ON THE SURFACE-LINK GROUPS

Akio KAWAUCHI

Department of Mathematics, Osaka City University

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@sci.osaka-cu.ac.jp

ABSTRACT

The set of the fundamental groups of n-dimensional manifold-links in S^{n+2} for $n > 2$ is equal to the set of the fundamental groups of surface-links in S^4. We consider the subset $\mathbb{G}^r_g(H)$ of this set consisting of the fundamental groups of r-component, total genus g surface-links with $H_2(G) \cong H$. We show that the set $\mathbb{G}^r_g(H)$ is a non-empty proper subset of $\mathbb{G}^r_{g+1}(H)$ for every integer $g \geq 0$ and every abelian group H generated by $2g$ elements. We also determine the set $\mathbb{G}^r_g(H)$ to which the fundamental group of every classical link belongs, and investigate the set $\mathbb{G}^r_g(H)$ to which the fundamental group of every virtual link belongs.

Keywords: Manifold-link group; Surface-link group; Classical link group; Virtual link group

1. Manifold-link groups

Let M be a closed oriented n-manifold with r components. An *M-link* (or an *M-knot* if $r = 1$) is the image of a locally-flat PL embedding $M \to S^{n+2}$. We are interested in the (fundamental) group of L: $G = G(L) = \pi_1(S^{n+2} \setminus L)$. Let $m = m(L) = \{m_1, m_2, \cdots, m_r\}$ be the meridian basis of L in $H_1(G) = H_1(S^{n+2} \setminus L) = Z^r$. We consider the set

$$\mathbb{G}^r[n] = \{G(L) \,|\, L \,\text{is an}\, M\text{-link}, \forall M\},$$

where we consider $G(L) = G(L')$ if there is an isomorphism $G(L) \to G(L')$ sending $m(L)$ to $m(L')$. Let $n = 2$. A *ribbon M-link* is an M-link obtained from a trivial 2-sphere link by surgeries along mutually disjoint embedded 1-handles in S^4 (see [11, p.52]). Let

$$R\mathbb{G}^r[2] = \{G(L) \,|\, L \,\text{is a ribbon}\, M\text{-link}, \forall M\}.$$

Then we have the following theorem:

Theorem 1.1. $R\mathbb{G}^r[2] = \mathbb{G}^r[2] = \mathbb{G}^r[3] = \mathbb{G}^r[4] = \cdots$.

Proof. The inclusion $\mathbb{G}^r[n] \subset \mathbb{G}^r[n+1]$ for every $n \geqq 2$ is proved by a spinning construction. In fact, given the group $G(L) \in \mathbb{G}^r[n]$ of an M-link L, then we choose an $(n+2)$-ball $B^{n+2} \subset S^{n+2}$ containing L and construct an $M \times S^1$-link

$$L^+ = L \times S^1 \subset B^{n+2} \times S^1 \cup \partial B^{n+2} \times B^2 = S^{n+3}.$$

Then we have $G(L) = G(L^+)$ in $\mathbb{G}^r[n+1]$. In [16], T. Yajima shows that if a group G has a Wirtinger presenation $\langle x_1, x_2, \cdots, x_k \,|\, r_1, r_2, \cdots, r_{k'} \rangle$ of deficiency $s = k - k'$ such that $r_j = w_j x_{u(j)} w_j^{-1} x_{v(j)}^{-1}$ for some generators $x_{u(j)}, x_{v(j)}$ and a word w_j on x_i $(i = 1, 2, \ldots, k)$, and a basis m for $H_1(G) \cong Z^r$ is given in x_i $(i = 1, 2, \ldots, k)$, then there is a ribbon F^r_g-link L with $g = r - s$ such that $G(L) = G$ and $m(L) = m$. Since S. Kamada shows in [3] that every $G(L) \in \mathbb{G}^r[n]$ has a Wirtinger presentation with $m(L)$ in the generators, we have $\mathbb{G}^r[n] \subset R\mathbb{G}^r[2]$. □

2. Grading the surface-link groups

Let $M = F^r_g = F^r_{g_1, g_2, \cdots, g_r}$ be a closed oriented 2-manifold with r components F_i of genus $g(F_i) = g_i$ $(i = 1, 2, \cdots, r)$, where $g = g_1 + g_2 + \cdots + g_r$ is the total genus of M. This M-link is called an $F^r_g(= F^r_{g_1, g_2, \cdots, g_r})$-*link*. Let $\mathbb{G}^r_g$ (or $\mathbb{G}^r_{g_1, g_2, \cdots, g_r}$) be the set of $G(L)$ such that L is an F^r_g-link (or $F^r_{g_1, g_2, \cdots, g_r}$-link). For a finitely generated abelian group H, let $\mathbb{G}^r_g(H)$ be the set of $G \in \mathbb{G}^r_g$ with $H_2(G) \cong H$. Then the following sequence of inclusions is obtained for every H by adding a trivial handle to a surface-link in S^4:

$$\mathbb{G}^r_0(H) \subset \mathbb{G}^r_1(H) \subset \mathbb{G}^r_2(H) \subset \cdots \subset \bigcup_{g=0}^{+\infty} \mathbb{G}^r_g(H) =: \mathbb{G}^r(H).$$

Similarly, letting $R\mathbb{G}^r_g$ be the set of $G(L)$ such that L is a ribbon F^r_g-link, and $R\mathbb{G}^r_g(H)$ the set of $G \in R\mathbb{G}^r_g$ with $H_2(G) \cong H$, we obtain the following sequence:

$$R\mathbb{G}^r_0(H) \subset R\mathbb{G}^r_1(H) \subset R\mathbb{G}^r_2(H) \subset \cdots \subset \bigcup_{g=0}^{+\infty} R\mathbb{G}^r_g(H) =: R\mathbb{G}^r(H).$$

Using $R\mathbb{G}^r(H) = \mathbb{G}^r(H)$ by Theorem 1.1, we obtain the following corollary.

Corollary 2.1. $R\mathbb{G}^r_g(H) \subset \mathbb{G}^r_g(H)$, and for every $G \in \mathbb{G}^r_g(H)$, there is an integer $h \geqq 0$ such that $G \in R\mathbb{G}^r_{g+h}(H)$.

Let Λ be the Laurent polynomial ring $Z[Z] = Z[t, t^{-1}]$. For a surface-link group $G = G(L)$, the homology $H_1(\mathrm{Ker}\chi)$ for the epimorphism $\chi : G \to Z$ sending every meridian to 1 forms a finitely generated Λ-module, which we call the *Alexander module* of G or L and denote by $A(G)$ or $A(L)$. The second part of the following theorem is a consequence of studies on the Alexander modules of surface-link groups in [10].

Theorem 2.2. Let $\mu(H)$ be the minimal number of generators of H. For $2g < \mu(H)$, we have $\mathbb{G}^r_g(H) = \emptyset$. For every $2g \geqq \mu(H)$ and every $h > 0$, we have

$$\mathbb{G}^r_g(H) \setminus (\mathbb{G}^r_{g-1}(H) \cup R\mathbb{G}^r_{g+h}(H)) \neq \emptyset.$$

Since $\mathbb{G}^1_0(0)$ is the set of S^2-knot groups and $\mathbb{G}^1(0) = \cup_{g=0}^{+\infty}\mathbb{G}^1_g(0)$ is the set of S^n-knot groups for every given $n \geqq 3$ (see M. A. Kervaire [12]), a weaker result of this theorem for $r = 1$ and $H = 0$ is found in [8, p.192].

Proof. The first claim is direct by Hopf's theorem saying that there is an epimorphism $H_2(S^4 \setminus L) = Z^{2g} \to H_2(G)$ for every $G = G(L) \in \mathbb{G}^r_g$, so that $\mu(H_2(G)) \leqq 2g$. For the second claim, we first observe by a result of R. Litherland [13] that $\mathbb{G}^r_g(H) \neq \emptyset$. For $G \in \mathbb{G}^r_g(H)$, we take the minimal $g_* \leqq g$ such that $G \in \mathbb{G}^r_{g_*}(H)$. Let L be an $F^r_{g_*}$-link with $G = G(L)$. Let L' be a non-ribbon S^2-knot with the Alexander module $A(L') = \Lambda/(t+1, 3)$ (e.g., the 2-twist-spun trefoil), and a ribbon T^2-knot L'' with $H_2(G(L'')) = 0$ and the Alexander module $A(L'') = \Lambda/(2t-1, 5)$ (see [2]). Let $L_{m',m''}$ be any connected sum of L, $m'(\geqq 0)$ copies of L', and $m''(\geqq 0)$ copies of L''. Then

$$H_2(G(L_{m',m''})) \cong H_2(G) \bigoplus H_2(G(L'))^{m'} \bigoplus H_2(G(L''))^{m''} \cong H_2(G) \cong H.$$

By [10, Theorems 3.2, 5.1], we have constants c', c'' such that $G(L_{m',m''}) \notin R\mathbb{G}^r_{g+h}(H)$ for every $m' \geqq c'$ and $m'' \geqq 0$ and $G(L_{m',m''}) \notin \mathbb{G}^r_{g-1}(H)$ for every $m' \geqq 0$ and $m'' \geqq c''$. Noting that $G(L_{m',m''}) \in \mathbb{G}^r_{g'}(H) \setminus \mathbb{G}^r_{g'-1}(H)$ implies

$$G(L_{m',m''+1}) \in (\mathbb{G}^r_{g'+1}(H) \setminus \mathbb{G}^r_{g'}(H)) \cup (\mathbb{G}^r_{g'}(H) \setminus \mathbb{G}^r_{g'-1}(H)),$$

we can find $(0 \leqq) m'' \leqq c''$ such that $G(L_{m',m''}) \in \mathbb{G}^r_g(H) \setminus \mathbb{G}^r_{g-1}(H)$. Thus, we can find $m' \geqq c'$ and $(0 \leqq) m'' \leqq c''$ such that $G(L_{m',m''}) \in \mathbb{G}^r_g(H) \setminus (\mathbb{G}^r_{g-1}(H) \cup R\mathbb{G}^r_{g+h}(H))$. □

3. Classical link groups

Let $\mathbb{G}^{r,s}[1]$ be the set of $G(L^1) \in \mathbb{G}^r[1]$ such that L^1 is a split union of s non-split links. For $G = G(L^1) \in \mathbb{G}^{r,s}[1]$, let L^1_j $(j = 1, 2, \ldots, s)$ be the non-split sublinks of L^1. The group G is the free product $G(L^1_1) * G(L^1_2) * \cdots * G(L^1_s)$ and we have

$$H_2(G) = \bigoplus_{j=1}^{s} H_2(G(L^1_j)) = \bigoplus_{j=1}^{s} H_2(E(L^1_j)) \cong Z^{r-s},$$

where $E(L^1_j)$ denotes the exterior of L^1_j. Let $\mathbb{G}^{r,s}_g(H)$ be the set of $G \in \mathbb{G}^r_g(H)$ which is realized by a split union of s non-split surface-links, which we call an $F^{r,s}_g$-*link*, and $R\mathbb{G}_g^{r,s}(H)$ the set of $G \in R\mathbb{G}_g^r(H)$ realized by a ribbon $F_g^{r,s}$-link. We show the following theorem:

Theorem 3.1. $\mathbb{G}^{r,s}[1] \subsetneqq R\mathbb{G}^{r,s}_{r-s}(Z^{r-s}) \setminus \mathbb{G}^{r,s}_{r-s-1}(Z^{r-s})$.

To prove this theorem, we need some preliminaries.

Lemma 3.2. Let M be a closed oriented $2n$-manifold, and X a compact polyhedron. Let $\tilde{f} : \tilde{M} \to \tilde{X}$ be a lift of a map $f : M \to X$ to an infinite cyclic covering. If $H_c^{2n}(\tilde{X}) = 0$, then the Λ-rank $\mathrm{rank}_\Lambda(\tilde{f})$ of the image of $\tilde{f}* : H_n(\tilde{M}) \to H_n(\tilde{X})$ has

$$\mathrm{rank}_\Lambda(\tilde{f}) \leqq \frac{1}{2}(\mathrm{rank}_\Lambda H_n(\tilde{M}) - |\sigma(M)|)$$

where $\sigma(M)$ denotes the signature of M (taking 0 when n is odd).

Proof. Let N_c be the image of the homomorphism $\tilde{f}^* : H_c^n(\tilde{X}) \to H_c^n(\tilde{M})$ on the cohomology with compact support, and N the image of N_c under the Poincaré duality $H_c^n(\tilde{M}) \cong H_n(\tilde{M})$. Since $H_c^{2n}(\tilde{X}) = 0$, we have the trivial cup product $u \cup v = 0$ and hence $\tilde{f}^*(u) \cup \tilde{f}^*(v) = \tilde{f}^*(u \cup v) = 0$ for all $u, v \in H_c^n(\tilde{X})$. This means that the Λ-intersection form

$$\mathrm{Int}_\Lambda : H_n(\tilde{M}) \times H_n(\tilde{M}) \longrightarrow \Lambda$$

has $\mathrm{Int}_\Lambda(N, N) = 0$. Since $\tilde{f}^* : H_c^n(\tilde{X}) \to H_c^n(\tilde{M})$ is equivalent to $\tilde{f}^* : H_\Lambda^n(\tilde{X}) \to H_\Lambda^n(\tilde{M})$ on the cohomology with Λ coefficients (see [6]), we see from the universal coefficient theorem over Λ in [6] that $\mathrm{rank}_\Lambda N_c = \mathrm{rank}_\Lambda N$ is equal to the Λ-rank of the image of the dual Λ-homomorphism

$$(\tilde{f}_*)^\# : \hom_\Lambda(H_n(\tilde{X}), \Lambda) \longrightarrow \hom_\Lambda(H_n(\tilde{M}), \Lambda)$$

of $\tilde{f}_* : H_n(\tilde{M}) \to H_n(\tilde{X})$, which is equal to $\mathrm{rank}_\Lambda(\tilde{f})$. Considering the Λ-intersection form Int_Λ over the quotient field $Q(\Lambda)$ of Λ to obtain a non-singular $Q(\Lambda)$-intersection form, we can see from [5] that

$$2\mathrm{rank}_\Lambda N + |\sigma(M)| \leqq \mathrm{rank}_\Lambda H_n(\tilde{M}).$$

□

Let $\Delta_G^T(t)$ be the *torsion Alexander polynomial* of a surface-link group G, that is a generator of the smallest principal ideal of the first elementary ideal of the Λ-torsion part Tor_ΛA(G) of the Alexander module $A(G)$ of G. Then the following lemma is known(cf. [9]).

Lemma 3.3. $\Delta_G^T(t)$ is symmetric for every $G \in \mathbb{G}^{r,s}[1]$.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Since every $G \in \mathbb{G}^{r,s}[1]$ has a Wirtinger presentation with deficiency s, we have $G \in R\mathbb{G}^{r,s}_{r-s}(Z^{r-s})$ and $\mathbb{G}^{r,s}[1] \subset R\mathbb{G}^{r,s}_{r-s}(Z^{r-s})$. We first show that $\mathbb{G}^{r,s}[1] \cap \mathbb{G}^{r,s}_{r-s-1}(Z^{r-s}) = \emptyset$. Let L be an F^r_g-link such that $G(L) = G(L^1) = G \in \mathbb{G}^{r,s}[1]$. Let $E = E(L)$, and E^1 the bouquet of the link exteriors $E(L^1_j)$ $(j = 1, 2, \ldots, s)$. Since E^1 is a $K(G,1)$-space, there is a PL map $f_E : E \to E^1$ inducing an isomorphism $(f_E)_\# : G(L) = \pi_1(E) \cong \pi_1(E^1) = G(L^1)$ sending the meridian basis of $H_1(E)$ to the meridian basis of L^1 in $H_1(E^1)$. For the components F_i $(i = 1, 2, \ldots, r)$ of F^r_g and handlebodies V_i with $F_i = \partial V_i$ $(i = 1, 2, \ldots, r)$, we construct a closed connected oriented 4-manifold $M = E \cup_{i=1}^r V_i \times S^1$ by attaching, for every i, the boundary component $F_i \times S^1$ of E to the boundary of $V_i \times S^1$. Construct a compact polyhedron $X = E^1 \cup_{i=1}^r V_i \times S^1$ by attaching $V_i \times S^1$ $(i = 1, 2, \ldots, r)$ to E^1 along the map $f_E|_{\partial E} : \partial E \to E^1$, so that f_E extends to a PL map $f : M \to X$. Let $\tilde{f}_E : \tilde{E} \to \tilde{E}^1$ be the infinite cyclic covering of $f_E : E \to E^1$ associated with the epimorphism $\chi : G \to Z$ sending every meridian to 1, which extends to an infinite cyclic covering $\tilde{f} : \tilde{M} \to \tilde{X}$ of $f : M \to X$. Noting that $V_i \times S^1$ lifts to $V_i \times R^1$ in $\tilde{M}$ and $\tilde{X}$, we see that $\mathrm{rank}_\Lambda H_2(\tilde{X}) = \mathrm{rank}_\Lambda H_2(\tilde{E})$ and $\mathrm{rank}_\Lambda H_2(\tilde{X}) = \mathrm{rank}_\Lambda H_2(\tilde{E})$. By Hopf's theorem, $(\tilde{f}_E)_* : H_2(\tilde{E}) \to H_2(\tilde{E}^1) = H_2(\mathrm{Ker}\chi)$ is onto, so that $\mathrm{rank}_\Lambda H_2(\tilde{E}^1) = \mathrm{rank}_\Lambda H_2(\tilde{X}) = \mathrm{rank}_\Lambda(\tilde{f})$. Let $\beta_j = \mathrm{rank}_\Lambda H_1(\tilde{E}^1_j)$. Then $\mathrm{rank}_\Lambda(\tilde{f}) = \mathrm{rank}_\Lambda H_2(\tilde{E}^1) = \sum_{j=1}^s \beta_j$ by [7]. By the compact support cohomology exact sequence for $(\tilde{X}, \tilde{X}_V)$ with $\tilde{X}_V = \cup_{i=1}^r V_i \times R^1$, we have the following exact sequence:

$$H^4_c(\tilde{X}, \tilde{X}_V) \to H^4_c(\tilde{X}) \longrightarrow H^4_c(\tilde{X}_V).$$

For the image $\tilde{X}_0$ of $\tilde{f}_E|_{\partial\tilde{E}} : \partial\tilde{E} \to \tilde{E}^1$, we have an excision isomorphism

$$H^4_c(\tilde{X}, \tilde{X}_V) \cong H^4_c(\tilde{E}^1, \tilde{X}_0) = 0,$$

since $(\tilde{E}^1, \tilde{X}_0)$ is a 3-dimensional complex pair. Also, by Poincaré duality we have

$$H^4_c(\tilde{X}_V) \cong H_0(\tilde{X}_V, \partial\tilde{X}_V) = 0.$$

Hence $H^4_c(\tilde{X}) = 0$. Since $\mathrm{rank}_\Lambda H_2(\tilde{E}) = 2(g + s - r + \sum_{j=1}^s \beta_j)$ by [10] and $\sigma(M) = 0$, it follows from Lemma 3.2 that $2(g + s - r + \sum_{j=1}^s \beta_j) \geqq 2(\sum_{j=1}^s \beta_j)$ and $g \geqq r - s$. Thus, $G^{r,s}[1] \cap \mathbb{G}^{r,s}_{r-s-1}(Z^{r-s}) \neq \emptyset$. Next, by a result of T. Yajima [16], the group $G_0 = \langle x_1, x_2 \,|\, x_2 = (x_2 x_1^{-1})^{-1} x_1 (x_2 x_1^{-1})\rangle$ with $\Delta^T_{G_0}(t) = 2 - t$ is represented by a ribbon S^2-knot L_0. For $G = G(L^1) \in \mathbb{G}^{r,s}[1]$, let L be an $F^{r,s}_{r-s}$-link with $G(L) = G$, and $L' = L \# L_0$ a connected sum of L and L_0. Then $G' = G(L') \in R\mathbb{G}^{r,s}_{r-s}(Z^{r-s})$. Since $\Delta^T_{G'}(t) = \Delta^T_G(t)\Delta^T_{G_0}(t)$ is not symmetric, we have $G' \notin \mathbb{G}^{r,s}[1]$ by Lemma 3.3. Let L'' be an $F^{r,s}_{g''}$-link with $G(L'') = G'$, and E'' the exterior L''. For the $K(G,1)$-space E^1 constructed from L^1 as above, we realize an epimorphism $G' \to G$ preserving the meridians by a PL map $f_{E''} : E'' \to E^1$, which is used to construct a PL map $f'' : M'' \to X$ from a closed 4-manifold $M'' = E'' \cup_{i=1}^r V_i \times S^1$ to $X = E^1 \cup_{i=1}^r V_i \times S^1$

in a similar way of the argument above. By a similar calculation using Lemma 3.2, we can conclude that $g'' \geqq r-s$ and $G' \notin \mathbb{G}^{r,s}[1] \cup \mathbb{G}^{r,s}_{r-s-1}(Z^{r-s})$. This completes the proof of Theorem 3.1. □

4. Virtual link groups

An *r-component*, *s-split* virtual link is a virtual link with r components which is represented by a split union of s diagrams of s non-split virtual links. The group of a virtual link diagram which is calculated in a similar way to a classical link diagram except that we do not count the virtual crossing points is an invariant of the virtual link (see L. H. Kauffman [4]). Let $V\mathbb{G}^{r,s}(H)$ be the set of the groups G of r-component, s-split virtual links with $H_2(G) \cong H$. Then we have the following theorem.

Theorem 4.1. $V\mathbb{G}^{r,s}(H) = R\mathbb{G}^{r,s}_{1,1,\ldots,1}(H)$ for every H and we have

$$\mathbb{G}^{r,s}[1] \subsetneqq V\mathbb{G}^{r,s}(Z^{r-s}) = R\mathbb{G}^{r,s}_{1,1,\ldots,1}(Z^{r-s}) \subset R\mathbb{G}^{r,s}_{r}(Z^{r-s}).$$

Proof. The first claim is observed in [10], coming essentially from a result of S. Satoh [15]. The inclusions of the second claim is obvious. For $G = G(L^1) \in \mathbb{G}^{r,s}[1]$, let L be an $F^{r,s}_{1,1,\ldots,1}$-link with $G(L) = G$, and $L' = L\#L_0$ a connected sum of L and a ribbon S^2-knot L_0 as in the proof of Theorem 3.1. Then $G' = G(L') \in R\mathbb{G}^{r,s}_{1,1,\ldots,1}(Z^{r-s}) = V\mathbb{G}^{r,s}(Z^{r-s})$. Since $\Delta^T_{G'}(t)$ is not symmetric as it is shown in the proof of Theorem 3.1, we have $G' \notin \mathbb{G}^{r,s}[1]$ by Lemma 3.3. Thus, $\mathbb{G}^{r,s}[1] \subsetneqq V\mathbb{G}^{r,s}(Z^{r-s})$.

Corollary 4.2. If $\mu(H) > r$, then we have $V\mathbb{G}^{r,s}(H) = \emptyset$. For $H = Z^u \oplus Z_2^v$ with $0 \leqq u+v \leqq r$, we have $V\mathbb{G}^{r,s}(H) \neq \emptyset$.

Proof. For $G \in R\mathbb{G}^r_g$, let L be a ribbon F^r_g-link, and E the exterior of L. By Hopf's theorem, there is an exact sequence

$$\pi_2(E,x) \longrightarrow H_2(E) \longrightarrow H_2(G) \to 0.$$

Since L has a Seifert hypersurface homeomorphic to a connected sum of a handlebody and some copies of $S^1 \times S^2$, we can represent a half basis of $H_2(E) \cong Z^{2g}$ by 2-spheres. Hence $\mu H_2(G) \leqq g$, showing the first claim. For the second claim, we first note that for every $r > 1$, there is a ribbon F^r_0-link L such that $G(L)$ is an indecomposable group by considering the spinning construction of an r-string tangle in the 3-ball with an indecomposable group (see[8, p.204]). Second, we note that any connected sum of this ribbon F^r_0-link L and any surface-knots L'_i $(i = 1, 2, \ldots, s)$ is a non-split surface-link. Then we take a ribbon $F^{r,s}_0$-link L whose non-split surface-sublinks have indecomposable groups, and a ribbon T^2-knot L_0 with $H_2(G(L_0)) \cong Z$ constructed by C. McA. Gordon [1] and a ribbon T^2-knot L_2 with $H_2(G(L_2)) \cong Z_2$ constructed by T. Maeda [14]. Let L' be a ribbon $F^{r,s}_{1,1,\ldots,1}$-link obtained by a connected sum of L, u copies of L_0, v copies of L_2, and $r-u-v$ copies of a trivial T^2-knot. Then $G(L') \in R\mathbb{G}^{r,s}_{1,1,\ldots,1}(H) = V\mathbb{G}^{r,s}(H)$ for $H = Z^u \oplus Z_2^v$. □

It is unknown *whether* $V\mathbb{G}^{r,s}(H) \neq \emptyset$ *for every* H *with* $\mu(H) \leqq r$.

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