,-2,1,-2,3,-2^2)$	327	$\frac{111606161}{500000000}$	0.223212322	$\frac{427}{512}$	0.833984375
$(1,-2,1,-2^2,3,-2,-4,3,-4)$	328	$\frac{108558053}{250000000}$	0.434232212	$\frac{365}{512}$	0.712890625
$(1,-2,1,-2^2,-3,2,4,-3,4)$	329	$\frac{108558053}{250000000}$	0.434232212	$\frac{157}{512}$	0.306640625
$(1,-2,1,-2,3,2^2,-4,3,-4)$	330	$\frac{108555803}{250000000}$	0.434223212	$\frac{325}{512}$	0.634765625
$(1,-2,1,-2,3,-2^2,-4,3,-4)$	331	$\frac{108555803}{250000000}$	0.434223212	$\frac{373}{512}$	0.728515625
$(1,-2,1,-2,-3,2^2,4,-3,4)$	332	$\frac{108555803}{250000000}$	0.434223212	$\frac{141}{512}$	0.275390625
$(1,-2,1,-2,-3,-2^2,4,-3,4)$	333	$\frac{108555803}{250000000}$	0.434223212	$\frac{189}{512}$	0.369140625
$(1,-2,1,3,2^3,-4,3,-4)$	334	$\frac{54277789}{125000000}$	0.434222312	$\frac{321}{512}$	0.626953125
$(1,-2,1,3,-2^3,-4,3,-4)$	335	$\frac{54277789}{125000000}$	0.434222312	$\frac{377}{512}$	0.736328125
$(1,-2^2,1,-2,3,-2,-4,3,-4)$	336	$\frac{217116061}{500000000}$	0.434232122	$\frac{363}{512}$	0.708984375
$(1,-2^2,1,-2,-3,2,4,-3,4)$	337	$\frac{217116061}{500000000}$	0.434232122	$\frac{155}{512}$	0.302734375
$(1,2^2,1,3,-2^2,-4,3,-4)$	338	$\frac{217111561}{500000000}$	0.434223122	$\frac{23}{32}$	0.71875
$(1,-2^2,1,3,-2^2,-4,3,-4)$	339	$\frac{217111561}{500000000}$	0.434223122	$\frac{371}{512}$	0.724609375
$(1,-2^2,1,-3,-2^2,4,-3,4)$	340	$\frac{217111561}{500000000}$	0.434223122	$\frac{187}{512}$	0.365234375
$(1,-2,3,-2,1,-2,-4,3,-2,-4)$	341	$\frac{26463827}{62500000}$	0.423421232	$\frac{437}{512}$	0.853515625
$(1,-2,-3,-2,1,-2,4,-3,-2,4)$	$\times$	$\frac{26463827}{62500000}$	0.423421232	$\frac{215}{512}$	0.419921875
$(1,-2,1,-2,3,-2,3,-4,3,-4)$	342	$\frac{108580803}{250000000}$	0.434323212	$\frac{341}{512}$	0.666015625
$(1,-2,1,-2,3,-2,-4,3^2,-4)$	$\times$	$\frac{108355803}{250000000}$	0.433423212	$\frac{309}{512}$	0.603515625
$(1,-2,1,3,-2^2,-4,3^2,-4)$	343	$\frac{54177789}{125000000}$	0.433422312	$\frac{313}{512}$	0.611328125
$(1,-2,1,3,-2,3,-2,-4,3,-4)$	344	$\frac{54279039}{125000000}$	0.434232312	$\frac{361}{512}$	0.705078125
$(1,-2,1,3,-2,-4,3,-2,-4,3)$	345	$\frac{42792789}{125000000}$	0.342342312	$\frac{217}{512}$	0.423828125
$(1,-2,1,3,-2,-4,3,5,-4,5)$	346	$\frac{5042789}{125000000}$	0.040342312	$\frac{153}{512}$	0.298828125

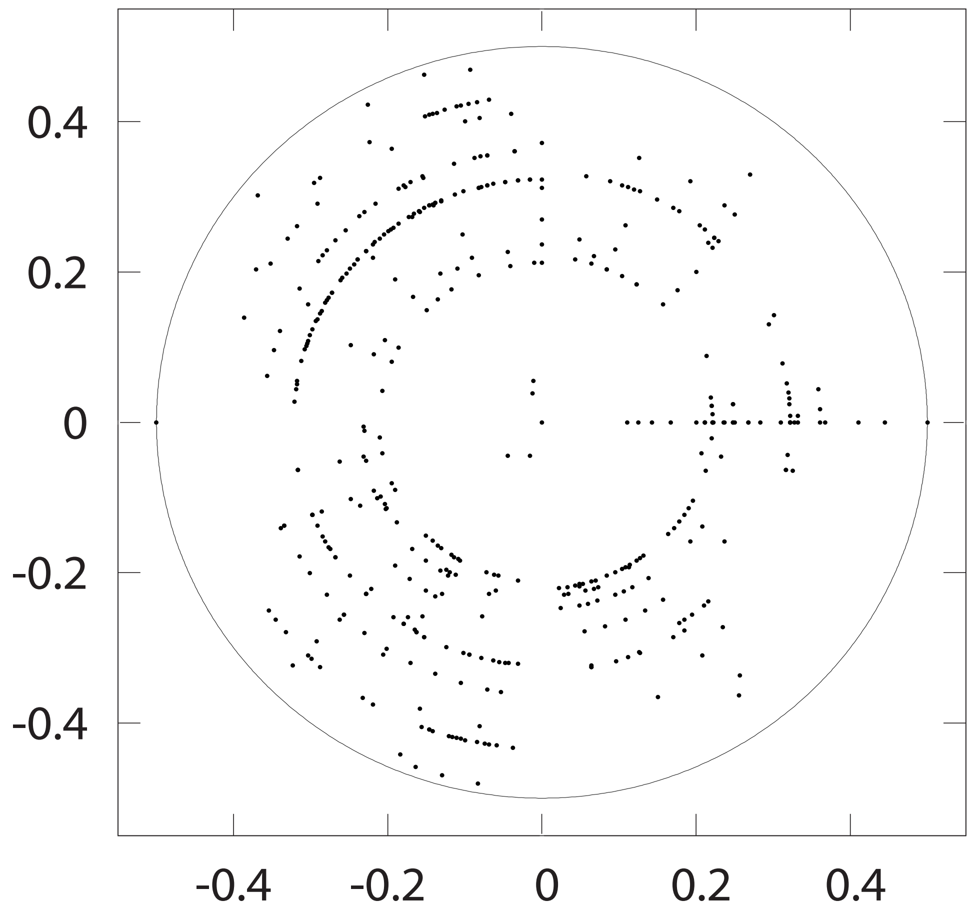

Figure 1: Distributive Situation of prime links of lengths up to 10

The distributive situation of the characteristic quantities of the prime links and the 3-manifolds of lengths up to 10 drawn in the complex number plane are respectively given in Fig. 1 and Fig. 2.

4. Constructing a characteristic holomorphic function for the ADelta set

An *ADelta subset* is a subset $\mathbb{S}$ of the ADelta set $A\Delta$ containing the lattice points $\emptyset$ and $\mathbf{0}$. For every integer $n \geqq 0$, the *n-fragment* of $\mathbb{S}$ is defined by

$$\mathbb{S}^{(n)} = \{\mathbf{x} \in \mathbb{S} |\, \ell(\mathbf{x}) \leqq n\}.$$

The n-fragment $\mathbb{S}^{(n)}$ is always a finite set. For complex numbers a, z with either

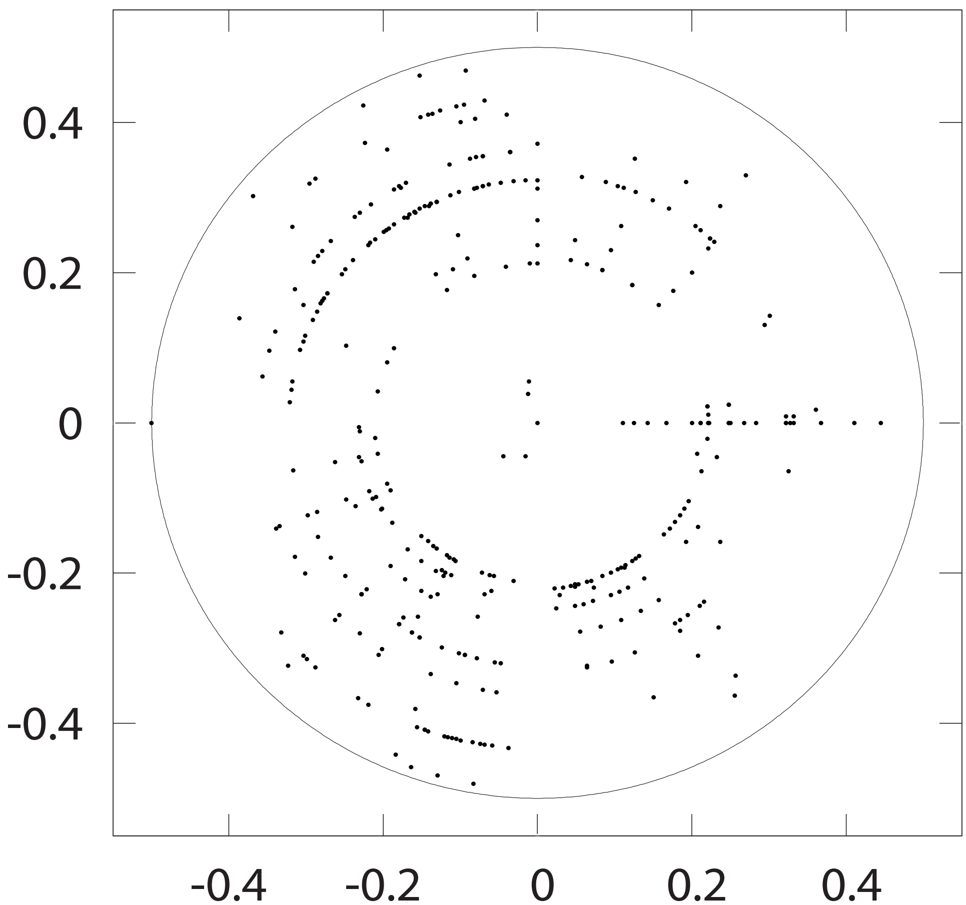

Figure 2: Distributive Situation of 3-manifolds of lengths up to 10

$|a| \leqq 1$ and $|z| < 1$ or $|a| < 1$ and $|z| \leqq 1$, the linear fraction

$$B(z; a) = \frac{z - a}{1 - \bar{a}z}$$

is considered, where $\bar{a}$ denotes the complex conjugation of a. Then we have $|B(z; a)| < 1$ for any z, a with $|a| < 1$ and $|z| < 1$, because

$$1 - |B(z; a)|^2 = \frac{(1 - |z|^2)(1 - |a|^2)}{|1 - \bar{a}z|^2}.$$

Further, if $|a| = 1$ or $|z| = 1$, then we have that $|B(z; a)| = 1$.

Noting that the characteristic quantities of $\emptyset$, $\mathbf{0}$ and 1^2 are not definite values as it is explained in Remark 2.2, we put the following definition for any $\mathbf{x} \in A\Delta$:

$$Q_{\mathbf{x}}(z) = \begin{cases} B(z; q(\mathbf{x})) & (\ell(\mathbf{x}) \geqq 3) \\ B(z; 1) = -1 & (\mathbf{x} = 1^2) \\ B(z; -1) = 1 & (\mathbf{x} = \emptyset, \mathbf{0}) \end{cases}$$

For every n-fragment $\mathbb{S}^{(n)}$ of an ADelta subset $\mathbb{S}$, let

$$Q_{\mathbb{S}}^{(n)}(z) = \prod_{\mathbf{x} \in \mathbb{S}^{(n)}} Q_{\mathbf{x}}(z).$$

The function $Q_{\mathbb{S}}^{(n)}(z)$ is called a finite *Blaschke product* whose zero's are precisely the quantities $q(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^{(n)}$ except $\emptyset$, $\mathbf{0}$ and 1^2 (cf. W. Blaschke [2]).

By the assumption of the set $\mathbb{S}$, we have

$$Q_{\mathbb{S}}^{(0)}(z) = Q_{\mathbb{S}}^{(1)}(z) = 1.$$

It is noted that we have $Q_{\mathbb{S}}^{(2)}(z) = -1$ or 1 respectively according to whether the lattice point 1^2 belongs to $\mathbb{S}$ or not.

For example, when we take $\mathbb{S} = \mathbb{L}^p$, the calculations of $Q_{\mathbb{L}^p}^{(n)}(z)$ for $n = 0, 1, 2, 3, 4, 5$ are made as follows:

$$\begin{aligned}
Q_{\mathbb{L}^p}^{(0)}(z) &= 1, \\
Q_{\mathbb{L}^p}^{(1)}(z) &= 1, \\
Q_{\mathbb{L}^p}^{(2)}(z) &= -1, \\
Q_{\mathbb{L}^p}^{(3)}(z) &= -Q_{1^3}(z) = -B(z; \frac{4}{9}), \\
Q_{\mathbb{L}^p}^{(4)}(z) &= -Q_{1^3}(z) Q_{1^4}(z) Q_{(1,-2,1,-2)}(z) = -B(z; \frac{4}{9}) B(z; \frac{21}{64}) B(z; \frac{\exp(\frac{5\pi \mathbf{i}}{4})}{16}), \\
Q_{\mathbb{L}^p}^{(5)}(z) &= -Q_{1^3}(z) Q_{1^4}(z) Q_{(1,-2,1,-2)}(z) Q_{1^5}(z) Q_{(1^2,-2,1,-2)}(z) \\
&= -B(z; \frac{4}{9}) B(z; \frac{21}{64}) B(z; \frac{\exp(\frac{5\pi \mathbf{i}}{4})}{16}) B(z; \frac{156}{625}) B(z; \frac{286}{625} \exp(\frac{5\pi \mathbf{i}}{4})).
\end{aligned}$$

We obtain the following theorem.

Theorem 4.1 For every ADelta subset $\mathbb{S}$, the series function

$$Q_{\mathbb{S}}(z) = \sum_{n=0}^{+\infty} Q_{\mathbb{S}}^{(n)}(z) z^n$$

is a holomorphic function with the unit open disk $D = \{z \in \mathbb{C} | \, |z| < 1\}$ as the convergence domain and characteristic defined for the set $\mathbb{S}$.

It is noted that the series function $Q_{\mathbb{S}}(z)$ does not converge for any z with $|z| = 1$. This is because

$$\lim_{n\to+\infty} |Q_{\mathbb{S}}^{(n)}(z) \cdot (z)^n| = 1 \neq 0$$

for any z with $|z| = 1$. The function $Q_{\mathbb{S}}(z)$ is called the *characteristic quantity function* defined for the ADelta subset $\mathbb{S}$. For example, for $\mathbb{S} = \{\emptyset, \mathbf{0}\}$, we have

$$Q_{\mathbb{S}}(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}.$$

For $\mathbb{S} = \{\emptyset, \mathbf{0}, 1^2\}$, we have

$$Q_{\mathbb{S}}(z) = 1 + z - (z^2 + z^3 + z^4 + \dots) = 1 + z - \frac{z^2}{1-z}.$$

For a finite set $\mathbb{S}$ with the maximal length n,

$$Q_{\mathbb{S}}(z) = \sum_{i=0}^{n-1} Q_{\mathbb{S}}^{(i)}(z) z^i + Q_{\mathbb{S}}^{(n)}(z) \frac{z^n}{1-z}.$$

By examining the constructions of the present characteristic quantity function and the characteristic genus function in [6], it is noted that if an ADelta subset $\mathbb{S} \subset A\Delta$ consists of $\emptyset, \mathbf{0}$ and some lattice points $\mathbf{x} \in A\Delta$ with

$$\max |\mathbf{x}| < \frac{\ell(\mathbf{x})}{2},$$

then $\mathbb{S}$ is also a subset of the PDelta set $P\Delta$ defining the characteristic genus function $G_{\mathbb{S}}(t)$ and then we have

$$Q_{\mathbb{S}}(t) = G_{\mathbb{S}}(t)$$

for any $t \in (-1, 1)$.

Proof of Theorem 4.1. Since $|Q_{\mathbb{S}}^{(n)}(z)| \leqq 1$ for any $n \geqq 2$, we have

$$|Q_{\mathbb{S}}(z)| \leqq \sum_{n=0}^{+\infty} |z|^n = \frac{1}{1-|z|}.$$

This means that the series $Q_{\mathbb{S}}(z)$ defined on D is uniformly convergent in the wide sense. Using that the function $Q_{\mathbb{S}}^{(n)}(z)$ ($z \in D$) is uniformly convergent in the wide sense, we see from the Weierstrass double series theorem that the series function $Q_{\mathbb{S}}(z)$ is a holomorphic function defined on D. To see that the function $Q_{\mathbb{S}}(z)$ is characteristic, it suffices to see by the induction on $n \geqq 2$ that the set $\mathbb{S}^{(n)} \setminus \mathbb{S}^{(n-1)}$ except $\emptyset$ and $\mathbf{0}$ is determined by the function $Q_{\mathbb{S}}(z)$. Let $n = 2$. According to whether

1^2 is in $\mathbb{S}$ or not, the second derivative $\frac{d^2}{dz^2}Q_{\mathbb{S}}(0)$ is -2 or 2, respectively. Thus, $\mathbb{S}^{(2)} \setminus \mathbb{S}^{(1)}$ is determined by the function $Q_{\mathbb{S}}(z)$. Assume that the set $\mathbb{S}^{(m)} \setminus \mathbb{S}^{(m-1)}$ except $\emptyset$ and $\mathbf{0}$ is determined by the function $Q_{\mathbb{S}}(z)$ for all $m \leqq n-1$. Let

$$\bar{Q}_{\mathbb{S}}^{(n)}(z) = Q_{\mathbb{S}}(z) - \sum_{i=0}^{n-1} Q_{\mathbb{S}}^{(i)}(z) z^i.$$

The function $\bar{Q}_{\mathbb{S}}^{(n)}(z)$ has the following splitting form:

$$\bar{Q}_{\mathbb{S}}^{(n)}(z) = Q_{\mathbb{S}}^{(n)}(z)\tilde{Q}(z)z^n,$$

where

$$\tilde{Q}(z) = 1 + \tilde{Q}_{\mathbb{S}}^{(n+1)}(z)z + \tilde{Q}_{\mathbb{S}}^{(n+2)}(z)z^2 + \tilde{Q}_{\mathbb{S}}^{(n+3)}(z)z^3 + \dots$$

for some finite Blaschke products $\tilde{Q}_{\mathbb{S}}^{(n+i)}(z)$ with

$$Q_{\mathbb{S}}^{(n)}(z)\tilde{Q}_{\mathbb{S}}^{(n+i)}(z) = Q_{\mathbb{S}}^{(n+i)}(z)$$

for all i ($i = 1, 2, 3, \dots$). Let $\frac{1}{2}D = \{z \in D | \, |z| < \frac{1}{2}\}$ be the half-unit open disk. The function $\tilde{Q}(z)$ is seen to have no zero's in $\frac{1}{2}D$. In fact,

$$|\tilde{Q}(z)| \geqq 1 - \sum_{i=1}^{+\infty} |z|^i = \frac{1-2|z|}{1-|z|} > 0$$

for any z with $|z| < \frac{1}{2}$. Thus, the characteristic quantities $q(\mathbf{x})$ for all lattice points $\mathbf{x} \in \mathbb{S}^{(n)}$ except $\emptyset$, $\mathbf{0}$ and 1^2 are characterized by the zero's of the function $\bar{Q}_{\mathbb{S}}^{(n)}(z)$ defined on $\frac{1}{2}D \setminus \{0\}$, so that the set $\mathbb{S}^{(n)} \setminus \mathbb{S}^{(n-1)}$ is determined by the function $Q_{\mathbb{S}}(z)$. □

For the subset $\mathbb{S} = \sigma(\mathbb{L}^p)$, we denote $Q_{\mathbb{S}}^{(n)}(z)$ and $Q_{\mathbb{S}}(z)$ by $Q_{\mathbb{L}^p}^{(n)}(z)$ and $Q_{\mathbb{L}^p}(z)$, respectively. The following corollary is direct from Theorem 4.1.

Corollary 4.2 The series function

$$\begin{aligned} Q_{\mathbb{L}^p}(z) &= \sum_{n=0}^{+\infty} Q_{\mathbb{L}^p}^{(n)}(z) z^n \\ &= 1 + z - z^2 - B(z; \frac{4}{9})z^3 - B(z; \frac{4}{9})B(z; \frac{21}{64})B(z; \frac{\exp(\frac{5\pi\mathbf{i}}{4})}{16})z^4 \\ &\qquad - B(z; \frac{4}{9})B(z; \frac{21}{64})B(z; \frac{\exp(\frac{5\pi\mathbf{i}}{4})}{16})B(z; \frac{156}{625})B(z; \frac{286}{625}\exp(\frac{5\pi\mathbf{i}}{4}))z^5 + \dots \end{aligned}$$

is a holomorphic function with the unit open disk $D = \{z | \, |z| < 1\}$ as the convergence domain and characteristic defined for the prime link set $\mathbb{L}^p$.

For example, let $\mathbb{L}(2,*)$ be the set of $(2,n)$-torus links regarding the $(2,0)$-torus link as the empty knot ϕ. Since

$$\sigma(\mathbb{L}(2,*)) = \{1^n |\, n = 0, 1, 2, 3, \dots\},$$

where $1^0 = \emptyset$, $1^1 = 0$ and $\tau(1^n) = \frac{1}{n-1} - \frac{1}{n^n - n^{n-1}}$ for $\geqq 3$, we have:

$$Q_{\mathbb{L}(2,*)}(z) = 1 + z - z^2 - \sum_{n=3}^{+\infty} \left(\prod_{k=3}^{n} B\left(z; \frac{1}{k-1} - \frac{1}{k^k - k^{k-1}}\right)\right) z^n \quad \text{(see [6])}.$$

For the subset $\mathbb{S} = \sigma_\alpha(\mathbb{M})$, we denote $Q_{\mathbb{S}}^{(n)}(z)$ and $Q_{\mathbb{S}}(z)$ by $Q_{\mathbb{M}}^{(n)}(z)$ and $Q_{\mathbb{M}}(z)$, respectively. Noting that the lattice point 1^2 is excluded from $\sigma(\mathbb{M})$ (by the reason that the empty lattice point $\emptyset$ is introduced), we have the following corollary obtained from Theorem 4.1.

Corollary 4.3 The series function

$$\begin{aligned} Q_{\mathbb{M}}(z) &= \sum_{n=0}^{+\infty} Q_{\mathbb{M}}^{(n)}(z) z^n \\ &= 1 + z + z^2 + B(z; \frac{4}{9}) z^3 + B(z; \frac{4}{9}) B(z; \frac{21}{64}) B(z; \frac{\exp(\frac{5\pi \mathbf{i}}{4})}{16}) z^4 \\ &\qquad + B(z; \frac{4}{9}) B(z; \frac{21}{64}) B(z; \frac{\exp(\frac{5\pi \mathbf{i}}{4})}{16}) B(z; \frac{156}{625}) B(z; \frac{286}{625} \exp(\frac{5\pi \mathbf{i}}{4})) z^5 + \dots. \end{aligned}$$

is a holomorphic function with the unit open disk $D = \{z |\, |z| < 1\}$ as the convergence domain and characteristic defined for the 3-manifold set $\mathbb{M}$.

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