Ribbonness of a stable-ribbon surface-link, II. General case

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ABSTRACT

It is shown that a handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link up to equivalences. This is a generalization of the result for the case of a stably trivial surface-link previously observed.

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1 Introduction

In this paper, a generalization of the result of the paper [9] on a trivial surface-link to a result on a ribbon surface-link is explained.

A surface-link is a closed oriented (possibly disconnected) surface $F$ embedded in the 4-space $\mathbb{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When $F$ is connected, it is also called a surface-knot. When a (possibly disconnected) closed surface $F$ is fixed, it is also called an $F$-link. If $F$ is the disjoint union of some copies of the 2-sphere $S^2$, then it is also called a 2-link. When $F$ is connected, it is also called a surface-knot, and a 2-knot for $F = S^2$. Two surface-links $F$ and $F'$ are equivalent
by an equivalence $f$ if $F$ is sent to $F'$ orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbb{R}^4 \to \mathbb{R}^4$. A trivial surface-link is a surface-link $F$ which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in $\mathbb{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, solid torus or a disk sum of some number of solid tori. A trivial surface-knot is also called an unknotted surface-knot. A trivial disconnected surface-link is also called an unknotted-unlinked surface-link. For any given closed oriented (possibly disconnected) surface $F$, a trivial $F$-link exists uniquely up to equivalences (see [3]). A ribbon surface-link is a surface-link $F$ which is obtained from a trivial $nS^2$-link $O$ for some $n$ (where $nS^2$ denotes the disjoint union of $n$ copies of the 2-sphere $S^2$) by the surgery along an embedded 1-handle system (see [4], [11, II]). A stabilization of a surface-link $F$ is a connected sum $\bar{F} = F \#_{k=1}^s T_k$ of $F$ and a system $T$ of trivial torus-knots $T_k (k = 1, 2, \ldots, s)$. By granting $s = 0$, we understand that a surface-link $F$ itself is a stabilization of $F$. The trivial torus-knot system $T$ is called the stabilizer with stabilizer components $T_k (k = 1, 2, \ldots, s)$ on the stabilization $\bar{F}$ of $F$. A stable-ribbon surface-link is a surface-link $F$ such that a stabilization $\bar{F}$ of $F$ is a ribbon surface-link.

For every surface-link $F$, there is a surface-link $F^*$ with minimal total genus such that $F$ is equivalent to a stabilization of $F^*$. The surface-link $F^*$ is called a handle-irreducible summand of $F$.

The following result called Stable-Ribbon Theorem is our main theorem.

**Theorem 1.1.** A handle-irreducible summand $F^*$ of every stable-ribbon surface-link $F$ is a ribbon surface-link which is determined uniquely from $F$ up to equivalences.

Since any stabilization of a ribbon surface-link is a ribbon surface-link, Theorem 1.1 implies the following corollary:

**Corollary 1.2.** Every stable-ribbon surface-link is a ribbon surface-link.

The following corollary of a ribbon surface-link is a standard consequence of Corollary 1.2, and contrasts with a behavior of a classical ribbon knot, for every classical knot is a connected summand of a ribbon knot.

**Corollary 1.3.** A connected sum $F = F_1 \# F_2$ of surface-links $F_i (i = 1, 2)$ is a ribbon surface-link if and only if the surface-links $F_i (i = 1, 2)$ are both ribbon surface-links.

**Proof of Corollary 1.3.** The ‘if’ part of Corollary 1.3 is seen from the definition of a ribbon surface-link. The proof of the ‘only if’ part of Corollary 1.3 uses an argument
of [3] showing the fact that every surface-link is made a trivial surface-knot by the surgery along a finite number of (possibly non-trivial) 1-handles. The connected summand $F_2$ is made a trivial surface-knot by the surgery along 1-handles within the 4-ball defining the connected sum, so that the surface-link $F$ changes into a new ribbon surface-link and hence $F_1$ is a stable-ribbon surface-link. By Corollary 1.2, $F_1$ is a ribbon surface-link. By interchanging the roles of $F_1$ and $F_2$, $F_2$ is also a ribbon surface-link. □

A stably trivial surface-link is a surface-link $F$ such that a stabilization $\tilde{F}$ of $F$ is a trivial surface-link. Since a trivial surface-link is a ribbon surface-link, Theorem 1.1 also implies the following corollary, which is a main result in [9]:

**Corollary 1.4.** A handle-irreducible summand of every stably trivial surface-link is a trivial 2-link.

This corollary implies that *every stably trivial surface-link is a trivial surface-link* as observed in [9]. See [9, 10] for further results on a trivial surface-link.

The plan for the proof of Theorem 1.1 is to show the following two theorems by an argument based on [9].

**Theorem 1.1.1** Any two handle-irreducible summands of any (not necessarily ribbon) surface-link are equivalent.

**Theorem 1.1.2** Any stable-ribbon surface-link is a ribbon surface-link.

The proofs of Theorem 1.1.1 and 1.1.2 are given in §2 and §3, respectively. The proof of Theorem 1.1 is completed by these theorems as follows:

**Proof of Theorem 1.1.** By Theorem 1.1.2, a handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link which is unique up to equivalences by Theorem 1.1.2. □

## 2 Proof of Theorem 1.1.1

A 2-handle on a surface-link $F$ in $\mathbb{R}^4$ is an embedded 2-handle $D \times I$ on $F$ with $D$ a chore disk such that $(D \times I) \cap F = (\partial D) \times I$, where $I$ denotes a closed interval containing 0 and $D \times 0$ is identified with $D$. An orthogonal 2-handle pair (or simply, an $O2$-handle pair) on $F$ is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I, D' \times I$ on $F$.
such that the core disks $D$ and $D'$ meet transversely at just one point $p$ in $F$ with

$$(D \times I) \cap (D' \times I) = (\partial D) \times I \cap (\partial D') \times I$$

which is homeomorphic to the square $Q = I \times I$ with $p$ the central point.

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link $F$. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from $F$ by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the bounded surface $F_D^c = \text{cl}(F \setminus ((\partial D) \times I \cup (\partial D') \times I))$ and the plumbed disk $\delta_D = D \times (\partial I) \cup Q \cup D' \times (\partial I)$. A compact once-punctured torus of a torus $T$ is simply called a punctured torus and denoted by $T^o$. A punctured torus $T^o$ in a 3-ball $B$ is trivial if $T^o$ is smoothly and properly embedded in $B$ and there is a solid torus $V$ in $B$ with $\partial V = T^o \cup \delta_B$ for a disk $\delta_B$ in $\partial B$.

A bump of a surface-link $F$ is a 3-ball $B$ in $\mathbb{R}^4$ with $F \cap B = T^o$ a trivial punctured torus in $B$. Let $F(B)$ be a surface-link $F^c \cup \delta_B$ for the surface $F_B^c = \text{cl}(F \setminus T^o)$ and a disk $\delta_B$ in $\partial B$ with $\partial \delta_B = \partial T^o$, where note that $F(B)$ is uniquely determined up to cellular moves on $\delta_B$ keeping $F^c$ fixed. For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link $F$, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbb{R}^4$ called the 2-handle union. By adding a boundary collar to the 2-handle union $\Delta$, we have a bump $B = B_D$ of $F$, which we call the associated bump of the O2-handle pair $(D \times I, D' \times I)$ (see [9, Fig. 2]).

An O2-handle pair and a bump on a surface-link are shown to be essentially equivalent notions in [9]. In particular, it is observed in [9] that for any O2-handle pair $(D \times I, D' \times I)$ on any surface-link $F$ and the associated bump $B$, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

A punctured torus $T^o$ in a 4-ball $A$ is trivial if $T^o$ is smoothly and properly embedded in $A$ and there is a solid torus $V$ in $A$ with $\partial \overline{V} = T^o \cup \delta_A$ for a disk $\delta_A$ in the 3-sphere $\partial A$. A 4D bump of a surface-link $F$ is a 4-ball $A$ in $\mathbb{R}^4$ with $F \cap A = T^o$ a trivial punctured torus in $A$. A 4D bump $A$ is obtained from a bump $B$ of a surface-link $F$ by taking a bi-collar $c(B \times [-1, 1])$ of $B$ in $\mathbb{R}^4$ with $c(B \times 0) = B$. The following lemma is proved by using a 4D bump $A$.

**Lemma 2.1.** For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link $F$, let $F(D \times I, D' \times I) = F_D^c \cup \delta_D$. Then for a trivial torus-knot $T$ with a spin loop basis $(\ell, \ell')$, there is an equivalence $f : \mathbb{R}^4 \to \mathbb{R}^4$ from the surface-link $F$ to a connected sum $F(D \times I, D' \times I) \# T$ keeping $F_D^c$ fixed such that

$$f(\partial D) = \ell \quad \text{and} \quad f(\partial D') = \ell'.$$
Proof of Lemma 2.1. Let $A$ be a 4D bump associated with the O2-handle pair $(D \times I, D' \times I)$ on $F$. Let $\delta_A$ be a disk in the 3-sphere $\partial A$ such that the union of $\delta_D$ and the trivial punctured torus $F \cap A = P$ bounds a solid torus $V$ in $A$. Then there is an equivalence $f' : F \cong F(D \times I, D' \times I) \# T$ by deforming $V$ in $A$ so that $P$ is isotopically deformed into the summand $T^0$ of a connected sum $\delta_D \# T$ in $A$. Then the spin loop pair $(\partial D, \partial D')$ on $F_1$ is sent to a spin loop basis $(\tilde{\ell}, \tilde{\ell}')$ of $T^0$. By [2] (see [9, (2.4.2)]), there is an orientation-preserving diffeomorphism $g : \mathbb{R}^4 \to \mathbb{R}^4$ with $g|_{\text{cl}(\mathbb{R}^4 \setminus A)} = 1$ such that

$$g(\tilde{\ell}, \tilde{\ell}') = (\ell, \ell').$$

By the composition $gf'$, we have a desired equivalence $f$. □

A surface-link $F$ has only unique O2-handle pair in the rigid sense if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on $F$ with $(\partial D) \times I = (\partial E') \times I$ and $(\partial D') \times I = (\partial E) \times I$, there is an equivalence $f : \mathbb{R}^4 \to \mathbb{R}^4$ from $F$ to $F$ such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$. It is shown in [9] that every surface-link $F$ has only unique O2-handle pair in the rigid sense with an additional condition that there is an ambient isotopy $f_t (t \in [0, 1])$ with $f_0 = 1$ and $f_1 = f$ keeping $F_0$ fixed.

A surface-link $F$ has only unique O2-handle pair in the soft sense if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on $F$ attached to the same connected component of $F$, there is an equivalence $f : \mathbb{R}^4 \to \mathbb{R}^4$ from $F$ to $F$ such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair in both the rigid and soft senses.

The following lemma shows that the uniqueness of an O2-handle pair in the soft sense is derived from the uniqueness of an O2-handle pair in the rigid sense.

Lemma 2.2. Every surface-link has only unique O2-handle pair in the soft sense.

Proof of Lemma 2.2. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be any two O2-handle pairs on a surface-link $F$ attached to the same connected component of $F$.

By Lemma 2.1, there is an equivalence $f : \mathbb{R}^4 \to \mathbb{R}^4$ from $F$ to $F(E \times I, E' \times I) \# T$ keeping $F_E$ fixed. Let $F_E = F(E \times I, E' \times I)$. Let $F_E(h)$ be a trivial surface-knot obtained from $F_E$ by the surgery along a system $h$ of mutually disjoint 1-handles $h_j (j = 1, 2, \ldots, s)$ on $F_E$.

Let $\hat{h}$ be the system of cylinders $\hat{h}_j = h_j \cap F_E(h) (j = 1, 2, \ldots, s)$, and $\bar{h}$ is the system of two disks $\bar{h}_j = \text{cl}(\partial h_j \setminus \hat{h}_j) (j = 1, 2, \ldots, s)$.

Let $(d \times I, d' \times I)$ be a standard O2-handle pair on $T^0$ in the 4-ball defining the connected summand $T^0$ in $F_E \# T$, and $(c, c') = (\partial d, \partial d')$ which is a spin loop basis...
of \( T^o \). By construction, the system \( h \) of 1-handles \( h_j \) \( (j = 1, 2, \ldots, s) \) is disjoint from the disk pair \((d, d')\). By an isotopic deformation of \( f \), we can assume that the system \( f^{-1}(h) \) of the 1-handles \( f^{-1}(h_j) \) \( (j = 1, 2, \ldots, s) \) on \( F \) is disjoint from \((D \times I, D' \times I)\).

By [2] (see [9, (2.4.2)]), there is an orientation-preserving diffeomorphism \( g : \mathbb{R}^4 \to \mathbb{R}^4 \) sending \( F_E(\hat{h})\#T \) to itself such that the spin loop pair \((gf(\partial D), gf(\partial D')) = (e, e')\) and the restriction of \( g \) to the system \( \hat{h} \) of the cylinders \( \hat{h}_j \) \( (j = 1, 2, \ldots, s) \) is the identity map. This last condition is assumed by a choice of a spin loop basis on \( F_E(\hat{h})\#T \).

By the uniqueness of an O2-handle pair in the rigid sense given in [9], there is an ambient isotopy \( i_t : \mathbb{R}^4 \to \mathbb{R}^4 \) \( (t \in [0,1]) \) keeping \((F_E(\hat{h})\#T)^c \) fixed such that \( i_0 \) is the identity and \( i_1g(f(D) \times I, f(D') \times I) = (d \times I, d' \times I) \). Let

\[
G^t = g^{-1}(\text{cl}(F_E(\hat{h}) \setminus \hat{h})\#T) \cup g^{-1}i_0g(\hat{h}) \quad (t \in [0,1])
\]

be a surface-link family with \( G^0 = F_E\#T \). There is an O2-handle pair

\[
(g^{-1}i_1g(f(D) \times I, f(D') \times I)
\]

on the surface-link \( G^t \), where

\[
g^{-1}i_0g(f(D) \times I, f(D') \times I) = (f(D) \times I, f(D') \times I),
\]

\[
g^{-1}i_1g(f(D) \times I, f(D') \times I) = g^{-1}(d \times I, d' \times I).
\]

Then the surface-link \( G^0(f(D) \times I, f(D') \times I) \) is given by

\[
G^0(f(D) \times I, f(D') \times I) = (F_E\#T)(f(D) \times I, f(D') \times I)
\]

\[
= F(D \times I, D' \times I)
\]

\[
= F_D.
\]

and the surface-link \( gG^1(d \times I, d' \times I) \) is given by

\[
gG^1(d \times I, d' \times I) = (\text{cl}(F_E(\hat{h}) \setminus \hat{h})\#T \cup i_1g(\hat{h}))(d \times I, d' \times I)
\]

\[
\cong (\text{cl}(F_E(\hat{h}) \setminus \hat{h})\#T \cup i_1g(\hat{h}))(i_1g(d \times I), i_1g(d' \times I))
\]

\[
= i_1g((F_E\#T)(d \times I, d' \times I))
\]

\[
\cong (F_E\#T)(d \times I, d' \times I)
\]

\[
\cong F_E,
\]

where the equivalence

\[
(\text{cl}(F_E(\hat{h}) \setminus \hat{h})\#T \cup i_1g(\hat{h}))(d \times I, d' \times I)
\]

\[
\cong (\text{cl}(F_E(\hat{h}) \setminus \hat{h})\#T \cup i_1g(\hat{h}))(i_1g(d \times I), i_1g(d' \times I))
\]

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is obtained from the uniqueness of an O2-handle pair in the rigid sense given in [9]. Since there is an equivalence

\[ G^0(f(D) \times I, f(D') \times I) \cong gG^1(d \times I, d' \times I), \]

there is an equivalence \( f' \) from \( F_D = F_D^0 \cup \delta_D \) to \( F_E = F_E^0 \cup \delta_E \) for disks \( \delta_D \) and \( \delta_E \). By a disk move, we can assume that \( f'(\delta_D) = \delta_E \). The map \( f' \) is isotopic to a diffeomorphism \( f'' : \mathbb{R}^4 \to \mathbb{R}^4 \) sending the associated bump \( B_D \) of \((D \times I, D' \times I)\) to the associated bump \( B_E \) of \((E \times I, E' \times I)\). The diffeomorphism \( f'' : \mathbb{R}^4 \to \mathbb{R}^4 \) is modified into an equivalence \( f''' : \mathbb{R}^4 \to \mathbb{R}^4 \) from \( F \) to \( F' \) such that \( f'''(D \times I) = E \times I \) and \( f'''(D' \times I) = E' \times I \) because the bumps \( B_D \) and \( B_E \) recover the unordered O2-handle pairs \((D \times I, D' \times I)\) and \((E \times I, E' \times I)\), respectively (cf. [9, Lemma 2.4]). Thus, every surface-link \( F \) has only unique O2-handle pair in the soft sense. □

We use the following corollary to Lemma 2.2.

**Corollary 2.3.** Let \( F, F' \) be surface-links with ordered components \( F_i, F'_i (i = 1, 2, \ldots, r) \), respectively, and \( F = F \#_iT, F' = F' \#_iT \) the stabilizations of \( F, F' \) with induced ordered components obtained by the connected sums \( F_i \#_iT, F'_i \#_iT \) of the \( i \)th components \( F_i, F'_i \) and a trivial torus-knot \( T \) for some \( i \), respectively. Assume that \( F \) is equivalent to \( F' \) by a component-order-preserving equivalence. Then \( F \) is equivalent to \( F' \) by a component-order-preserving equivalence.

**Remark 2.4.** Corollary 2.3 for ribbon surface-links \( F, F' \) has a different proof using the result of [8].

The proof of Theorem 1.1.1 is done as follows.

**Proof of Theorem 1.1.1.** A surface-link \( F \) with \( r \) ordered components is \emph{kth-handle-reducible} if \( F \) is equivalent to a stabilization \( F' \#_k n_k T \) of a surface-link \( F' \) for a positive integer \( n_k \), where \( \#_k n_k T \) denotes the stabilizer components \( n_k T \) attaching to the \( k \)th component of \( F' \). Otherwise, the surface-link \( F \) is \emph{kth-handle-irreducible}. Note that if a \( k \)th-handle-irreducible surface-link \( F \) is component-order-preserving equivalent to a surface-link \( G \), then \( G \) is also \( k \)th-handle-irreducible.

Let \( F \) and \( G \) be ribbon surface-links with components \( F_i (i = 1, 2, \ldots, r) \) and \( G_i (i = 1, 2, \ldots, r) \), respectively. Let \( F^* \) and \( G^* \) be handle-irreducible summands of \( F \) and \( G \), respectively.

Assume that there is an equivalence \( f \) from \( F \) to \( G \). Then we show that \( F^* \) and \( G^* \) are equivalent. Changing the indexes if necessary, we assume that \( f \) sends \( F_i \) to
\( G_i \) for every \( i \). Let
\[
F = F^* \#_1 n_1 T \#_2 n_2 T \#_3 \ldots \#_r n_r T,
\]
\[
G = G^* \#_1 n'_1 T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T.
\]

Taking the inverse equivalence \( f^{-1} \) instead of \( f \) if necessary, we may assume that \( n'_1 \geq n_1 \). If \( n'_1 > n_1 \), then by (*) , there is an equivalence \( f^{(1)} \) from the first-handle-irreducible surface-link
\[
F^{(1)} = F^* \#_2 n_2 T \#_3 \ldots \#_r n_r T
\]
to the first-handle-reducible surface-link
\[
G^* \#_1 (n'_1 - n_1) T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T,
\]
which has a contradiction. Thus, \( n'_1 = n_1 \) and the first-handle-irreducible surface-link \( F^{(1)} \) is equivalent to the first-handle-irreducible ribbon surface-link
\[
G^{(1)} = G^* \#_2 n'_2 T \#_3 \ldots \#_r n'_r T.
\]

By continuing this process, it is shown that \( F^* \) is equivalent to \( G^* \). This completes the proof of Theorem 1.1.1. \( \square \)

### 3 Proof of Theorem 1.1.2

A chord graph is a pair \((o, \alpha)\) of a trivial ink \( o \) and an arc system \( \alpha \) attaching to \( o \) in the 3-space \( \mathbb{R}^3 \), where \( o \) and \( \alpha \) are called a based loop system and a chord system, respectively. A chord diagram is a diagram \( C(o, \alpha) \) in the plane \( \mathbb{R}^2 \) of a chord graph \((o, \alpha)\) as a spatial graph. Let \( D^+ \) be a proper disk system in the upper half-space \( \mathbb{R}^4_+ \) obtained from a disk system \( d^+ \) in \( \mathbb{R}^3 \) bounded by \( o \) by pushing the interior into \( \mathbb{R}^4_+ \). Similarly, let \( D^- \) be a proper disk system in the lower half-space \( \mathbb{R}^4_- \) obtained from a disk system \( d^- \) in \( \mathbb{R}^3 \) bounded by \( o \) by pushing the interior into \( \mathbb{R}^4_- \). Let \( O \) be the union of \( D^+ \) and \( D^- \) which is a trivial \( nS^2 \)-link in the 4-space \( \mathbb{R}^4 \), where \( n \) is the number of components of \( o \). The union \( O \cup \alpha \) is called a chorded sphere system constructed from a chord graph \((o, \alpha)\).

By using the Horibe-Yanagawa lemma in [11, I], the chorded sphere system \( O \cup \alpha \) up to orientation-preserving diffeomorphisms of \( \mathbb{R}^4 \) is independent of choices of \( d^+ \) and \( d^- \) and uniquely determined by the chord graph \((o, \alpha)\). A ribbon surface-link \( F = F(o, \alpha) \) is uniquely constructed from the chorded sphere system \( O \cup \alpha \) so that \( F \) is the surgery of \( O \) along a 2-handle system \( N(\alpha) \) on \( O \) with core arc system \( \alpha \) (see
[5, 6, 7, 8]), where note by [3] that the surface-link $F$ up to equivalences is unaffected by choices of the 2-handle $N(\alpha)$.

A semi-unknotted punctured handlebody system (or simply a SUPH system) for a surface-link $F$ is a punctured handlebody system $V$ in $\mathbb{R}^4$ such that the boundary $\partial V$ of $V$ is a union $F \cup O$ of $F$ and a trivial $S^2$-link $O$ with $F \cap O = \emptyset$. The following lemma is a characterization of a ribbon surface-link (cf. [11, II], Yanagawa [12]).

**Lemma 3.1.** A surface-link $F$ is a ribbon surface-link if and only if there is a punctured SUPH system $V$ for $F$.

**Proof of Lemma 3.1.** Given a ribbon surface-link, a SUPH system $V$ is constructed by a thickening $O \times I$ of $O$ in $\mathbb{R}^4$ by attaching a 1-handle system. Conversely, given a SUPH system $V$ in $\mathbb{R}^4$ such that $\partial V = F \cup O$ for a trivial $S^2$-link $O$ with $F \cap O = \emptyset$, there is a chord system $\alpha$ in $V$ attaching to $O$ such that the frontier of the regular neighborhood of $O \cup \alpha$ in $V$ is parallel to $F$, showing that $F$ is a ribbon surface-link. □

The following lemma is basic to the proof of Theorem 1.1.2.

**Lemma 3.2.** The following (1) and (2) hold.

1. For a surface-link $F$ and a trivial torus-knot $T$, if a connected sum $F \# T$ is a ribbon surface-link, then $F$ is a ribbon surface-link.
2. If $F$ is a ribbon surface-link and $(D \times I, D' \times I)$ is an $O2$-handle pair on $F$, then $F(D \times I, D' \times I)$ is a ribbon surface-link.

Theorem 1.1.2 is a consequence of Lemma 3.2 as follows:

**Proof of Theorem 1.1.2.** If a stabilization $\bar{F}$ of a surface-link $F$ is a ribbon surface-link, then $F$ is a ribbon surface-link by an inductive use of Lemma 3.2 (1). □

We are in a position to show Lemma 3.2.

**Proof of Lemma 3.2.** The assertion (1) $\Rightarrow$ (2) holds. In fact, by Lemma 2.1, there is a connected sum splitting $F \cong F(D \times I, D' \times I) \# T$ for a trivial torus-knot $T$. Thus, if $F$ is a ribbon surface-link, then $F(D \times I, D' \times I)$ is a ribbon surface-link by (1).

We show (1). Let $F \# T = F_1 \# T \cup F_2 \cup \cdots \cup F_r$ be a ribbon surface-link for a trivial torus-knot $T$. The following claim (3.2.1) is shown later.
(3.2.1) There is a stabilization \( \bar{F} = \bar{F}_1 \cup \bar{F}_2 \cup \cdots \cup \bar{F}_r \) of \( F\#T \) with \( \bar{F}_1 = F_1\#T\#_{i=1}^{2m} T_i \) such that the following conditions (i) and (ii) hold:

(i) There is an O2-handle pair \((D \times I, D' \times I)\) on \( \bar{F} \) attached to \( \bar{F}_1 \) such that the surface-link \( \bar{F}(D \times I) \) is a ribbon surface-link admitting a SUPH system with the 1-handles \( h_i' (i = 1, 2, \ldots, 2m) \) trivially attached.

(ii) There is an O2-handle pair \((E \times I, E' \times I)\) on \( \bar{F} \) attached to \( \bar{F}_1 \) such that the surface-link \( \bar{F}(E \times I) \) is the surface-link \( F \) with the 1-handles \( h_i'' (i = 1, 2, \ldots, 2m) \) trivially attached.

By assuming (3.2.1), the proof of Lemma 3.2 is completed as follows.

By (i), the surface-link \( F'' = \bar{F}(D \times I, D' \times I) \cong \bar{F}(D \times I) \) is a ribbon surface-link and further the surface-link \( F^* \) obtained from \( F'' \) by the surgery on O2-handle pairs of all the trivial 1-handles \( h_i' (i = 1, 2, \ldots, 2m) \) is also a ribbon surface-link. By (ii), the surface-link \( F(E \times I, E' \times I) \cong F(E \times I) \) is the surface-link \( F \) with the 1-handles \( h_i'' (i = 1, 2, \ldots, 2m) \) trivially attached. By an inductive use of Lemma 2.2 (or Theorem 1.1.1), the surface-link \( F \) is equivalent to the ribbon surface-link \( F^* \). Hence \( F \) is a ribbon surface-link, obtaining (3). Thus, the proof of Lemma 3.2 is completed except for the proof of (3.2.1). □

We are in a position to prove the claim (3.2.1).

Proof of (3.2.1). Let \( V \) be a SUPH system for \( F\#T \) by Lemma 3.1. Let the component of the SUPH system \( V \) containing \( F_1\#T \) be a disk sum \( U\#_p W \) for a punctured 3-ball \( U \) and a handlebody \( W \). Let \( A \) be a 4D bump defining the connected sum \( F\#T \) with \((F\#T) \cap A = T^o \).

We proceed the proof by assuming the following claim (3.2.2) shown later.

(3.2.2) There are a spin loop basis \((\ell, \ell')\) for \( T^o \) and a spin simple loop \( \tilde{\ell}' \) in \( F\#T \) such that Int(\( \ell, \tilde{\ell}' \)) = 1 and \( \tilde{\ell}' \) bounds a disk \( D' \) in \( W \).

By assuming (3.2.2), the proof of (3.2.1) is completed as follows.

Let \( p_i (i = 0, 1, \ldots, 2m) \) be the intersection points of \( \ell \) and \( \tilde{\ell}' \). For every \( i > 0 \), let \( \alpha_i \) be an arc neighborhood of \( p_i \) in \( \ell \), and \( h_i \) a 1-handle on \( F\#T \) with a core arc \( \tilde{\alpha}_i \) obtained by pushing the interior of \( \alpha_i \) into \( \mathbb{R}^4 \setminus V \). Let \( \tilde{\alpha}_i \) be a proper arc in \( \partial h_i = \text{cl} (\partial h_i \setminus h_i \cap F\#T) \) parallel to \( \alpha_i \) in \( h_i \) with \( \partial \tilde{\alpha}_i = \partial \alpha_i \).

Let \( \bar{F} = F\#T\#_{i=1}^{2m} T_i \) be a stabilization of \( F \) associated with the system of mutually disjoint trivial 1-handles \( h_i (i = 1, 2, \ldots, 2m) \).

Let \( \bar{\ell} \) be a simple loop obtained from \( \ell \) by replacing \( \alpha_i \) with \( \tilde{\alpha}_i \) for every \( i > 0 \). The loop \( \bar{\ell} \) is taken to be a spin loop in \( \bar{F} \) meeting \( \ell' \) transversely in just one point.
Let $A$ be a 4D bump of the associated bump $B$ of an O2-handle pair $(E \times I, E' \times I)$ on $F \# T$ in $\mathbb{R}^4$ attached to $T^o$ with $(\ell, \ell') = (\partial E, \partial E')$. Then the loop $\ell$ and the trivial 1-handles $h_i (i = 1, 2, \ldots, 2m)$ are taken in $A$.

Let $W^+(D')$ be the handlebody obtained from the handlebody $W^+ = W \cup_{i=1}^{2m} h_i$ by splitting along a thickened disk $D' \times I$ of $D'$. Then the manifold $V^+(D')$ obtained from $V^+ = V \cup_{i=1}^{2m} h_i$ by replacing $W^+$ with $W^+(D')$ is a SUPH system.

The SUPH system $V^+$ is ambient isotopic in $\mathbb{R}^4$ to a SUPH system $\hat{V}^+$ which is the union of $V^+(D')$ and a solid torus $W_1$ in $A$ connected by a 1-handle $h_W$ in $A$, where the solid torus $W_1$ has a deformed disk $\hat{D}'$ of $D'$ as a meridian disk and the loop $\hat{\ell}$ as a longitude. Since the trivial 1-handles $h_i (i = 1, 2, \ldots, 2m)$ are taken in the bump $B$, the solid torus $W_1$ is moved into a 4-ball disjoint from $T^o \cup_{i=1}^{2m} h_i$ and hence the loop $\hat{\ell}$ bounds a disk $\hat{D}$ in $A$ not meeting $T^o, h_i$ for all $i > 0$ and $h_W$. By putting back the ambient isotopy from the SUPH system $\hat{V}^+$ to the SUPH system $V^+$, we see that there is an O2-handle pair $(D \times I, D' \times I)$ on the surface-link $\hat{F}$ such that $\hat{F}(D \times I)$ is a ribbon surface-link admitting trivial 1-handles $h'_i (i = 1, 2, \ldots, 2m)$. This shows (i).

On the other hand, the 1-handles $h_i (i = 1, 2, \ldots, 2m)$ on $F \# T$ are isotopically deformed in $A$ into 1-handles $h''_i (i = 1, 2, \ldots, 2m)$ on $F \# T$ disjoint from the disk pair $(E, E')$ such that the core arcs of the 1-handles $h_i (i = 1, 2, \ldots, 2m)$ are deformed into simple arcs in $F \# T$ away from the disk pair $(E, E')$ in $A$. Hence the surface-link $\tilde{F}(E \times I, E' \times I)$ is the surface-link $\tilde{F}$ with the trivial 1-handles $h''_i (i = 1, 2, \ldots, 2m)$ attached. This shows (ii). Thus, the proof of (3.2.1) is completed except for the proof of (3.2.2). □

The proof of (3.2.2) is given as follows:

**Proof of (3.2.2)** Consider a disk sum decomposition of the handlebody $W$ into a 3-ball $B_0$ and solid tori $V_j = S^1 \times D^2_j (j = 1, 2, \ldots, g)$ pasting along mutually disjoint disks in $\partial B_0$. Let $(\ell_j, m_j)$ be a longitude-meridian pair of the solid torus $V_j$ for all $j$. By [1] (see [9, (2.4.1)]), the loop basis $(\ell_j, m_j)$ for $\partial V_j$ is taken as a spin loop basis in $\mathbb{R}^4$ for all $j$. The homology $H_1(\partial W; \mathbb{Z})$ has the basis $[\ell_j], [m_j], (j = 1, 2, \ldots, g)$.

For a loop basis $(\ell, \ell')$ of $T^o$ with the intersection number $\text{Int}(\ell, \ell') = 1$ in $T^o$, the image $I(T^o)$ and the kernel $K(T^o)$ of the natural homomorphism $\iota_* : H_1(T^o; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ are infinite cyclic groups. Let $x$ be an element of $H_1(T^o; \mathbb{Z})$ such that the image $\iota_*(x)$ is a generator of $I(T^o)$, and $x'$ a generator of $K(T^o)$. By noting that the intersection number $\text{Int}(x, x') = 1$ in $T^o$, let $x = a[\ell] + b[\ell']$ and $x' = a'[\ell] + b'[\ell']$ for coprime integral pairs $(a, b)$ and $(a', b')$ with $ab' - a'b = 1$. Let $(\ell'', \ell''')$ be a loop basis for $T^o$ such that $[\ell''] = x$ and $[\ell'''] = x'$.
The homology class $[\ell'''] \in H_1(\partial W; \mathbb{Z})$ is written as the sum

$$[\ell'''] = \sum_{j=1}^g b_j[m_j]$$

for an integral system $b_j (j = 1, 2, \ldots, g)$. Since $\iota_*([\ell''']) \neq 0$, there is a non-zero integer in the integers $b_j (j = 1, 2, \ldots, g)$. By changing the orientations of $\ell_j$ and the indexes of the solid tori $V_j$ if necessary, assume that $a_j \geq 0$ for all $j$ and $a_1$ is the smallest non-zero integer in the integral system $a_j (j = 1, 2, \ldots, g)$. For $j \geq 2$, let

$$a_j = n_j a_1 + r_j$$

for an integer $r_j$ with $0 \leq r_j < a_1$. By handle slides of $W$, we have a new disk sum decomposition of $W$ into a 3-ball $B_0$ and solid tori $V_j = S^1 \times D^2_j (j = 1, 2, \ldots, g)$ such that

$$[\ell''] = a_1[\ell_1] + b_1[m_1] + \sum_{j=2}^g r_j[\ell_j] + \tilde{b}_j[m_j]$$

for some integers $\tilde{b}_j (j = 2, 3, \ldots, g)$. By repeating this process, we have a disk sum decomposition of $W$ into a 3-ball $B_0$ and solid tori $V_j = S^1 \times D^2_j (j = 1, 2, \ldots, g)$ such that

$$[\ell''] = a[\ell_1] + \sum_{j=1}^g \tilde{b}_j[m_j]$$

for some integers $\tilde{b}_j (j = 1, 2, \ldots, g)$, where $a$ is the greatest common divisor of the integers $a_j (j = 1, 2, \ldots, g)$.

Let

$$[\ell''''] = \sum_{j=1}^g b_j'[m_j]$$

for an integral system $b_j' (j = 1, 2, \ldots, g)$. Since the intersection number $\text{Int}(\ell'', \ell''') = 1$ in $\partial W$, we have $ab_1' = 1$ and hence $a = 1$. By [1] (see [9, (2.4.1)]), the loop basis $(\ell'', \ell''')$ of $T^o$ is taken spin if we consider $x + x'$ instead of $x$ if necessary since $\ell'''$ is a spin loop. Since the intersection number $\text{Int}(\ell'', m_1) = 1$ in $F \# T$, we can take $(\ell'', \ell''')$, $m_1$ and a meridian disk of $m_1$ in $V_1$ as $(\ell, \ell')$, $\tilde{\ell}'$ and $D'$ in (3.2.2), respectively. Thus, the proof of (3.2.2) is completed. □

This completes the proof of Lemma 3.2. □

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References


