Ribbonness of a stable-ribbon surface-link, II. General case

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ABSTRACT

It is shown that a handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link up to equivalences. This is a generalization of the result for the case of a stably trivial surface-link previously observed.

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1 Introduction

In this paper, a generalization of the result of the paper [9] on a trivial surface-link to a result on a ribbon surface-link is explained.

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When F is connected, it is also called a *surface-knot*. When a (possibly disconnected) closed surface $\mathbf{F}$ is fixed, it is also called an $\mathbf{F}$-*link*. If $\mathbf{F}$ is the disjoint union of some copies of the 2-sphere S^2, then it is also called a 2-*link*. When $\mathbf{F}$ is connected, it is also called a *surface-knot*, and a 2-*knot* for $\mathbf{F} = S^2$. Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbf{R}^4 \to \mathbf{R}^4$. A *trivial* surface-link is a surface-link F which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in $\mathbf{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, solid torus or a disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted-unlinked* surface-link. For any given closed oriented (possibly disconnected) surface $\mathbf{F}$, a trivial $\mathbf{F}$-link exists uniquely up to equivalences (see [3]). A *ribbon* surface-link is a surface-link F which is obtained from a trivial nS^2-link O for some n (where nS^2 denotes the disjoint union of n copies of the 2-sphere S^2) by the surgery along an embedded 1-handle system (see [4], [11, II]). A *stabilization* of a surface-link F is a connected sum $\bar{F} = F\#_{k=1}^{s} T_k$ of F and a system T of trivial torus-knots $T_k\,(k = 1, 2, \dots, s)$. By granting $s = 0$, we understand that a surface-link F itself is a stabilization of F. The trivial torus-knot system T is called the *stabilizer* with *stabilizer components* $T_k\,(k = 1, 2, ..., s)$ on the stabilization $\bar{F}$ of F. A *stable-ribbon* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a ribbon surface-link.

For every surface-link F, there is a surface-link F^* with minimal total genus such that F is equivalent to a stabilization of F^*. The surface-link F^* is called a *handle-irreducible summand* of F.

The following result called *Stable-Ribbon Theorem* is our main theorem.

Theorem 1.1. A handle-irreducible summand F^* of every stable-ribbon surface-link F is a ribbon surface-link which is determined uniquely from F up to equivalences.

Since any stabilization of a ribbon surface-link is a ribbon surface-link, Theorem 1.1 implies the following corollary:

Corollary 1.2. Every stable-ribbon surface-link is a ribbon surface-link.

The following corollary of a ribbon surface-link is a standard consequence of Corollary 1.2, and contrasts with a behavior of a classical ribbon knot, for every classical knot is a connected summand of a ribbon knot.

Corollary 1.3. A connected sum $F = F_1\#F_2$ of surface-links $F_i\,(i = 1, 2)$ is a ribbon surface-link if and only if the surface-links $F_i\,(i = 1, 2)$ are both ribbon surface-links.

Proof of Corollary 1.3. The 'if' part of Corollary 1.3 is seen from the definition of a ribbon surface-link. The proof of the 'only if' part of Corollary 1.3 uses an argument of [3] showing the fact that every surface-link is made a trivial surface-knot by the surgery along a finite number of (possibly non-trivial) 1-handles. The connected summand F_2 is made a trivial surface-knot by the surgery along 1-handles within the 4-ball defining the connected sum, so that the surface-link F changes into a new ribbon surface-link and hence F_1 is a stable-ribbon surface-link. By Corollary 1.2, F_1 is a ribbon surface-link. By interchanging the roles of F_1 and F_2, F_2 is also a ribbon surface-link. □

A *stably trivial* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a trivial surface-link. Since a trivial surface-link is a ribbon surface-link, Theorem 1.1 also implies the following corollary, which is a main result in [9]:

Corollary 1.4. A handle-irreducible summand of every stably trivial surface-link is a trivial 2-link.

This corollary implies that *every stably trivial surface-link is a trivial surface-link* as observed in [9]. See [9, 10] for further results on a trivial surface-link.

The plan for the proof of Theorem 1.1 is to show the following two theorems by an argument based on [9].

Theorem 1.1.1 Any two handle-irreducible summands of any (not necessarily ribbon) surface-link are equivalent.

Theorem 1.1.2 Any stable-ribbon surface-link is a ribbon surface-link.

The proofs of Theorem 1.1.1 and 1.1.2 are given in § 2 and § 3, respectively. The proof of Theorem 1.1 is completed by these theorems as follows:

Proof of Theorem 1.1. By Theorem 1.1.2, a handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link which is unique up to equivalences by Theorem 1.1.2. □

2 Proof of Theorem 1.1.1

A 2-*handle* on a surface-link F in $\mathbf{R}^4$ is an embedded 2-handle $D \times I$ on F with D a chore disk such that $(D \times I) \cap F = (\partial D) \times I$, where I denotes a closed interval containing 0 and $D \times 0$ is identified with D. An *orthogonal 2-handle pair* (or simply, an *O2-handle pair*) on F is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$, $D' \times I)$ on F

such that the core disks D and D' meet transversely at just one point p in F with

$$(D \times I) \cap (D' \times I) = (\partial D) \times I \cap (\partial D') \times I$$

which is homeomorphic to the square $Q = I \times I$ with p the central point.

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link F. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from F by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the bounded surface $F_D^c = \mathrm{cl}(F \setminus ((\partial D) \times I \cup (\partial D') \times I))$ and the plumbed disk $\delta_D = D \times (\partial I) \cup Q \cup D' \times (\partial I)$. A compact once-punctured torus of a torus T is simply called a *punctured torus* and denoted by T^o. A punctured torus T^o in a 3-ball B is *trivial* if T^o is smoothly and properly embedded in B and there is a solid torus V in B with $\partial V = T^o \cup \delta_B$ for a disk δ_B in ∂B.

A *bump* of a surface-link F is a 3-ball B in $\mathbf{R}^4$ with $F \cap B = T^o$ a trivial punctured torus in B. Let $F(B)$ be a surface-link $F^c \cup \delta_B$ for the surface $F_B^c = \mathrm{cl}(F \setminus T^o)$ and a disk δ_B in ∂B with $\partial \delta_B = \partial T^o$, where note that $F(B)$ is uniquely determined up to cellular moves on δ_B keeping F^c fixed. For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $\Delta = D \times I \cup D' \times I$ is a 3-ball in $\mathbf{R}^4$ called the *2-handle union*. By adding a boundary collar to the 2-handle union Δ, we have a bump $B = B_D$ of F, which we call the *associated bump* of the O2-handle pair $(D \times I, D' \times I)$ (see [9, Fig. 2]).

An O2-handle pair and a bump on a surface-link are shown to be essentially equivalent notions in [9]. In particular, it is observed in [9] that for any O2-handle pair $(D \times I, D' \times I)$ on any surface-link F and the associated bump B, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

A punctured torus T^o in a 4-ball A is *trivial* if T^o is smoothly and properly embedded in A and there is a solid torus V in A with $\partial V = T^o \cup \delta_A$ for a disk δ_A in the 3-sphere ∂A. A *4D bump* of a surface-link F is a 4-ball A in $\mathbf{R}^4$ with $F \cap A = T^o$ a trivial punctured torus in A. A 4D bump A is obtained from a bump B of a surface-link F by taking a bi-collar $c(B \times [-1, 1])$ of B in $\mathbf{R}^4$ with $c(B \times 0) = B$. The following lemma is proved by using a 4D bump A.

Lemma 2.1. For an O2-handle pair $(D \times I, D' \times I)$ on a surface-link F, let $F(D \times I, D' \times I) = F_D^c \cup \delta_D$. Then for a trivial torus-knot T with a spin loop basis (ℓ, ℓ'), there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from the surface-link F to a connected sum $F(D \times I, D' \times I) \# T$ keeping F_D^c fixed such that

$$f(\partial D) = \ell \quad \text{and} \quad f(\partial D') = \ell'.$$

Proof of Lemma 2.1. Let A be a 4D bump associated with the O2-handle pair $(D \times I, D' \times I)$ on F. Let δ_A be a disk in the 3-sphere ∂A such that the union of δ_D and the trivial punctured torus $F \cap A = P$ bounds a solid torus V in A. Then there is an equivalence $f' : F \cong F(D \times I, D' \times I)\#T$ by deforming V in A so that P is isotopically deformed into the summand T^o of a connected sum $\delta_D\#T$ in A. Then the spin loop pair $(\partial D, \partial D')$ on F_1 is sent to a spin loop basis $(\tilde{\ell}, \tilde{\ell}')$ of T^o. By [2] (see [9, (2.4.2)]), there is an orientation-preserving diffeomorphism $g : \mathbf{R}^4 \to \mathbf{R}^4$ with $g|_{\mathrm{cl}(\mathbf{R}^4 \setminus A)} = 1$ such that

$$g(\tilde{\ell}, \tilde{\ell}) = (\ell, \ell').$$

By the composition gf', we have a desired equivalence f. □

A surface-link F has *only unique O2-handle pair in the rigid sense* if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F with $(\partial D) \times I = (\partial E) \times I$ and $(\partial D') \times I = (\partial E') \times I$, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to F such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$. It is shown in [9] that every surface-link F has only unique O2-handle pair in the rigid sense with an additional condition that there is an ambient isotopy $f_t\,(t \in [0,1])$ with $f_0 = 1$ and $f_1 = f$ keeping F_D^c fixed.

A surface-link F has *only unique O2-handle pair in the soft sense* if for any O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ on F attached to the same connected component of F, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to F such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair in both the rigid and soft senses.

The following lemma shows that the uniqueness of an O2-handle pair in the soft sense is derived from the uniqueness of an O2-handle pair in the rigid sense.

Lemma 2.2. Every surface-link has only unique O2-handle pair in the soft sense.

Proof of Lemma 2.2. Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be any two O2-handle pairs on a surface-link F attached to the same connected component of F.

By Lemma 2.1, there is an equivalence $f : \mathbf{R}^4 \to \mathbf{R}^4$ from F to to $F(E \times I, E' \times I)\#T$ keeping F_E^c fixed. Let $F_E = F(E \times I, E' \times I)$. Let $F_E(\dot{h})$ be a *trivial* surface-knot obtained from F_E by the surgery along a system h of mutually disjoint 1-handles $h_j\,(j = 1, 2, \ldots, s)$ on F_E.

Let $\dot{h}$ be the system of cylinders $\dot{h}_j = h_j \cap F_E(\dot{h})\,(j = 1, 2, \ldots, s)$, and $\ddot{h}$ is the system of two disks $\ddot{h}_j = \mathrm{cl}(\partial h_j \setminus \dot{h}_j)\,(j = 1, 2, \ldots, s)$.

Let $(d \times I, d' \times I)$ be a standard O2-handle pair on T^0 in the 4-ball defining the connected summand T^o in $F_E\#T$, and $(e, e') = (\partial d, \partial d')$ which is a spin loop basis of T^o. By construction, the system h of 1-handles h_j ($j = 1, 2, \ldots, s$) is disjoint from the disk pair (d, d'). By an isotopic deformation of f, we can assume that the system $f^{-1}(h)$ of the 1-handles $f^{-1}(h_j)$ ($j = 1, 2, \ldots, s$) on F is disjoint from $(D \times I, D' \times I)$.

By [2] (see [9, (2.4.2)]), there is an orientation-preserving diffeomorphism $g : \mathbf{R}^4 \to \mathbf{R}^4$ sending $F_E(\dot{h})\#T$ to itself such that the spin loop pair $(gf(\partial D), gf(\partial D')) = (e, e')$ and the restriction of g to the system $\dot{h}$ of the cylinders $\dot{h}_j$ ($j = 1, 2, \ldots, s$) is the identity map. This last condition is assumed by a choice of a spin loop basis on $F_E(\dot{h})\#T$.

By the uniqueness of an O2-handle pair in the rigid sense given in [9], there is an ambient isotopy $i_t : \mathbf{R}^4 \to \mathbf{R}^4$ ($t \in [0, 1]$) keeping $(F_E(\dot{h})\#T)^c$ fixed such that i_0 is the identity and $i_1 g(f(D) \times I, f(D') \times I) = (d \times I, d' \times I)$. Let

$$G^t = g^{-1}(\mathrm{cl}(F_E(\dot{h}) \setminus \dot{h})\#T) \cup g^{-1} i_t g(\ddot{h}) \quad (t \in [0, 1])$$

be a surface-link family with $G^0 = F_E\#T$. There is an $O2$-handle pair

$$(g^{-1} i_t g(f(D) \times I, f(D') \times I)$$

on the surface-link G^t, where

$$\begin{aligned} g^{-1} i_0 g(f(D) \times I, f(D') \times I) &= (f(D) \times I, f(D') \times I), \\ g^{-1} i_1 g(f(D) \times I, f(D') \times I) &= g^{-1}(d \times I, d' \times I). \end{aligned}$$

Then the surface-link $G^0(f(D) \times I, f(D') \times I$ is given by

$$\begin{aligned} G^0(f(D) \times I, f(D') \times I) &= (F_E\#T)(f(D) \times I, f(D') \times I) \\ &\cong F(D \times I, D' \times I) \\ &= F_D, \end{aligned}$$

and the surface-link $gG^1(d \times I, d' \times I)$ is given by

$$\begin{aligned} gG^1(d \times I, d' \times I) &= (\mathrm{cl}(F_E(\dot{h}) \setminus \dot{h})\#T \cup i_1 g(\ddot{h}))(d \times I, d' \times I) \\ &\cong (\mathrm{cl}(F_E(\dot{h}) \setminus \dot{h})\#T \cup i_1 g(\ddot{h}))(i_1 g(d \times I), i_1 g(d' \times I)) \\ &= i_1 g((F_E\#T)(d \times I, d' \times I)) \\ &\cong (F_E\#T)(d \times I, d' \times I) \\ &\cong F_E, \end{aligned}$$

where the equivalence

$$\begin{aligned} &(\mathrm{cl}(F_E(\dot{h}) \setminus \dot{h})\#T \cup i_1 g(\ddot{h}))(d \times I, d' \times I) \\ &\qquad \cong (\mathrm{cl}(F_E(\dot{h}) \setminus \dot{h})\#T \cup i_1 g(\ddot{h}))(i_1 g(d \times I), i_1 g(d' \times I)) \end{aligned}$$

is obtained from the uniqueness of an O2-handle pair in the rigid sense given in [9]. Since there is an equivalence

$$G^0(f(D)\times I, f(D')\times I) \cong gG^1(d\times I, d'\times I),$$

there is an equivalence f' from $F_D = F_D^c \cup \delta_D$ to $F_E = F_E^c \cup \delta_E$ for disks δ_D and δ_E. By a disk move, we can assume that $f'(\delta_D) = \delta_E$. The map f' is isotopic to a diffeomorphism $f'' : \mathbf{R}^4 \to \mathbf{R}^4$ sending the associated bump B_D of $(D\times I, D'\times I)$ to the associated bump B_E of $(E\times I, E'\times I)$. The diffeomorphism $f'' : \mathbf{R}^4 \to \mathbf{R}^4$ is modified into an equivalence $f''' : \mathbf{R}^4 \to \mathbf{R}^4$ from F to F such that $f'''(D\times I) = E\times I$ and $f'''(D'\times I) = E'\times I$ because the bumps B_D and B_E recover the unordered O2-handle pairs $(D\times I, D'\times I)$ and $(E\times I, E'\times I)$, respectively (cf. [9, Lemma 2.4]). Thus, every surface-link F has only unique O2-handle pair in the soft sense. □

We use the following corollary to Lemma 2.2.

Corollary 2.3. Let F, F' be surface-links with ordered components $F_i, F'_i\,(i = 1, 2, \ldots, r)$, respectively, and $\bar{F} = F\#_i T, \bar{F}' = F'\#_i T$ the stabilizations of F, F' with induced ordered components obtained by the connected sums $F_i\#T, F'_i\#T$ of the ith components F_i, F'_i and a trivial torus-knot T for some i, respectively. Assume that $\bar{F}$ is equivalent to $\bar{F}'$ by a component-order-preserving equivalence. Then F is equivalent to F' by a component-order-preserving equivalence.

Remark 2.4. Corollary 2.3 for ribbon surface-links F, F' has a different proof using the result of [8].

The proof of Theorem 1.1.1 is done as follows.

Proof of Theorem 1.1.1. A surface-link F with r ordered components is *kth-handle-reducible* if F is equivalent to a stabilization $F'\#_k n_k T$ of a surface-link F' for a positive integer n_k, where $\#_k n_k T$ denotes the stabilizer components $n_k T$ attaching to the kth component of F'. Otherwise, the surface-link F is *kth-handle-irreducible.* Note that if a kth-handle-irreducible surface-link F is component-order-preserving equivalent to a surface-link G, then G is also kth-handle-irreducible.

Let F and G be ribbon surface-links with components $F_i\,(i = 1, 2, \ldots, r)$ and $G_i\,(i = 1, 2, \ldots, r)$, respectively. Let $F^*$and G^* be handle-irreducible summands of F and G, respectively.

Assume that there is an equivalence f from F to G. Then we show that F^* and G^* are equivalent. Changing the indexes if necessary, we assume that f sends F_i to

G_i for every i. Let

$$\begin{aligned} F &= F^* \#_1 n_1 T \#_2 n_2 T \#_3 \ldots \#_r n_r T, \\ G &= G^* \#_1 n'_1 T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T. \end{aligned}$$

Taking the inverse equivalence f^{-1} instead of f if necessary, we may assume that $n'_1 \geq n_1$. If $n'_1 > n_1$, then by (*), there is an equivalence $f^{(1)}$ from the first-handle-irreducible surface-link

$$F^{(1)} = F^* \#_2 n_2 T \#_3 \ldots \#_r n_r T$$

to the first-handle-reducible surface-link

$$G^* \#_1 (n'_1 - n_1) T \#_2 n'_2 T \#_3 \ldots \#_r n'_r T,$$

which has a contradiction. Thus, $n'_1 = n_1$ and the first-handle-irreducible surface-link $F^{(1)}$ is equivalent to the first-handle-irreducible ribbon surface-link

$$G^{(1)} = G^* \#_2 n'_2 T \#_3 \ldots \#_r n'_r T.$$

By continuing this process, it is shown that F^* is equivalent to G^*. This completes the proof of Theorem 1.1.1. □

3 Proof of Theorem 1.1.2

A *chord graph* is a pair (o, α) of a trivial ink o and an arc system α attaching to o in the 3-space $\mathbf{R}^3$, where o and α are called a *based loop system* and a *chord system*, respectively. A *chord diagram* is a diagram $C(o, \alpha)$ in the plane $\mathbf{R}^2$ of a chord graph (o, α) as a spatial graph. Let D^+ be a proper disk system in the upper half-space $\mathbf{R}^4_+$ obtained from a disk system d^+ in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_+$. Similarly, let D^- be a proper disk system in the lower half-space $\mathbf{R}^4_-$ obtained from a disk system d^- in $\mathbf{R}^3$ bounded by o by pushing the interior into $\mathbf{R}^4_-$. Let O be the union of D^+ and D^- which is a trivial nS^2-link in the 4-space $\mathbf{R}^4$, where n is the number of components of o. The union $O \cup \alpha$ is called a *chorded sphere system* constructed from a chord graph (o, α).

By using the Horibe-Yanagawa lemma in [11, I], the chorded sphere system $O \cup \alpha$ up to orientation-preserving diffeomorphisms of $\mathbf{R}^4$ is independent of choices of d^+ and d^- and uniquely determined by the chord graph (o, α). A ribbon surface-link $F = F(o, \alpha)$ is uniquely constructed from the chorded sphere system $O \cup \alpha$ so that F is the surgery of O along a 2-handle system $N(\alpha)$ on O with core arc system α (see

[5, 6, 7, 8]), where note by [3] that the surface-link F up to equivalences is unaffected by choices of the 2-handle $N(\alpha)$.

A *semi-unknotted punctured handlebody system* (or simply a *SUPH system*) for a surface-link F is a punctured handlebody system V in $\mathbf{R}^4$ such that the boundary ∂V of V is a union $F \cup O$ of F and a trivial S^2-link O with $F \cap O = \emptyset$. The following lemma is a characterization of a ribbon surface-link (cf. [11, II], Yanagawa [12]).

Lemma 3.1. A surface-link F is a ribbon surface-link if and only if there is a punctured SUPH system V for F.

Proof of Lemma 3.1. Given a ribbon surface-link, a SUPH system V is constructed by a thickening $O \times I$ of O in $\mathbf{R}^4$ by attaching a 1-handle system. Conversely, given a SUPH system V in $\mathbf{R}^4$ such that $\partial V = F \cup O$ for a trivial S^2-link O with $F \cap O = \emptyset$, there is a chord system α in V attaching to O such that the frontier of the regular neighborhood of $O \cup \alpha$ in V is parallel to F, showing that F is a ribbon surface-link. □

The following lemma is basic to the proof of Theorem 1.1.2.

Lemma 3.2. The following (1) and (2) hold.

(1) For a surface-link F and a trivial torus-knot T, if a connected sum $F\#T$ is a ribbon surface-link, then F is a ribbon surface-link.

(2) If F is a ribbon surface-link and $(D \times I, D' \times I)$ is an O2-handle pair on F, then $F(D \times I, D' \times I)$ is a ribbon surface-link.

Theorem 1.1.2 is a consequence of Lemma 3.2 as follows:

Proof of Theorem 1.1.2. If a stabilization $\bar{F}$ of a surface-link F is a ribbon surface-link, then F is a ribbon surface-link by an inductive use of Lemma 3.2 (1). □

We are in a position to show Lemma 3.2.

Proof of Lemma 3.2. The assertion (1) $\Rightarrow$ (2) holds. In fact, by Lemma 2.1, there is a connected sum splitting $F \cong F(D \times I, D' \times I)\#T$ for a trivial torus-knot T. Thus, if F is a ribbon surface-link, then $F(D \times I, D' \times I)$ is a ribbon surface-link by (1).

We show (1). Let $F\#T = F_1\#T \cup F_2 \cup \cdots \cup F_r$ be a ribbon surface-link for a trivial torus-knot T. The following claim (3.2.1) is shown later.

(3.2.1) There is a stabilization $\bar{F} = \bar{F}_1 \cup F_2 \cup \cdots \cup F_r$ of $F\#T$ with $\bar{F}_1 = F_1\#T\#_{i=1}^{2m}T_i$ such that the following conditions (i) and (ii) hold:

(i) There is an O2-handle pair $(D \times I, D' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(D \times I)$ is a ribbon surface-link admitting a SUPH system with the 1-handles $h'_i\,(i = 1, 2, \ldots, 2m)$ trivially attached.

(ii) There is an O2-handle pair $(E \times I, E' \times I)$ on $\bar{F}$ attached to $\bar{F}_1$ such that the surface-link $\bar{F}(E \times I)$ is the surface-link F with the 1-handles $h''_i\,(i = 1, 2, \ldots, 2m)$ trivially attached.

By assuming (3.2.1), the proof of Lemma 3.2 is completed as follows.

By (i), the surface-link $F'' = \bar{F}(D \times I, D' \times I) \cong \bar{F}(D \times I)$ is a ribbon surface-link and further the surface-link F^* obtained from F'' by the surgery on O2-handle pairs of all the trivial 1-handles $h'_i\,(i = 1, 2, \ldots, 2m)$ is also a ribbon surface-link. By (ii), the surface-link $\bar{F}(E \times I, E' \times I) \cong \bar{F}(E \times I)$ is the surface-link F with the 1-handles $h''_i\,(i = 1, 2, \ldots, 2m)$ trivially attached. By an inductive use of Lemma 2.2 (or Theorem 1.1.1), the surface-link F is equivalent to the ribbon surface-link F^*. Hence F is a ribbon surface-link, obtaining (3). Thus, the proof of Lemma 3.2 is completed except for the proof of (3.2.1). □

We are in a position to prove the claim (3.2.1).

Proof of (3.2.1). Let V be a SUPH system for $F\#T$ by Lemma 3.1. Let the component of the SUPH system V containing $F_1\#T$ be a disk sum $U\#_\partial W$ for a punctured 3-ball U and a handlebody W. Let A be a 4D bump defining the connected sum $F\#T$ with $(F\#T) \cap A = T^o$.

We proceed the proof by assuming the following claim (3.2.2) shown later.

(3.2.2) There are a spin loop basis (ℓ, ℓ') for T^o and a spin simple loop $\tilde{\ell}'$ in $F\#T$ such that $\mathrm{Int}(\ell, \tilde{\ell}') = 1$ and $\tilde{\ell}'$ bounds a disk D' in W.

By assuming (3.2.2), the proof of (3.2.1) is completed as follows.

Let $p_i\,(i = 0, 1, \ldots, 2m)$ be the intersection points of ℓ and $\tilde{\ell}'$. For every $i > 0$, let α_i be an arc neighborhood of p_i in ℓ, and h_i a 1-handle on $F\#T$ with a core arc $\hat{\alpha}_i$ obtained by pushing the interior of α_i into $\mathbf{R}^4 \setminus V$. Let $\tilde{\alpha}_i$ be a proper arc in $\dot{\partial} h_i = \mathrm{cl}(\partial h_i \setminus h_i \cap F\#T)$ parallel to $\hat{\alpha}_i$ in h_i with $\partial\tilde{\alpha}_i = \partial\alpha_i$.

Let $\bar{F} = F\#T\#_{i=1}^{2m}T_i$ be a stabilization of F associated with the system of mutually disjoint trivial 1-handles $h_i\,(i = 1, 2, \ldots, 2m)$.

Let $\tilde{\ell}$ be a simple loop obtained from ℓ by replacing α_i with $\tilde{\alpha}_i$ for every $i > 0$. The loop $\tilde{\ell}$ is taken to be a spin loop in $\bar{F}$ meeting ℓ' transversely in just one point.

Let A be a 4D bump of the associated bump B of an O2-handle pair $(E\times I, E'\times I)$ on $F\#T$ in $\mathbf{R}^4$ attached to T^o with $(\ell,\ell')=(\partial E,\partial E')$. Then the loop $\tilde{\ell}$ and the trivial 1-handles $h_i\,(i=1,2,\dots,2m)$ are taken in A.

Let $W^+(D')$ be the handlebody obtained from the handlebody $W^+=W\cup_{i=1}^{2m} h_i$ by splitting along a thickened disk $D'\times I$ of D'. Then the manifold $V^+(D')$ obtained from $V^+=V\cup_{i=1}^{2m} h_i$ by replacing W^+ with $W^+(D')$ is a SUPH system.

The SUPH system V^+ is ambient isotopic in $\mathbf{R}^4$ to a SUPH system $\tilde{V}^+$ which is the union of $V^+(D')$ and a solid torus W_1 in A connected by a 1-handle h_W in A, where the solid torus W_1 has a deformed disk $\tilde{D}'$ of D' as a meridian disk and the loop $\tilde{\ell}$ as a longitude. Since the trivial 1-handles $h_i\,(i=1,2,\dots,2m)$ are taken in the bump B, the solid torus W_1 is moved into a 4-ball disjoint from $T^o\cup_{i=1}^{2m} h_i$ and hence the loop $\tilde{\ell}$ bounds a disk $\tilde{D}$ in A not meeting T^o, h_i for all $i>0$ and h_W. By putting back the ambient isotopy from the SUPH system $\tilde{V}^+$ to the SUPH system V^+, we see that there is an O2-handle pair $(D\times I, D'\times I)$ on the surface-link $\bar{F}$ such that $\bar{F}(D\times I)$ is a ribbon surface-link admitting trivial 1-handles $h'_i\,(i=1,2,\dots,2m)$. This shows (i).

On the other hand, the 1-handles $h_i\,(i=1,2,\dots,2m)$ on $F\#T$ are isotopically deformed in A into 1-handles $h''_i\,(i=1,2,\dots,2m)$ on $F\#T$ disjoint from the disk pair (E,E') such that the core arcs of the 1-handles $h_i\,(i=1,2,\dots,2m)$ are deformed into simple arcs in $F\#T$ away from the disk pair (E,E') in A. Hence the surface-link $\bar{F}(E\times I, E'\times I)$ is the surface-link F with the trivial 1-handles $h''_i\,(i=1,2,\dots,2m)$ attached. This shows (ii). Thus, the proof of (3.2.1) is completed except for the proof of (3.2.2). $\square$

The proof of (3.2.2) is given as follows:

Proof of (3.2.2) Consider a disk sum decomposition of the handlebody W into a 3-ball B_0 and solid tori $V_j=S^1\times D^2_j\,(j=1,2,\dots,g)$ pasting along mutually disjoint disks in ∂B_0. Let (ℓ_j,m_j) be a longitude-meridian pair of the solid torus V_j for all j. By [1] (see [9, (2.4.1)]), the loop basis (ℓ_j,m_j) for ∂V_j is taken as a spin loop basis in $\mathbf{R}^4$ for all j. The homology $H_1(\partial W;\mathbb{Z})$ has the basis $[\ell_j],[m_j]$, $(j=1,2,\dots,g)$.

For a loop basis (ℓ,ℓ') of T^o with the intersection number $\mathrm{Int}(\ell,\ell')=1$ in T^o, the image $I(T^o)$ and the kernel $K(T^o)$ of the natural homomorphism $\iota_*: H_1(T^o;\mathbb{Z})\to H_1(W;\mathbb{Z})$ are infinite cyclic groups. Let x be an element of $H_1(T^o;\mathbb{Z})$ such that the image $\iota_*(x)$ is a generator of $I(T^o)$, and x' a generator of $K(T^o)$. By noting that the intersection number Int(x,x')=1 in T^o, let $x=a[\ell]+b[\ell']$ and $x'=a'[\ell]+b'[\ell']$ for coprime integral pairs (a,b) and (a',b') with $ab'-a'b=1$. Let (ℓ'',ℓ''') be a loop basis for T^o such that $[\ell'']=x$ and $[\ell''']=x'$.

The homology class $[\ell''] \in H_1(\partial W;\mathbb{Z})$ is written as the sum

$$[\ell''] = \sum_{j=1}^{g} a_j[\ell_j] + b_j[m_j]$$

for an integral system $a_j, b_j\,(j = 1, 2, \ldots, g)$. Since $\iota_*([\ell'']) \neq 0$, there is a non-zero integer in the integers $a_j\,(j = 1, 2, \ldots, g)$. By changing the orientations of ℓ_j and the indexes of the solid tori V_j if necessary, assume that $a_j \geq 0$ for all j and a_1 is the smallest non-zero integer in the integral system $a_j\,(j = 1, 2, \ldots, g)$. For $j \geq 2$, let

$$a_j = n_j a_1 + r_j$$

for an integer r_j with $0 \leq r_j < a_1$. By handle slides of W, we have a new disk sum decomposition of W into a 3-ball B_0 and solid tori $V_j = S^1 \times D_j^2\,(j = 1, 2, \ldots, g)$ such that

$$[\ell''] = a_1[\ell_1] + b_1[m_1] + \sum_{j=2}^{g} r_j[\ell_j] + \tilde{b}_j[m_j]$$

for some integers $\tilde{b}_j\,(j = 2, 3, \ldots, g)$ By repeating this process, we have a disk sum decomposition of W into a 3-ball B_0 and solid tori $V_j = S^1 \times D_j^2\,(j = 1, 2, \ldots, g)$ such that

$$[\ell''] = a[\ell_1] + \sum_{j=1}^{g} \tilde{b}_j[m_j]$$

for some integers $\tilde{b}_j\,(j = 1, 2, \ldots, g)$, where a is the greatest common divisor of the integers $a_j\,(j = 1, 2, \ldots, g)$.

Let

$$[\ell'''] = \sum_{j=1}^{g} b'_j[m_j]$$

for an integral system $b'_j\,(j = 1, 2, \ldots, g)$. Since the intersection number $\mathrm{Int}(\ell'', \ell''') = 1$ in ∂W, we have $ab'_1 = 1$ and hence $a = 1$. By [1] (see [9, (2.4.1)]), the loop basis (ℓ'', ℓ''') of T^o is taken spin if we consider $x + x'$ instead of x if necessary since ℓ''' is a spin loop. Since the intersection number $\mathrm{Int}(\ell'', m_1) = 1$ in $F\#T$, we can take (ℓ'', ℓ'''), m_1 and a meridian disk of m_1 in V_1 as (ℓ, ℓ'), $\tilde{\ell}'$ and D' in (3.2.2), respectively. Thus, the proof of (3.2.2) is completed. □

This completes the proof of Lemma 3.2. □

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