Ribbonness of a stable-ribbon surface-link, I. A stably trivial link

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ABSTRACT

There is a question asking whether a handle-irreducible summand of every stable-ribbon surface-link is a unique ribbon surface-link. This question for the case of a trivial surface-link is affirmatively answered. That is, a handle-irreducible summand of every stably trivial surface-link is only a trivial 2-link. By combining this result with an old result of F. Hosowaka and the author that every surface-knot with infinite cyclic fundamental group is a stably trivial surface-knot, it is concluded that every surface-knot with infinite cyclic fundamental group is a trivial (i.e., an unknotted) surface-knot.

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1 Introduction

A surface-link is a closed oriented (possibly disconnected) surface $F$ embedded in the 4-space $\mathbb{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When a
(possibly disconnected) closed surface $F$ is fixed, it is also called an $F$-link. If $F$ is the disjoint union of some copies of the 2-sphere $S^2$, then it is also called a 2-link. When $F$ is connected, it is also called a surface-knot, and a 2-knot for $F = S^2$.

Two surface-links $F$ and $F'$ are equivalent by an equivalence $f$ if $F$ is sent to $F'$ orientation-preservingly by an orientation-preserving diffeomorphism (or piecewise-linear homeomorphism) $f : \mathbb{R}^4 \to \mathbb{R}^4$. The notation $F \cong F'$ is used for equivalent surface-links $F$, $F'$. A trivial surface-link is a surface-link $F$ which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in $\mathbb{R}^4$, where a handlebody is a 3-manifold which is a 3-ball, a solid torus or a boundary-disk sum of some number of solid tori. A trivial surface-knot is also called an unknotted surface-knot. A trivial disconnected surface-link is also called an unknotted and unlinked surface-link. For any given closed oriented (possibly disconnected) surface $F$, a trivial $F$-link exists uniquely up to equivalences (see [6]). A ribbon surface-link is a surface-link $F$ which is obtained from a trivial 2-link $O$ by the surgery along an embedded 1-handle system (see [10, 11, 12, 13], [16, II]). A stabilization of a surface-link $F$ is a connected sum $\tilde{F} = F \#_{k=1}^s T_k$ of $F$ and a system $T$ of trivial torus-knots $T_k (k = 1, 2, \ldots, s)$. By granting $s = 0$, we understand that a surface-link $F$ itself is a stabilization of $F$. The trivial torus-knot system $T$ is called the stabilizer on the stabilization $\tilde{F}$ of $F$ with stabilizer components $T_k (k = 1, 2, \ldots, s)$.

A stable-ribbon surface-link is a surface-link $F$ such that a stabilization $\tilde{F}$ of $F$ is a ribbon surface-link. For every surface-link $F$, there is a surface-link $F^*$ with minimal total genus such that $F$ is equivalent to a stabilization of $F^*$. The surface-link $F^*$ is called a handle-irreducible summand of $F$. The following question is a central question.

**Question 1.0.** A handle-irreducible summand of every stable-ribbon surface-link is a ribbon surface-link which is unique up to equivalences?

A stably trivial surface-link is a surface-link $F$ such that a stabilization $\tilde{F}$ of $F$ is a trivial surface-link.

In this paper, the following theorem is shown answering affirmatively this question for the case of a stably trivial surface-link. This question in the general case will be answered affirmatively in [15].

**Theorem 1.1.** Any handle-irreducible summand of every stably trivial surface-link is a trivial 2-link.

The following corollary is directly obtained from Theorem 1.1:
Corollary 1.2. Every stably trivial surface-link is a trivial surface-link.

If a surface-knot $F$ has an infinite cyclic fundamental group, then $F$ is a TOP-trivial surface-knot, which was shown by Freedman for a 2-knot and by [3, 9] for a higher genus surface-knot. In the case of a piecewise linear surface-knot (equivalent to a smooth surface-knot), it is known by [6] that a stabilization $\tilde{F}$ of the surface-knot $F$ is a trivial surface-knot, namely the surface-knot $F$ is a stably trivial surface-knot. Hence the following corollary is directly obtained from Corollary 1.2 answering the problem [17, Problem 1.55(A)] on unknotting of a 2-knot positively (see [14] for the surface-link version):

Corollary 1.3. A surface-knot $F$ is a trivial surface-knot if the fundamental group $\pi_1(\mathbb{R}^4 \setminus F)$ is an infinite cyclic group.

The exterior of a surface-knot $F$ is the 4-manifold $E = \text{cl}(\mathbb{R}^4 \setminus N(F))$ for a tubular neighborhood $N(F)$ of $F$ in $\mathbb{R}^4$. Then the boundary $\partial E$ is a trivial circle bundle over $F$. A surface-knot $F$ is of Dehn’s type if there is a section $F' \subset F$ in the bundle $\partial E$ such that the inclusion $F' \to E$ is homotopic to a constant map. By [3, Corollary 4.2], the fundamental group $\pi_1(\mathbb{R}^4 \setminus F)$ of a surface-knot $F$ of Dehn’s type is an infinite cyclic group. Thus, we have the following corollary (answering the problem [17, Problem 1.51]) on unknotting of a 2-knot of Dehn’s type positively):

Corollary 1.4. A surface-knot of Dehn’s type is a trivial surface-knot.

Unknotting Conjecture asks whether an $n$-knot $K^n$ (i.e., a smooth embedding image of the $n$-sphere $S^n$ in the $(n+2)$-sphere $S^{n+2}$) is unknotted (i.e., bounds a smooth $(n + 1)$-ball in $S^{n+2}$) if and only if the complement $S^{n+2} \setminus K^n$ is homotopy equivalent to $S^1$ (see [8] for example). This conjecture was previously known to be true for $n > 3$ by [18], for $n = 3$ by [20] and for $n = 1$ by [5, 19]. The conjecture for $n = 2$ was known only in the TOP category by [1](see also [2]). Corollary 1.3 answers this finally remained smooth unknotting conjecture affirmatively and hence Unknotting Conjecture can be changed into the following:

Unknotting Theorem. A smooth $S^n$-knot $K^n$ in $S^{n+2}$ is unknotted if and only if the complement $S^{n+2} \setminus K^n$ is homotopy equivalent to $S^1$ for every $n \geq 1$.

A main idea in our argument is to use the surgery of a surface-link on an orthogonal 2-handle pair, which is much different from the surgery of a surface-link on a single 2-handle. It is known that every surface-link $F$ in $\mathbb{R}^4$ is obtained from a higher genus
trivial surface-knot $F'$ by the surgery of $F'$ on a system of mutually disjoint 2-handles, because a handlebody in $\mathbb{R}^4$ is obtained from a connected Seifert hypersurface of $F$ by removing mutually disjoint 1-handles (see [6]). Thus, for example, every 2-twist spun 2-bridge knot in [21] is obtained from a trivial torus-knot $T$ in $\mathbb{R}^4$ by the surgery of $T$ on a single 2-handle, because it bounds a once-punctured lens space as a Seifert hypersurface.

In Section 2, it is shown that every stably trivial surface-link is a trivial surface-link if and only if the uniqueness of an orthogonal 2-handle pair on every trivial surface-link holds. In Section 3, the uniqueness of every orthogonal 2-handle pair on every surface-link is shown, by which Theorem 1.1 is obtained.

2 A triviality condition on a stably trivial surface-link

A 2-handle on a surface-link $F$ in $\mathbb{R}^4$ is an embedded 2-handle $D \times I$ on $F$ with $D$ a core disk such that $(D \times I) \cap F = (\partial D) \times I$, where $I$ denotes a closed interval containing 0 and $D \times 0$ is identified with $D$. If $D$ is an immersed disk, then call it an immersed 2-handle. An orthogonal 2-handle pair (or simply, an O2-handle pair) on $F$ is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I, D' \times I$ on $F$ such that

$$(D \times I) \cap (D' \times I) = (\partial D) \times I \cap (\partial D') \times I$$

and $(\partial D) \times I$ and $(\partial D') \times I$ meet orthogonally on $F$ (that is, $\partial D$ and $\partial D'$ meet transversely at one point $p$ and the intersection $(\partial D) \times I \cap (\partial D') \times I$ is homeomorphic to the square $Q = p \times I \times I$) (see Fig. 1).

Let $(D \times I, D' \times I)$ be an O2-handle pair on a surface-link $F$. Let $F(D \times I)$ and $F(D' \times I)$ be the surface-links obtained from $F$ by the surgeries along $D \times I$ and $D' \times I$, respectively. Let $F(D \times I, D' \times I)$ be the surface-link which is the union of the surface

$$F^c = \text{cl}(F \setminus ((\partial D) \times I \cup (\partial D') \times I))$$

and the plumbed disk

$$\delta = D \times (\partial I) \cup Q \cup D' \times (\partial I).$$

A once-punctured torus $T^o$ in a 3-ball $B$ is trivial if $T^o$ is smoothly and properly embedded in $B$ which splits $B$ into two solid tori. A bump of a surface-link $F$ is a 3-ball $B$ in $\mathbb{R}^4$ with $F \cap B = T^o$ a trivial once-punctured torus in $B$. Let $F(B)$ be a surface-link $F^c \cup \delta_B$ for the surface $F^c = \text{cl}(F \setminus T^o)$ and a disk $\delta_B$ in $\partial B$ with
\[ \partial \delta_B = \partial T^o, \] where note that \( F(B) \) is uniquely determined up to cellular moves on \( \delta_B \) keeping \( F^c \) fixed. Here, a cellular move of a surface \( P \) in \( \mathbb{R}^4 \) is a surface \( \tilde{P} \) in \( \mathbb{R}^4 \) such that the complements \( d = \text{cl}(P \setminus P_0) \) and \( \tilde{d} = \text{cl}(\tilde{P} \setminus P_0) \) of the intersection \( P_0 = P \cap P' \) are disks in the interiors of \( P \) and \( \tilde{P} \), respectively and the union \( d \cup \tilde{d} \) is a 2-sphere bounding a 3-ball smoothly embedded in \( \mathbb{R}^4 \) and not meeting \( P_0 \setminus \partial d = P_0 \setminus \partial \tilde{d} \).

For an O2-handle pair \((D \times I, D' \times I)\) on a surface-link \( F \), let \( \Delta = D \times I \cup D' \times I \) is a 3-ball in \( \mathbb{R}^4 \) called the 2-handle union. Consider the 3-ball \( \Delta \) as a Seifert hypersurface of the trivial \( S^2 \)-knot \( K = \partial \Delta \) in \( \mathbb{R}^4 \) to construct a 3-ball \( B_\Delta \) obtained from \( \Delta \) by adding an outer boundary collar. This 3-ball \( B_\Delta \) is a bump \( B_\Delta \) of \( F \), which we call the associated bump of the O2-handle pair \((D \times I, D' \times I)\). When the 3-ball \( \Delta \) and a boundary collar of \( F^c \) are deformed into the 3-space \( \mathbb{R}^3 \), this associated bump \( B_\Delta \) is also considered as a regular neighborhood of \( \Delta \) in \( \mathbb{R}^3 \) (see Fig. 2).

The following lemma shows that giving an O2-handle unordered pair on a surface-link \( F \) is the same as giving a bump of \( F \).

**Lemma 2.1.** An O2-handle unordered pair \((D \times I, D' \times I)\) on a surface-link \( F \) is uniquely constructed from any given bump \( B \) of \( F \) in \( \mathbb{R}^4 \) with \( F(D \times I, D' \times I) \cong F(B) \).

**Proof of Lemma 2.1.** For a bump \( B \) of \( F \), the set of two solid tori bounded by \( T^o = F \cap B \) is unique, whose meridian-longitude disk pair is an O2-handle pair. \( \square \)

The following lemma represents the unique nature of the surgery of a surface-link \( F \) on an O2-handle pair.

**Lemma 2.2.** For any O2-handle pair \((D \times I, D' \times I)\) on any surface-link \( F \) and the
associated bump $B$, there are equivalences

$$F(B) \cong F(D \times I, D' \times I) \cong F(D \times I) \cong F(D' \times I).$$

Further, these equivalences are attained by cellular moves keeping $F^c$ fixed.

**Proof of Lemma 2.2.** By definition, we have $F(B) \cong F(D \times I, D' \times I)$. The surface-link $F(D \times I, D' \times I)$ is equivalent to $F(D \times I)$ and $F(D' \times I)$ by cellular moves on the 3-balls $D' \times I$ and $D \times I$, respectively. □

The following lemma shows how an O2-handle pair on $F$ is obtained from an immersed 2-handle and a 2-handle on a surface-link $F$ whose attaching annuli meet orthogonally on $F$.

**Lemma 2.3.** Let $D \times I$ be an immersed 2-handle on a surface-link $F$ such that $(D \times I) \cap F = (\partial D) \times I$. Let $D' \times I$ be a 2-handle on $F$ such that the annuli $(\partial D) \times I$ and $(\partial D') \times I$ meet orthogonally on $F$. Then there is a 2-handle $D^* \times I$ on $F$ with $(\partial D^*) \times I = (\partial D) \times I$ such that $(D^* \times I, D' \times I)$ is an O2-handle pair on $F$.

**Proof of Lemma 2.3.** Assume that the immersed disk $D$ has only transversely intersecting double points. Let $a = \partial D \cap (\partial D' \times I)$, which is an arc. Let $N(a)$ be a
thin regular neighborhood of \(a\) in \(D' \times I\). Add a small embedded disk \(A\) in \(D' \times I\) to \(D\) along a part of the arc \(a\). Then take mutually disjoint simple arcs in \(D\) from the double points of \(D\) to the interior of \(A\). Slide the double points of \(D\) into \(A\) along the arcs by finger moves. Forget this additional disk \(A\) to obtain an embedded disk \(D''\) with \(\partial D'' = \partial D\). The free arc \(a\) can be replaced by \(N(a)\) without meeting the interior of \(D''\). Enlarge \(N(a)\) to \(D' \times I\). Then we see that by an ambient isotopic deformation, the pair \((D'' \times I, N(a))\) deforms into an O2-handle pair \((D^* \times I, D' \times I)\) on \(F\). □

Lemma 2.3 is applied for the following result containing an essential property to our argument.

**Lemma 2.4.** Let \(D \times I\) be an immersed 2-handle on a surface-link \(F\). Let \(D'_i \times I\) \((i = 1, 2)\) be any 2-handles on \(F\) such that \((\partial D'_1) \times I = (\partial D'_2) \times I\), and this annulus and \((\partial D) \times I\) meet orthogonally on \(F\). Then there is a 2-handle \(D^* \times I\) on \(F\) with \((\partial D^*) \times I = (\partial D) \times I\) such that the pairs \((D^* \times I, D'_i \times I)\) \((i = 1, 2)\) are O2-handle pairs on \(F\).

**Proof of Lemma 2.4.** A bi-collar of a 3-manifold \(M\) smoothly embedded in \(\mathbb{R}^4\) is the image \(c(M \times [-r, r])\) of a smooth embedding \(c : M \times [-r, r] \to \mathbb{R}^4\) such that \(c(M \times 0) = M\) for a positive number \(r\). In this proof, the 3-manifold \(M\) is taken as the 2-handle \(D'_1 \times I\) which is a 3-ball. Using a uniqueness of a regular neighborhood, we can assume that \(M\) is in the the hyperplane \(\mathbb{R}^3[0]\) of \(\mathbb{R}^4\) with the fourth coordinate \(x_4 = 0\), and

\[
c(M \times [-1, 1]) = M[-1, 1] = \{(x, y) \in \mathbb{R}^4 | x \in M, -1 \leq y \leq 1\}
\]

by identifying the hyperplane \(\mathbb{R}^3[0]\) with the 3-space \(\mathbb{R}^3\). Assume that the immersed disk \(D\) meets \(M[-1, 1]\) with the arc

\[
a = \partial D \cap M[-1, 1] = \partial D \cap (\partial D'_1 \times I)
\]

and the immersed 2-handle \(D \times I\) meets \(M[-1, 1]\) with the square

\[
Q = D \times I \cap M[-1, 1] = (\partial D \times I) \cap (\partial D'_1 \times I).
\]

Slide the loop \(\partial D_2'\) through \(F\) to be disjoint from \((\partial D'_1) \times I\). Assume that the interior of the disk \(D'_2\) meets the interior of the disk \(D'_1\) with only finitely many transverse double points. Thus, the intersection \(J = D'_2 \cap M[-1, 1]\) is the disjoint union of some number of proper disks in the 4-ball \(M[-1, 1]\) and \(J\) is disjoint from \(\partial D'_2\). Let \(N(a)\) be a thin regular neighborhood of \(a\) in \(D'_1 \times I\), and \(\hat{N}(a) = F(D'_1 \times I) \cap N(a)\) the

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Thus, the desired O2-handle pairs \((D, \partial D)\) then slide the loop \(\partial D\) for a sufficiently small \(\varepsilon > 0\). Enlarge \(N(a)[\varepsilon, \varepsilon]\) and \(f_1(\partial J)\) is in one component of \(\tilde{N}(a)[\varepsilon, \varepsilon]\) for a sufficiently small \(\varepsilon > 0\).

To construct \(f_t(0 \leq t \leq 1)\), move the trivial link \(\partial J\) in the 3-sphere \(\partial(M[−1, 1])\) into a 3-ball in \(c((D'_1 \times \partial I) \times [−1, 1])\) by a smooth isotopy avoiding \(N(a)\) and \(Q[-1, 1]\) and then deform \(M[−1, 1]\) into \(N(a)[−\varepsilon, \varepsilon]\) by a strong deformation retract.

The embedded disk \(D''\) with \(\partial D'' = \partial D\) constructed from \(D\) on the free arc \(a\) in Lemma 2.3 is assumed that

\[D'' \subset D \times I \cup M[−1, 1]\quad \text{and}\quad D'' \cap (N(a)[−\varepsilon, \varepsilon] \setminus a) = \emptyset\]

for a sufficiently small \(\varepsilon > 0\). Enlarge \(N(a)[−\varepsilon, \varepsilon]\) to \(M[−1, 1]\) by the inverse isotopy \(f_t^{-1}(0 \leq t \leq 1)\) of \(f_t\). Then we see that by an ambient isotopic deformation, the pair \((D'' \times I, f_1(J))\) and \(f_1(F)\) deform into the pair \((D' \times I, J)\) and \(F\), respectively, and then slide the loop \(\partial D'_2\) through \(F\) to be the center loop of the annulus \((\partial D'_1) \times I\).

Thus, the desired O2-handle pairs \((D' \times I, D'_2 \times I) (i = 1, 2)\) on \(F\) are obtained. □

A surface-link \(F\) has only unique O2-handle pair if for any O2-handle pairs \((D \times I, D' \times I)\) and \((E \times I, E' \times I)\) on \(F\) with \((\partial D) \times I = (\partial E) \times I\) and \((\partial D') \times I = (\partial E') \times I\), there is an equivalence \(f : \mathbb{R}^4 \rightarrow \mathbb{R}^4\) sending \(F\) to \(F\) such that \(f(D \times I) = E \times I\) and \(f(D' \times I) = E' \times I\). A surface-link not admitting any O2-handle pair is understood as a surface-link with only unique O2-handle pair. We have the following characterization on a stably trivial surface-link.

**Lemma 2.5.** The following (1)-(3) are mutually equivalent.

1. If a connected sum \(F \# T\) of a surface-link \(F\) and a trivial torus-knot \(T\) is a trivial surface-link, then \(F\) is a trivial surface-link.

2. If \(F\) is a trivial surface-link and \((D \times I, D' \times I)\) is an O2-handle pair on \(F\), then \(F(D \times I, D' \times I)\) is a trivial surface-link.

3. Any trivial surface-link has only unique O2-handle pair.

**Proof of Lemma 2.5.** (1) \(\Rightarrow\) (2): Let \(B\) be the associated bump of the O2-handle pair \((D \times I, D' \times I)\). A 4-ball \(A\) obtained by taking a bi-collar \(c(B \times [−1, 1])\) of \(B\) in
$\mathbb{R}^4$ with $c(B \times 0) = B$ gives a connected sum decomposition $F \cong F(D \times I, D' \times I) \# T$. By (1), $F(D \times I, D' \times I)$ is a trivial surface-link.

(2) $\Rightarrow$ (3): Let $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ be O2-handle pairs with $(\partial D) \times I = (\partial E) \times I$ and $(\partial D') \times I = (\partial E') \times I$. Let $F(D \times I, D' \times I) = F^c \cup \delta_D$ and $F(E \times I, E' \times I) = F^c \cup \delta_E$ be trivial surface-links for disks $\delta_D$ and $\delta_E$ in the boundaries $\partial B_{\Delta(D \times I, D' \times I)}$ and $\partial B_{\Delta(E \times I, E' \times I)}$ of the associated bumps $B_{\Delta(D \times I, D' \times I)}$ and $B_{\Delta(E \times I, E' \times I)}$ of the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$, respectively. Let $F_D$ and $F_E$ be the components of $F(D \times I, D' \times I)$ and $F(E \times I, E' \times I)$ containing the loop $\partial \delta_D = \partial \delta_E$, respectively, which are made split from the other components in $\mathbb{R}^4$ because all the components of every trivial surface-link are split in $\mathbb{R}^4$. Since $F_D$ and $F_E$ are trivial surface-knots of the same genus, there is an equivalence $f : \mathbb{R}^4 \to \mathbb{R}^4$ sending $F_D$ to $F_E$ orientation-preservingly and the other components identically. By a cellular move of $\delta_D$ in $F_D$, this map $f$ is modified to have $f(\delta_D) = \delta_E$. Further, this map $f$ is modified to send $F^c \cup \delta_D$ to $F^c \cup \delta_E$ by sending all the components except for $F_D$ and $F_E$ identically. Thus, we have an equivalence $f$ with $f(F^c) = F^c$ and $f(\delta_D) = \delta_E$. The map $f$ is isotopic to a diffeomorphism $f' : \mathbb{R}^4 \to \mathbb{R}^4$ sending the associated bump $B_{\Delta(D \times I, D' \times I)}$ of $(D \times I, D' \times I)$ to the associated bump $B_{\Delta(E \times I, E' \times I)}$ of $(E \times I, E' \times I)$ by regarding $B_{\Delta(D \times I, D' \times I)}$ and $B_{\Delta(E \times I, E' \times I)}$ as collars of $\delta_D$ and $\delta_E$, respectively. The diffeomorphism $f' : \mathbb{R}^4 \to \mathbb{R}^4$ is modified into an equivalence $f'' : \mathbb{R}^4 \to \mathbb{R}^4$ from $F$ to $F$ such that $f(D \times I) = E \times I$ and $f(D' \times I) = E' \times I$.

(3) $\Rightarrow$ (1): Let $F_i (i = 0, 1, \ldots, r)$ be the components of $F$, and $F \# T = F_0 \# T \cup F_1 \cup \cdots \cup F_r$ a trivial surface-link. Let $V$ be the disjoint union of handlebodies $V_i (i = 0, 1, \ldots, r)$ in $\mathbb{R}^4$ such that $\partial V_0 = F_0 \# T$ and $\partial V_i = F_i (i = 1, 2, \ldots, r)$.

A loop basis of $F_0 \# T$ of genus $g + 1$ is a system of oriented simple loop pairs $(e_j, e'_j) (j = 0, 1, 2, \ldots, g)$ on $F_0 \# T$ representing a basis for $H_1(F_0 \# T; \mathbb{Z})$ such that $e_j \cap e'_j = e'_j \cap e'_j = e_j \cap e'_j = \emptyset$ for all distinct $j, j'$ and $e_j \cap e'_j$ is one point with the intersection number Int$(e_j, e'_j) = +1$ in $F_0 \# T$ for all $j$. A loop basis $(e_j, e'_j) (j = 0, 1, 2, \ldots, g)$ of $F_0 \# T$ is spin if the $\mathbb{Z}_2$-quadratic function $q : H_1(F_0 \# T; \mathbb{Z}_2) \to \mathbb{Z}_2$ associated with the surface-knot $F_0 \# T$ has $q(e_j) = q(e'_j) = 0$ for all $j$. The following result is obtained from [3, Lemma 2.2] where a non-oriented spin loop basis $(e_j, e'_j) (j = 0, 1, 2, \ldots, g)$ of $F_0 \# T$ is constructed.

(2.5.1) For a surface-knot $F_0 \# T$ of genus $g + 1$ in $\mathbb{R}^4$, there is a spin loop basis $(e_j, e'_j) (j = 0, 1, 2, \ldots, g)$ of $F_0 \# T$. In particular, for a trivial surface-knot $F_0 \# T$ bounded by a handlebody $V_0$ in $\mathbb{R}^4$, every loop basis $(e_j, e'_j) (j = 0, 1, 2, \ldots, g)$ on $\partial V_0$ with $e'_j (j = 0, 1, 2, \ldots, g)$ a meridian loop system of $V_0$ has $q(e'_j) = 0$ and either $q(e_j) = 0$ or $q(e_j + e'_j) = 0$ for all $j$, where $e_j + e'_j$ denotes a Dehn twist of $e_j$ along $e'_j$.

The following result is obtained from [4]:
For any two loop bases \((e_j, e'_j) (j = 0, 1, 2, \ldots, g)\) and \((\tilde{e}_j, \tilde{e}'_j) (j = 0, 1, 2, \ldots, g)\) on a trivial genus \(g\) surface-knot \(F_0 \# T\) with \(q(e_j) = q(\tilde{e}_j)\) and \(q(e'_j) = q(\tilde{e}'_j)\) for all \(j\), there is an orientation-preserving diffeomorphism \(f : \mathbb{R}^4 \to \mathbb{R}^4\) with \(f(F_0 \# T) = F_0 \# T\) such that \(f(e_j) = \tilde{e}_j\) and \(f(e'_j) = \tilde{e}'_j\) for all \(j\).

Let \((D \times I, D' \times I)\) be an O2-handle pair on \(F \# T\) in \(\mathbb{R}^4\) attached to \(T^o\) such that \((F \# T)(D \times I, D' \times I) \cong F\). By (2.5.1), there is a spin loop basis for \(F_0 \# T\) containing the pair \((\partial D, \partial D')\). Also, let \((e_i, e'_i) (i = 0, 1, 2, \ldots, g)\) be a spin loop basis for \(F_0 \# T\) such that \(e_0\) bounds a disk \(d\) in \(\mathbb{R}^4\) with \(d \cap V = e_0\) and \(e'_0\) bounds a meridian disk \(d'\) of \(V_0\). Since the handlebodies \(V_i (i = 0, 1, \ldots, r)\) are splittable in \(\mathbb{R}^4\) by [6], we see from (2.5.2) that there is an orientation-preserving diffeomorphism \(f : \mathbb{R}^4 \to \mathbb{R}^4\) with \(f(F_0 \# T) = F_0 \# T\) and \(f|_{V_i} = 1 (i = 1, 2, \ldots, r)\) such that \(f(\partial D) = e_0\) and \(f(\partial D') = e'_0\). A thickening pair \((d \times I, d' \times I)\) of the disk pair \((d, d')\) is an O2-handle pair with \((F \# T)(d \times I, d' \times I)\) is a trivial surface-knot. Since \((f(D) \times I, f(D') \times I)\) is an O2-handle pair on \(F \# T\), we obtain from (3) that

\[
F \cong (F \# T)(D \times I, D' \times I) \\
\cong (F \# T)(f(D) \times I, f(D') \times I) \\
\cong (F \# T)(d \times I, d' \times I).
\]

Thus, \(F\) is a trivial surface-link. \(\Box\)

3 Uniqueness of an orthogonal 2-handle pair

The following theorem is our main result.

**Theorem 3.1.** Any two O2-handle pairs \((D \times I, D' \times I)\) and \((E \times I, E' \times I)\) on any (not necessarily trivial) surface-link \(F\) with \((\partial D) \times I = (\partial E) \times I\) and \((\partial D') \times I = (\partial E') \times I\) are moved into each other by a finite number of cellular moves on \(D \times I\) and \(D' \times I\) keeping \(F^c\) fixed. In particular, any (not necessarily trivial) surface-link has only unique O2-handle pair.

Theorem 3.1 and Lemma 2.5 implies Theorem 1.1.

**Proof of Theorem 1.1.** Let \(F\) be a stably trivial link. That is, assume that a stabilization \(\tilde{F} = F \# \sum_{k=1}^{s} T_k\) of \(F\) is a trivial link for some \(s \geq 1\). By Theorem 3.1 and Lemma 2.5, \(F \# \sum_{k=1}^{s} T_k\) is a trivial surface-link. Inductively, \(F\) is a surface-link, so that any handle-irreducible summand \(F^*\) of \(F\) is a trivial \(S^2\)-link. \(\Box\)
Throughout the remainder of this section, the proof of Theorem 3.1 will be done.

**Proof of Theorem 3.1.** Let \((D \times I, D' \times I)\) and \((E \times I, E' \times I)\) be O2-handle pairs on a surface-link \(F\) with \((\partial D) \times I = (\partial E') \times I\) and \((\partial D') \times I = (\partial E) \times I\).

The following lemma is actually a version of Lemma 2.4.

**Lemma 3.2.** There is an immersed 2-handle \(\tilde{D} \times I\) on \(F\) with \((\partial \tilde{D}) \times I = (\partial D) \times I\) such that the interior of \(\tilde{D}\) does not meet the interiors of \(E\) and \(E'\).

**Proof of Lemma 3.2.** Let \(M\) be the 2-handle \(E \times I\) which is a 3-ball. Assume that \(M\) is in the the hyperplane \(R^3[0]\) of \(R^4\) with the fourth coordinate \(x_4 = 0\), and

\[c(M \times [-1, 1]) = M[-1, 1] = \{(x, y) \in R^4 | x \in M, -1 \leq y \leq 1\}\]

identified by the hyperplane \(R^3[0]\) with the 3-space \(R^3\). Also, assume that the disk \(E'\) meets \(M[-1, 1]\) with the arc

\[a = \partial E' \cap M[-1, 1] = \partial E' \cap (\partial E \times I)\]

and the 2-handle \(E' \times I\) meets \(M[-1, 1]\) with the square

\[Q = E' \times I \cap M[-1, 1] = (\partial E' \times I) \cap (\partial E \times I)\]

Further, assume that \(\partial D' = \partial E'\) and \(\partial E \times I \cap M[-1, 1] = Q\). Slide the loop \(\partial D\) through \(F\) to be disjoint from \((\partial E') \times I\) and the loop \(\partial D'\) through \(F\) to be disjoint from \((\partial E) \times I\). Let \(J\) be the intersection between \(M[-1, 1]\) and the union of the interiors of the disks \(D\) and \(D'\). Since the interiors of the disks \(D\) and \(D'\) meets the interiors of the disks \(E\) and \(E'\) with only finitely many transverse double points, the set \(J\) is the disjoint union of some number of proper disks in the 4-ball \(M[-1, 1]\), which is disjoint from \(\partial D\) and the double points among the interiors of \(D, D'\) and \(E'\). Let \(\tilde{N}(a)\) be a thin regular neighborhood of \(a\) in \(E \times I\), and \(\tilde{N}(a) = F(E \times I) \cap N(a)\) the disjoint union of two disks.

Apply the method similar to Lemma 2.4 for every double point between the interiors of \(D\) and \(E'\) instead of every immersed double point of \(D\). To understand this situation simply as in Lemma 2.3, consider \(M[-1, 1]\) as the arc \(a\) attached to the boundary \(\partial E'\) so that the boundaries \(\partial D\) and \(\partial J\) are regarded as an endpoint of the arc \(a\).

Since \(\partial E' = \partial D'\), the disk \(D\) is deformed so that every double point between \(D\) and \(E'\) is moved through the immersed sphere \(E' \cup D'\) into a new double point.
between $D$ and $D'$ while avoiding the double points among the interiors of $D$, $D'$ and $E'$. Let $D''$ be the disk constructed from $D$ in this way.

Interchanging the role of $E$ and $E'$, we can also deform the disk $D''$ so that every double point between the disk $D''$ and the disk $E$ is moved into a new self-double point of the disk $D''$ through the immersed sphere $D'' \cup E$ while avoiding the double points between the interiors of $D''$ and $E$. The immersed disk $\tilde{D}$ constructed from the disk $D''$ in this way forms a desired immersed 2-handle $\tilde{D} \times I$ on $F$. □

Note that for the immersed 2-handle $\tilde{D} \times I$ on $F$ constructed in the proof of Lemma 3.2, the pair $(\tilde{D} \times I, D' \times I)$ does not always an O2-handle pair on $F$ because the interiors of the immersed disk $\tilde{D}$ and the disk $D'$ may meet. We show the following modification lemma for the pair $(\tilde{D} \times I, D' \times I)$.

Lemma 3.3. There is an O2-handle pair $(\tilde{D} \times I, D' \times I)$ on $F$ with $\tilde{D}$ an immersed disk such that $\partial \tilde{D} = \partial D = \partial D$ and the interior of $\tilde{D}$ does not meet the interiors of $E$ and $E'$.

Proof of Lemma 3.3. Let $B_{E,E'}$ be an associated bump of the O2-handle pair $(E \times I, E' \times I)$. Since the attachments of the O2-handle pairs $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ to $F$ are equal, we may consider that some neighborhoods of the attaching parts of $(D \times I, D' \times I)$ and $(E \times I, E' \times I)$ in $\mathbb{R}^4$ are consistent although we assume that the interiors of $D$, $D'$, $E$ and $E'$ transversely intersect each other in a finite number of double points. Thus, from the construction of the immersed disk $\tilde{D}$ in Lemma 3.2, it is assumed that the double points between the interiors of the immersed disk $\tilde{D}$ and the disk $D'$ as well as the self double points of $\tilde{D}$ do not belong to the bump $B_{E,E'}$. Then the intersection $B_{E,E'} \cap \tilde{D}$ is considered as an annulus $\tilde{A}$, so that $\tilde{D} = \text{cl}(\tilde{D} \setminus \tilde{A})$ is a disk. Let $\tilde{\partial} = \partial \tilde{D}$ be the boundary loop. The intersection $B_{E,E'} \cap D'$ is the disjoint union of an annulus $A'$ and some number of proper arcs $\{a_j\}$ in $B_{E,E'}$, so that

$$D'_B = D' \setminus (A' \cup \{a_j\}) \cup (\partial A' \setminus \partial D')$$

is a (non-compact) connected surface with connected boundary $\partial' = \partial A' \setminus \partial D'$ be the loop. Let $S$ be the boundary sphere $\partial B_{E,E'}$. The loops $\tilde{\partial}$ and $\partial'$ are disjoint and connected by a band $\beta$ in $S \setminus ((F \cap S) \cup \{\partial a_j\})$. Thus, every double point between $\tilde{D}_B$ and $D'_B$ is moved into a new self double point of $\tilde{D}_B$ through the immersed surface $\tilde{D}_B \cup \beta \cup D'_B$ disjoint from $F$ to obtain an immersed disk $\tilde{D}'_B$ disjoint from the surface $D'_B$. Let $\tilde{D}' = \tilde{D}'_B \cup \tilde{A}$ be an immersed disk, from which we have a desired immersed O2-handle pair $(\tilde{D}' \times I, D' \times I)$. □
By Lemma 3.3, immersed O2-handle pairs \((\tilde{D}^b \times I, D' \times I)\) and \((\tilde{D}^b \times I, E' \times I)\) on \(F\) are obtained. By Lemma 2.4, there is an embedded 2-handle \(D^* \times I\) on \(F\) such that the pairs \((D^* \times I, D' \times I)\) and \((D^* \times I, E' \times I)\) are O2-handle pairs on \(F\). By Lemma 2.2, we have
\[
F(D \times I, D' \times I) \cong F(D' \times I) \cong F(D^* \times I) \cong F(E' \times I) \cong F(E \times I, E' \times I).
\]
Each equivalence is attained by a finite number of cellular moves keeping \(F^c\) fixed, as it is observed in Lemma 2.2. This completes the proof of Theorem 3.1.

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