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On the Signature Invariants of Infinite Cyclic Coverings of Even Dimensional Manifolds

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§ 0. Introduction

We consider a compact oriented *topological* $2m$-manifold W with boundary M (which may be ϕ). Let $\gamma \in H^1(W; Z)$ and $\dot{\gamma}=\gamma|M \in H^1(M; Z)$. Let $\tilde{W}$ be the infinite cyclic covering space of W associated with γ, whose covering transformation group is infinite cyclic and denoted by $\langle t\rangle$ with a specified generator t (cf. [K3, § 0]). The boundary $\tilde{M}$ of $\tilde{W}$ is the infinite cyclic covering space of M associated with $\dot{\gamma}$ (if it is not ϕ), and we have the signature invariants $\sigma_a^{\dot{\gamma}}(M)$, $a \in [-1, 1]$, of $(M, \dot{\gamma})$ (cf. [K2], [K3]). These signature invariants were defined as a result of a duality on the cohomology ring $H^*(\tilde{M})$. This duality was first observed by Milnor [M], under the restriction that $H^*(\tilde{M})$ is finitely generated over a field. This restriction was removed in [K1]. Neumann [N2] has independently shown it by modifying the Blanchfield linking form. [*Remark*: In [M], [K1] and [N2], it was assumed that M is triangulated, but one can find a proof of its topological version in [K3, Appendix B].] In [K3], the author could compute these signature invariants by using a certain linking matrix on $(M, \dot{\gamma})$. By convention, $\sigma_a^{\dot{\gamma}}(M)=0$ if $M=\phi$. The purpose of this paper is to introduce and compute signature invariants, $\tau_{a\pm 0}^{\gamma}(W)$ of (W, γ), defined for all $a\pm 0$*) $\in [-1, 1]$ (cf. § 1). It turns out that the set $\{\tau_{a\pm 0}^{\gamma}(W) - \mathrm{sign}\, W \mid a\pm 0 \in [-1, 1]\}$ and $\{\sigma_a^{\dot{\gamma}}(M) \mid a \in [-1, 1],\ a \neq -\varepsilon(m)\}$ determine each other, where $\varepsilon(m)=(-1)^m$ and sign W denotes the usual signature of W (By convention, sign $W=0$ if $\varepsilon(m)=-1$). Moreover, we shall show that $\sigma_{-\varepsilon(m)}^{\dot{\gamma}}(M)$ can be written in terms of $\tau_{-\varepsilon(m)+\varepsilon(m)0}^{\gamma}(W)$, sign W and a certain signature invariant, $\mathrm{sign}_{\gamma} W$ of (W, γ). Thus, we can see that the signature invariants $\sigma_a^{\dot{\gamma}}(M)$, $a \in [-1, 1]$, are all *peripheral invariants* (the terms due to Neumann [N1]), such as an invariant of Atiyah/Singer [A/S], called α-invariant by Hirzebruch/Zagier [H/Z] and an invariant of Atiyah/Patodi/Singer [A/P/S], called γ-invariant by Neumann [N1], [N2]. The

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*) We use the convention $a+0 \in [-1, 1]$ for $a \in [-1, 1)$ and $a-0 \in [-1, 1]$ for $a \in (-1, 1]$.

differences $\tau^{\gamma}_{a\pm 0}(W)-\operatorname{sign} W$ appear to be closely related to some kind of γ-invariants (cf. [N1], [N2]), but we do not discuss any relationship between them in this paper.

In Section 1 we state our main results together with the definitions of $\tau^{\gamma}_{a\pm 0}(W), \sigma^{\hat{t}}_{a}(M)$ and $\operatorname{sign}_{\gamma} W$. In Section 2 several properties on $\tau^{\gamma}_{a\pm 0}(W)$ are given. In Section 3 we compute $\tau^{\gamma}_{a\pm 0}(W)$ and $\operatorname{sign}_{\gamma} W$ for a special pair (W, γ) constructed from a given pair $(M, \hat{\gamma})$. In the final section (§ 4), we prove Theorems I, II and the Proposition, stated in Section 1.

Throughout the paper, coefficients of homology and cohomology will be taken in the real number field R, unless otherwise specified. Since we intend to depend heavily on the preceding paper [K3], it will be better to note that "m" in [K3] means "$m-1$" of this paper.

§ 1. Definitions and main results

We note first that the cohomology with compact support $H^*_c(\tilde{W}, \tilde{M})$ forms a finitely generated $R\langle t\rangle$-module. In fact, the Poincaré duality $\cap[\tilde{W}]: H^*_c(\tilde{W}, \tilde{M}) \cong H_{2m-*}(\tilde{W})$ stated in [K3, Appendix A] gives a t-anti isomorphism [In fact, $(tu)\cap[\tilde{W}]=t^{-1}(u\cap t[\tilde{W}])=t^{-1}(u\cap[\tilde{W}])$] and $H_*(\tilde{W})$ is finitely generated over $R\langle t\rangle$ by [K/S] (cf. [K3]), where $[\tilde{W}]\in H^c_{2m}(\tilde{W}, \tilde{M})$ denotes the fundamental class of $\tilde{W}$ in the sense of [K3, Appendix A]. By using the cup product pairing $\cup: H^m_c(\tilde{W}, \tilde{M})\times H^m_c(\tilde{W}, \tilde{M})\to H^{2m}_c(\tilde{W}, \tilde{M})$, we define a form

$$\tilde{I}: H^m_c(\tilde{W}, \tilde{M})\times H^m_c(\tilde{W}, \tilde{M})\longrightarrow R\langle t\rangle$$

by the identity

$$\tilde{I}(u, v)=\sum_{i=-\infty}^{+\infty} \varepsilon_{\tilde{W}}[(u\cup t^i v)\cap[\tilde{W}]]t^i,$$

where $\varepsilon_{\tilde{W}}: H_0(\tilde{W})\to R$ denotes the augmentation map. For $x=u\cap[\tilde{W}]$ and $y=v\cap[\tilde{W}]$, we have the identity

$$\varepsilon_{\tilde{W}}[(u\cup t^i v)\cap[\tilde{W}]]=\operatorname{Int}_{\tilde{W}}(x, t^{-i}y)$$

by [K3, A.4], and the latter is 0 except a finite number of i by the definition of the intersection pairing $\operatorname{Int}_{\tilde{W}}$. Hence $\tilde{I}$ is well defined. The following two properties are easily established:

1.1. $\tilde{I}(fu, v)=f\tilde{I}(u, v)=\tilde{I}(u, \bar{f}v)$ for $f\in R\langle t\rangle$.

1.2. $\tilde{I}(v, u)=\varepsilon(m)\overline{\tilde{I}(u, v)}$.

Here $\bar{\ }$ denotes the involution on $R\langle t\rangle$ sending t to t^{-1}. We define a

t-Hermitian $R\langle t\rangle$-form

$$\tilde{S}\colon H_c^m(\tilde{W},\tilde{M})\times H_c^m(\tilde{W},\tilde{M})\longrightarrow R\langle t\rangle$$

by the identity $\tilde{S}(u,v)=\tilde{I}(u,v)$ (if $\varepsilon(m)=1$) or $\tilde{S}(u,v)=\tilde{I}(u,(t-t^{-1})v)$ (if $\varepsilon(m)=-1$). Let $T_c^m(\tilde{W},\tilde{M})=\mathrm{Tor}_{R\langle t\rangle}H_c^m(\tilde{W},\tilde{M})$ and $B_c^m(\tilde{W},\tilde{M})=H_c^m(\tilde{W},\tilde{M})/T_c^m(\tilde{W},\tilde{M})$. Since $R\langle t\rangle$ is a principal ideal domain, $B_c^m(\tilde{W},\tilde{M})$ is $R\langle t\rangle$-free of finite rank. By (1.1), $\tilde{I}$ and $\tilde{S}$ induce forms

$$B_c^m(\tilde{W},\tilde{M})\times B_c^m(\tilde{W},\tilde{M})\longrightarrow R\langle t\rangle,$$

also denoted by $\tilde{I}$ and $\tilde{S}$, respectively. Let $A(t)$ be an $R\langle t\rangle$-matrix, which is t-Hermitian, representing the t-Hermitian form $\tilde{S}$ on $B_c^m(\tilde{W},\tilde{M})$. For $x\in[-1,1]$, let ω_x be the complex number $x+(1-x^2)^{1/2}i$ of norm 1. As in [K3, § 5], we define

$$\tau_{a+0}(A(t))=\lim_{x\to a+0}\operatorname{sign}A(\omega_x)$$

for $a+0\in[-1,1]$ and

$$\tau_{a-0}(A(t))=\lim_{x\to a-0}\operatorname{sign}A(\omega_x)$$

for $a-0\in[-1,1]$. It is easy to see that $\tau_{a\pm0}(A(t))$ are idenpendent of a choice of $A(t)$ representing $\tilde{S}$.

Definition 1.3. $\tau^{r}_{a\pm0}(W)=\tau_{a\pm0}(A(t))$ for all $a\pm0\in[-1,1]$.

We also define forms

$$I\colon H_m(\tilde{W})\times H_m(\tilde{W})\longrightarrow R\langle t\rangle$$

and

$$S\colon H_m(\tilde{W})\times H_m(\tilde{W})\longrightarrow R\langle t\rangle$$

by the identities $I(x,y)=\tilde{I}(u,v)$ and $S(x,y)=\tilde{S}(u,v)$ for $x=u\cap[\tilde{W}]$, $y=v\cap[\tilde{W}]$. Then we have

$$I(x,y)=\sum_{i=-\infty}^{+\infty}\mathrm{Int}_{\tilde{W}}(x,t^{-i}y)t^{i}=\sum_{i=-\infty}^{+\infty}\mathrm{Int}_{\tilde{W}}(x,t^{i}y)t^{-i},$$

which is $\tilde{\mathrm{I}}\mathrm{nt}_{\tilde{W}}(x,y)$ in [K3, Appendix C] and $S(x,y)=I(x,y)$ (if $\varepsilon(m)=1$) or $S(x,y)=I(x,(t^{-1}-t)y)$ (if $\varepsilon(m)=-1$). Noting that $I(\bar{f}x,y)=fI(x,y)=I(x,fy)$ for $f\in R\langle t\rangle$, we see that I and S also induce forms

$$B_m(\tilde{W})\times B_m(\tilde{W})\longrightarrow R\langle t\rangle,$$

also denoted by I and S, repectively. Since $\cap[\tilde{W}]$ induces a t-anti isomorphism $B_c^m(\tilde{W}, \tilde{M}) \cong B_m(\tilde{W})$, it follows that $\tau_{a\pm 0}^{\gamma}(W) = \tau_{a\pm 0}(A(t))$ for any t-Hermitian $R\langle t\rangle$-matrix $A(t)$ representing the form S on $B_m(\tilde{W})$. The invariants $\sigma_a^{\dot{\gamma}}(M)$ are briefly defined as follows (see [K3] for details): Given $(M, \dot{\gamma})$, we have an $\varepsilon(m-1)$-symmetric pairing $T^{m-1}(\tilde{M}) \times T^{m-1}(\tilde{M}) \xrightarrow{\cup} T^{2m-2}(\tilde{M})$ induced from the cup product pairing $H^{m-1}(\tilde{M}) \times H^{m-1}(\tilde{M}) \xrightarrow{\cup} H^{2m-2}(\tilde{M})$ and a t-invariant unique homomorphism $\tilde{\mu}\colon T^{2m-2}(\tilde{M}) \to R$, where $T^*(\tilde{M}) = \mathrm{Hom}_R[T_*(\tilde{M}), R]$, $T_*(\tilde{M}) = \mathrm{Tor}_{R\langle t\rangle} H_*(\tilde{M})$. The quadratic form $\tilde{b}\colon T^{m-1}(\tilde{M}) \times T^{m-1}(\tilde{M}) \to R$ is defined by the identity $\tilde{b}(u, v) = \tilde{\mu}(u \cup v)$ (if $\varepsilon(m) = -1$) or $\tilde{b}(u, v) = \tilde{\mu}(u \cup (t - t^{-1})v)$ (if $\varepsilon(m) = 1$). Let $T^{m-1}(\tilde{M})_a$ be the $p_a(t)$-component of $T^{m-1}(\tilde{M})$ with $p_a(t)$ being $t^2 - 2at + 1$ (if $a \in (-1, 1)$) or $t \mp 1$ (if $a = \pm 1$). Then $\sigma_a^{\dot{\gamma}}(M)$ is defined to be the signature of $\tilde{b} \mid T^{m-1}(\tilde{M})_a$. Let $\sigma^{\dot{\gamma}}(M)$ be the signature of $\tilde{b}$. Then we have $\sigma^{\dot{\gamma}}(M) = \sum_{a \in [-1,1]} \sigma_a^{\dot{\gamma}}(M)$. We shall prove the following:

Theorem I. *For all* $a \in (-1, 1)$, $\sigma_a^{\dot{\gamma}}(M) = \tau_{a-0}^{\gamma}(W) - \tau_{a+0}^{\gamma}(W)$, *and* $\sigma_{\varepsilon(m)}^{\dot{\gamma}}(M) = \varepsilon(m)(\tau_{\varepsilon(m)-\varepsilon(m)0}^{\gamma}(W) - \text{sign } W)$.

Note that the invariant $\sigma_{-\varepsilon(m)}^{\dot{\gamma}}(M)$ does not appear in Theorem I. Since $\tau_{a\pm 0}^{\gamma}(W)$ are locally constant on a except a finite number of a (cf. [K3, § 5]), we see that Theorem I is equivalent to the following:

Theorem I′. *For all* $a \pm 0 \in [-1, 1]$, $\tau_{a-0}^{\gamma}(W) - \text{sign } W = \sum_{x \in [a,1]} \sigma_x^{\dot{\gamma}}(M)$ (*if* $\varepsilon(m) = 1$) *or* $-\sum_{x \in [-1,a)} \sigma_x^{\dot{\gamma}}(M)$ (*if* $\varepsilon(m) = -1$), *and* $\tau_{a+0}^{\gamma}(W) - \text{sign } W = \sum_{x \in (a,1]} \sigma_x^{\dot{\gamma}}(M)$ (*if* $\varepsilon(m) = 1$) *or* $-\sum_{x \in [-1,a]} \sigma_x^{\dot{\gamma}}(M)$ (*if* $\varepsilon(m) = -1$).

Here are two remarks on the invariant $\sigma_{-\varepsilon(m)}^{\dot{\gamma}}(M)$.

Remark 1.4. When $\varepsilon(m) = -1$, we have $\sigma^{\dot{\gamma}}(M) = 0$, so that by Theorem I

$$\sigma_1^{\dot{\gamma}}(M) = -\sum_{x \in [-1,1)} \sigma_x^{\dot{\gamma}}(M) = \tau_{1-0}^{\gamma}(W) = \tau_{1-0}^{\gamma}(W) - \text{sign } W.$$

To see that $\sigma^{\dot{\gamma}}(M) = 0$, we first assume that γ has a *leaf* U in W (see [K3] for this terminology) so that $V = \partial U$ is a leaf of $\dot{\gamma}$ in M. Then $\sigma^{\dot{\gamma}}(M) = \text{sign } V$ (cf. [K3]), which is clearly 0. If γ has no leaf in W, we consider the product $(W, M) \times CP^2 = (W_P, M_P)$ and $\gamma_P \in H^1(W_P; Z)$ corresponding to γ. Then by transversality on a map $f_P\colon W_P \to S^1$ respresenting γ_P (cf. Kirby/Siebenmann [K/S]), γ_P has a leaf in W_P whose boundary is a leaf of $\dot{\gamma}_P$ in M_P. Hence $\sigma^{\dot{\gamma}_P}(M_P) = 0$. By [K3, Lemma 1.2], $\sigma^{\dot{\gamma}_P}(M_P) = \sigma^{\dot{\gamma}}(M)$, so that $\sigma^{\dot{\gamma}}(M) = 0$.

This construction (W_P, M_P, γ_P) from (W, M, γ) will be used later.

Remark 1.5. When $\varepsilon(m)=1$, $\tau^{\gamma}_{a\pm 0}(W)$ and sign W do not restrict $\sigma^{\dot{\gamma}}_{-1}(M)$. In fact, we have the following:

Proposition. *For each integer $s\neq 0$ and each $m>0$ with $\varepsilon(m)=1$, there is a compact connected oriented $2m$-manifold W with an element $\gamma\in H^1(W;Z)$ such that* sign $W=\tau^{\gamma}_{a\pm 0}(W)=0$ *for all $a\pm 0\in[-1,1]$, but $\sigma^{\dot{\gamma}}_{-1}(M)=s$.*

To capture the invariant $\sigma^{\dot{\gamma}}_{-1}(M)$ in the case $\varepsilon(m)=1$, we define $\operatorname{sign}_\gamma W$ to be the signature of the double covering space of W associated with the (mod 2) reduction $\gamma(2)\in H^1(W;Z_2)$ of γ. By convention, $\operatorname{sign}_\gamma W=0$ if $\varepsilon(m)=-1$. We shall obtain the following:

Theorem II. *When $\varepsilon(m)=1$, $\sigma^{\dot{\gamma}}_{-1}(M)=\operatorname{sign}_\gamma W-\operatorname{sign} W-\tau^{\gamma}_{-1+0}(W)$.*

Before concluding this section, we give a note from the bordism theory.

Remark 1.6. We consider a pair $(M^{2m-1},\dot{\gamma})$ which may not be a boundary. If some multiple $N(M,\dot{\gamma})$ $(N>0)$ is the boundary of a pair (W,γ), then we can see that $\sigma^{\dot{\gamma}}_a(M)$, $a\in[-1,1]$, are still peripheral invariants. In fact, the resulting identities can be obtained as the identities in Theorems I, II with the right hand sides divided by N. When $\varepsilon(m)=1$, some multiple of $(M,\dot{\gamma})$ is a boundary. In fact, we have a natural isomorphism $\Omega^{\mathrm{Diff}}_*(S^1)\otimes Q\cong\Omega^{\mathrm{Top}}_*(S^1)\otimes Q$ by Wall [W, p. 190] and $\Omega^{\mathrm{Diff}}_*(S^1)\cong[\Omega^{\mathrm{Diff}}_{*-1}\otimes H_1(S^1;Z)]\oplus[\Omega^{\mathrm{Diff}}_*\otimes H_0(S^1;Z)]$ by Conner/Floyd [C/F]. $\Omega^{\mathrm{Diff}}_*\otimes Q$ is well-known by Thom to be the algebra on generators represented by CP^{2i}, $i=0,1,2,\cdots$. In case $\varepsilon(m)=1$, we have $\Omega^{\mathrm{Top}}_{2m-1}(S^1)\otimes Q=0$, implying the above assertion. When $\varepsilon(m)=-1$, the same assertion does not hold in general. In this case, note that $\sigma^{\dot{\gamma}}(M)$ is a bordism invariant (cf. Remark 1.4) and if $m>3$, then by [K/S] $\dot{\gamma}$ has a leaf V in M. Then we can see that some multiple of $(M,\dot{\gamma})$ is a boundary if and only if $\sigma^{\dot{\gamma}}(M)=0$ (when $m=1,3$) or V represents 0 in $\Omega^{\mathrm{Top}}_{2m-2}\otimes Q$ (when $m>3$ and $\varepsilon(m)=-1$).

§ 2. Several properties on the signature invariants

Lemma 2.1. *Assume that W is closed and (W,γ) is the boundary of a pair (X,γ_X) with X a compact oriented manifold and $\gamma_X\in H^1(X;Z)$. Then $\tau^{\gamma}_{a\pm 0}(W)=0$ for all $a\pm 0\in[-1,1]$.*

Proof. Consider the following exact part (obtained from the exact sequence of the infinite cyclic covering space pair $(\tilde{X},\tilde{W})$):

$$H_{m+1}(\tilde{X},\tilde{W})_{R(t)}\xrightarrow{\partial}H_m(\tilde{W})_{R(t)}\xrightarrow{i_*}H_m(\tilde{X})_{R(t)},$$

where $R(t)$ is the quotient field of $R\langle t\rangle$ and $H_*(\ )_{R(t)}=H_*(\ )\otimes_{R\langle t\rangle}R(t)$. By the Blanchfield duality (cf. [K3, Appendix C]), the form I on $B_m(\tilde{W})$ is non-singular. Hence the extension of I to $H_m(\tilde{W})_{R(t)}$ (also denoted by I) is non-singular. Similarly, we obtain the non-singular form

$$I_{\tilde{X}}: H_{m+1}(\tilde{X}, \tilde{W})_{R(t)}\times H_m(\tilde{X})_{R(t)}\longrightarrow R(t)$$

induced from $\tilde{\mathrm{Int}}_{\tilde{X}}$ in [K3, Appendix C]. Note that $I(\partial x, y)=I_{\tilde{X}}(x, i_*(y))$ for $x\in H_{m+1}(\tilde{X}, \tilde{W})_{R(t)}$ and $y\in H_m(\tilde{W})_{R(t)}$. Then $\mathrm{Im}\,\partial=(\mathrm{Im}\,\partial)^{\perp}$ with respect to I on $H_m(\tilde{W})_{R(t)}$. Let $A^+(t)$ be a t-Hermitian $R(t)$-matrix representing the extension of S to $H_m(\tilde{W})_{R(t)}$. It is easy to see that

$$\tau_{a\pm 0}(A^+(t))=\lim_{x\to a\pm 0}\ \mathrm{sign}\ A^+(\omega_x)$$

are well defined and independent of a choice of $A^+(t)$ and, in particular, equal to $\tau^{\gamma}_{a\pm 0}(W)$. Since $\mathrm{Im}\,\partial=(\mathrm{Im}\,\partial)^{\perp}$, we have that $\tau_{a\pm 0}(A^+(t))=0$, so that $\tau^{\gamma}_{a\pm 0}(W)=0$, for all $a\pm 0\in[-1, 1]$. This completes the proof.

Lemma 2.2. *When $\gamma=0$ or W is closed, $\tau^{\gamma}_{a\pm 0}(W)=\mathrm{sign}\ W$ for all $a\pm 0\in[-1, 1]$.*

Proof. When $\gamma=0$, the identities $\tau^{\gamma}_{a\pm 0}(W)=\mathrm{sign}\ W$ follow from the definition of the form I. When W is closed, we see from the bordism theory (cf. Remark 1.6) that some multiple $N(W, \gamma)$ is bordant to a pair (W_0, γ_0) with $\gamma_0\doteq 0$. By Lemma 2.1 and the above remark, $N\tau^{\gamma}_{a\pm 0}(W)=\tau^{\gamma_0}_{a\pm 0}(W_0)=\mathrm{sign}\ W_0=N\ \mathrm{sign}\ W$, so that $\tau^{\gamma}_{a\pm 0}(W)=\mathrm{sign}\ W$. This completes the proof.

The following is an infinite cyclic covering version of the Novikov addition theorem (cf. [A/S, Proposition 7.1]):

Lemma 2.3. *Assume that W is splitted into two compact submanifolds W_1, W_2 by a closed orientable $(2m-1)$-submanifold M_0 in* Int W. *Let $\gamma_i=\gamma\,|\,W_i$, $i=1, 2$. Then we have*

$$\tau^{\gamma}_{a\pm 0}(W)=\tau^{\gamma_1}_{a\pm 0}(W_1)+\tau^{\gamma_2}_{a\pm 0}(W_2)$$

for all $a\pm 0\in[-1, 1]$.

Proof. Let $B_i=\mathrm{Im}\,[H_m(\tilde{W}_i)_{R(t)}\to H_m(\tilde{W})_{R(t)}]$, $i=1, 2$. By the Mayer/Vietoris sequence for $(\tilde{W}; \tilde{W}_1, \tilde{W}_2; \tilde{M}_0)$ and the Blanchfield duality, we have an orthogonal splitting $H_m(\tilde{W})_{R(t)}=B_1^*\perp B_2^*\perp(C\oplus D)$ with respect to the (extended) form I on $H_m(\tilde{W})_{R(t)}$ such that B_i^* is a maximal non-singular $R(t)$-subspace in B_i and $C=\mathrm{Im}\,[H_m(\tilde{M}_0)_{R(t)}\oplus H_m(\tilde{M})_{R(t)}\to$

$H_m(\tilde{W})_{R(t)}]$ $(\tilde{M}=\partial\tilde{W})$ and $\partial|D\colon D\cong \operatorname{Im}\partial$ for the Mayer/Vietoris boundary $\partial\colon H_m(\tilde{W})_{R(t)}\to H_{m-1}(\tilde{M}_0)_{R(t)}$. Using the $R(t)$-extension I_0 of the Blanchfield duality pairing $\tilde{\operatorname{Int}}_{\tilde{M}_0}\colon B_{m-1}(\tilde{M}_0)\times B_m(\tilde{M}_0)\to R\langle t\rangle$ in [K3, Appendix C], we see that for each $x\neq 0$ in D, there is y_0 in $H_m(\tilde{M}_0)_{R(t)}$ such that $I_0(\partial x, y_0)\neq 0$. But, $I_0(\partial x, y_0)=I(x, y)$ for the image y of y_0 under $H_m(\tilde{M}_0)_{R(t)}\to H_m(\tilde{W})_{R(t)}$. Hence we have an $R(t)$-subspace $C_0\subset C$ such that $I|(C_0\oplus D)$ is non-singular. Since $I|C=0$, we have an orthogonal splitting $(C_0\oplus D)\perp C_1$ of $C\oplus D$ for some C_1 in C, so that the signature invariants $\tau_{a\pm 0}$ of $I|(C\oplus D)$ are 0. Clearly, the signature invariants $\tau_{a\pm 0}$ of $I|B_i^*$ are equal to $\tau_{a\pm 0}^{\gamma_i}(W_i)$. The result follows.

Lemma 2.4. *Let (W_i,γ_i), $i=1, 2$, have the same boundary (M,γ). Then we have*

$$\tau_{a\pm 0}^{\gamma_1}(W_1)-\operatorname{sign} W_1=\tau_{a\pm 0}^{\gamma_2}(W_2)-\operatorname{sign} W_2$$

for all $a\pm 0\in[-1, 1]$.

Proof. It is direct from Lemmas 2.2 and 2.3.

Lemma 2.5. *Let $(W_P, M_P)=(W, M)\times CP^2$ and $\gamma_P\in H^1(W_P\colon Z)$ correspond to $\gamma\in H^1(W; Z)$. Then we have $\tau_{a\pm 0}^{\gamma_P}(W_P)=\tau_{a=0}^{\gamma}(W)$ for all $a\pm 0\in[-1, 1]$.*

Proof. Note that

$$B_c^{m+2}(\tilde{W}_P, \tilde{M}_P)=[B_c^{m-2}(\tilde{W}, \tilde{M})\otimes H^4(CP^2)]\oplus[B_c^m(\tilde{W}, \tilde{M})\otimes H^2(CP^2)]$$
$$\oplus[B_c^{m+2}(\tilde{W}, \tilde{M})\otimes H^0(CP^2)]$$

and the form $\tilde{I}_P$ on $B_c^{m+2}(\tilde{W}_P, \tilde{M}_P)$ vanishes on the first and third summands, and the second summand is orthogonal to the first and third. The restriction $\tilde{I}_P|B_c^m(\tilde{W}, \tilde{M})\otimes H^2(CP^2)$ is clearly isomorphic to the form $\tilde{I}$ on $B_c^m(\tilde{W}, \tilde{M})$. Thus the result follows easily.

§ 3. A special construction and the signature invariants

We consider a pair (M,γ) (which may not be boundary) such that M is a closed oriented $(2m-1)$-manifold and $\gamma\in H^1(M; Z)$ has a leaf V. We orient the product $M\times[-1, 1]$ so that $M\times 1$ with the induced orientation is identified with M. Let N_V be a bicollar neighborhood of V in M. Let $W_c=\operatorname{cl}(M\times[-1, 1]-N_V\times[-1/2, 0])$ and $U=V\times[0, 1]$. By using the product framing of $N_V\times[-1/2, 0]$, we identify $(N_V\times[-1/2, 0], \partial(N_V\times[-1/2, 0]), V\times(-1/4))$ with $(V\times D^2, V\times S^1, V\times 0)$. Note that $M\times[-1, 1]=W_c\cup V\times D^2$ and $\partial W_c=M\times(-1)+V\times S^1+M$. By the

Pontrjagin/Thom construction, we have an element $\gamma_c \in H^1(W_c; Z)$ such that U is a leaf of γ_c, $\gamma_c | M = \dot{\gamma}$, $\gamma_c | M \times (-1) = 0$ and $\gamma_c | V \times S^1$ is represented by the projection $V \times S^1 \to S^1$. Let A be a linking matrix on $K_{m-1}(V) = \mathrm{Ker}\,[i_*: H_{m-1}(V) \to H_{m-1}(M)]$ (cf. [K3]). Let $A^{\varepsilon(m-1)}(t)$ be the associated t-Hermitian matrix $[(1-t^{-1}) - \varepsilon(m-1)(1-t)][(1-t)A - \varepsilon(m-1)(1-t^{-1})A'] = [(1-t^{-1}) + \varepsilon(m)(1-t)][(1-t)A + \varepsilon(m)(1-t^{-1})A']$.

Lemma 3.1. $\tau^{\gamma_c}_{a\pm 0}(W_c) = \varepsilon(m)\tau_{a\pm 0}(A^{\varepsilon(m-1)}(t))$ *for all* $a \pm 0 \in [-1, 1]$.

Proof. Let W'_c be the manifold obtained from W_c by splitting along U. Let U^+ and U^- be copies of U in $\partial W'_c$ so that $U^{\pm} \cong \pm U$. The infinite cyclic covering space $\tilde{W}_c$ of W_c associated with γ_c is constructed from the topological sum of copies $(W'_c)_j$, $j \in Z$, of W'_c by pasting U^-_{j-1} to U^+_j, so that t translates each $(W'_c)_{j-1}$ to $(W'_c)_j$. By the Mayer/Vietoris sequence, we have the following $R\langle t\rangle$-exact sequence (cf. Levine [L]):

$$\longrightarrow H_m(W'_c)\otimes R\langle t\rangle \xrightarrow{J} H_m(\tilde{W}_c) \xrightarrow{\partial} H_{m-1}(U)\otimes R\langle t\rangle \xrightarrow{I^-_* - tI^+_*} H_{m-1}(W'_c)\otimes R\langle t\rangle \longrightarrow,$$

where $I^{\pm}: U \cong U^{\mp} \subset W'_c$ and J is induced by the natural map from the topological sum of $(W'_c)_j$'s to $\tilde{W}_c$. Letting $\partial W'_c = M^* + M \times (-1)$, we have a homeomorphism

$$h: (W'_c; M^*, M\times(-1)) \cong (M\times[-1, 1], M\times 1, M\times(-1))$$

such that $h | M\times(-1)$ is the identity. Note that the following square

$$\begin{array}{ccc}
H_{m-1}(V)\otimes R\langle t\rangle & \xrightarrow{(1-t)i_*} & H_{m-1}(M)\otimes R\langle t\rangle \\
\cong\big\downarrow & & \big\downarrow\cong \\
H_{m-1}(U)\otimes R\langle t\rangle & \xrightarrow{I^-_* - tI^+_*} & H_{m-1}(W'_c)\otimes R\langle t\rangle
\end{array}$$

is commutative, where the left vertical map is induced from the inclusion $V = V \times 1 \subset U$ and the right vertical map is induced from the composite $M = M \times 1 \underset{h^{-1}}{\cong} M^* \subset W'_c$. Then we see that the above exact sequence is reduced to the following exact sequence with J', ∂' induced from J, ∂:

$$(\sharp) \quad H_m(W'_c)\otimes R\langle t\rangle \xrightarrow{J'} B_m(\tilde{W}_c) \xrightarrow{\partial'} K_{m-1}(V)\otimes R\langle t\rangle \longrightarrow 0.$$

Let $e_1 = \{c_1\}, \cdots, e_s = \{c_s\}$ be a basis for $K_{m-1}(V)$. Let $c^{(+)}_i$ and $c^{(-)}_i$ be copies of the cycle $c_i \times (1/2)$ in U^+ and U^-, respectively. Let $\bar{c}^{(+)}_i$ and $\bar{c}^{(-)}_i$ be m-chains in W'_c such that $\partial \bar{c}^{(\pm)}_i = c^{(\pm)}_i$. Let $s(c_i)$ be the m-cycle

$t\bar{c}_i^{(-)}-\bar{c}_i^{(+)}$ in $\tilde{W}_c$, $i=1,\cdots,s$, where we identify W'_c with $(W'_c)_0\subset\tilde{W}_c$. By the sequence (♯), there is an $R\langle t\rangle$-basis $\bar{e}_1,\cdots,\bar{e}_s,\bar{e}_{s+1},\cdots,\bar{e}_n$ of $B_m(\tilde{W}_c)$ such that $\bar{e}_i=\{s(c_i)\}$ for $i\leq s$ and $\bar{e}_i\in\operatorname{Im}J'$ for $i\geq s+1$. If i or j is $\geq s+1$, then clearly $I(\bar{e}_i,\bar{e}_j)=0$, since $\operatorname{Im}J'$ is represented by cycles in $M\times(-1)$. Assume that both i and j are $\leq s$. Then

$$\begin{aligned}I(\bar{e}_i,\bar{e}_j)&=\operatorname{Int}_{\tilde{W}_c}(s(c_i),t^{-1}s(c_j))t+\operatorname{Int}_{\tilde{W}_c}(s(c_i),s(c_j))+\operatorname{Int}_{\tilde{W}_c}(s(c_i),ts(c_j))t^{-1}\\&=-\operatorname{Int}_{W'_c}(\bar{c}_i^{(+)},\bar{c}_j^{(-)})t+\operatorname{Int}_{W'_c}(\bar{c}_i^{*(-)},\bar{c}_j^{(-)})+\operatorname{Int}_{W'_c}(\bar{c}_i^{*(+)},\bar{c}_j^{(+)})\\&\quad-\operatorname{Int}_{W'_c}(\bar{c}_i^{(-)},\bar{c}_j^{(+)})t^{-1},\end{aligned}$$

where $\bar{c}_i^{*(\pm)}$ are m-chains similar to $\bar{c}_i^{(\pm)}$ but beginning with the cycle $c_i\times(1/4)$ in place of $c_i\times(1/2)$. Let $A=(a_{ij})$ with $a_{ij}=\operatorname{Link}_M(c_i^+,c_j)$ (cf. [K3, § 0]). Noting that hU^- is a translation (with opposite orientation) of hU^+ in M in the positive normal direction, we have

$$\begin{aligned}&\operatorname{Int}_{W'_c}(\bar{c}_i^{(+)},\bar{c}_j^{(-)})=\varepsilon(m)\operatorname{Link}_M(c_i,c_j^+)=\operatorname{Link}_M(c_j^+,c_i)=a_{ji},\\&\operatorname{Int}_{W'_c}(\bar{c}_i^{*(-)},\bar{c}_j^{(-)})=\varepsilon(m)\operatorname{Link}_M(c_i,c_j^+)=a_{ji},\\&\operatorname{Int}_{W'_c}(\bar{c}_i^{*(+)},\bar{c}_j^{(+)})=\varepsilon(m)\operatorname{Link}_M(c_i^+,c_j)=\varepsilon(m)a_{ij},\quad\text{and}\\&\operatorname{Int}_{W'_c}(\bar{c}_i^{(-)},\bar{c}_j^{(+)})=\varepsilon(m)\operatorname{Link}_M(c_i^+,c_j)=\varepsilon(m)a_{ij}.\end{aligned}$$

That is, we have $I(\bar{e}_i,\bar{e}_j)=(1-t)a_{ji}+\varepsilon(m)(1-t^{-1})a_{ij}$. Hence the form I on $B_m(\tilde{W}_c)$ is represented by the block sum of $(1-t)A'+\varepsilon(m)(1-t^{-1})A$ and a zero matrix. When $\varepsilon(m)=1$, it is easy to see that

$$\tau^{\gamma_c}_{a\pm0}(W_c)=\tau_{a\pm0}((1-t)A'+(1-t^{-1})A)=\tau_{a\pm0}(A^{-1}(t)).$$

When $\varepsilon(m)=-1$,

$$\tau^{\gamma_c}_{a\pm0}(W_c)=\tau_{a\pm0}((t^{-1}-t)[(1-t)A'-(1-t^{-1})A])=-\tau_{a\pm0}(A^1(t)).$$

This completes the proof.

Combining Lemma 3.1 with the Main Theorem of [K3], we obtain the following:

Lemma 3.2. *For all* $a\in(-1,1)$, $\sigma^{\dot{\gamma}}_a(M)=\tau^{\gamma_c}_{a-0}(W_c)-\tau^{\gamma_c}_{a+0}(W_c)$ *and* $\sigma^{\dot{\gamma}}_{\varepsilon(m)}(M)=\varepsilon(m)\tau^{\gamma_c}_{\varepsilon(m)-\varepsilon(m)0}(W_c)$.

Let $W_c^{(2)}$ be the double covering space over W_c associated with $\gamma_c(2)\in H^1(W_c;Z_2)$. Let $\hat{W}^{(2)}=W_c^{(2)}\cup V\times D^2$ be the double branched covering space over $M\times[-1,1]=W_c\cup V\times D^2$ with branch set $V\times 0\subset V\times D^2$.

Lemma 3.3. *When* $\varepsilon(m)=1$, sign $W_2^{(2)}=$ sign $\hat{W}^{(2)}=\sigma^{\dot{\gamma}}(M)$.

Proof. The first identity is a result of the Novikov addition theorem, for sign $V \times D^2 = 0$. We show the second identity. Let $\hat{W}'$ be the manifold obtained from $M \times [-1, 1]$ by splitting along $V \times (0, 1]$. Let U^+ and U^+ be copies of $V \times [0, 1]$ in $\hat{W}'$ with $U^+ \cap U^- = V \times 0$. Let $\bar{U} = U^+ \cup U^-$. Taking another copy $(\hat{W}'_1, \bar{U}_1 = U_1^+ \cup U_1^-)$ of $(\hat{W}', \bar{U} = U^+ \cup U^-)$, we can consider that $\hat{W}^{(2)}$ is the union $\hat{W}' \cup \hat{W}'_1$ identifying $\bar{U}$ with $\bar{U}_1$ so that $U^+ \equiv U_1^-$ and $U^- \equiv U_1^+$. Note that $(\hat{W}', \bar{U}) \cong (M \times [-1, 1], N_v \times 1)$. Let $e_1 = \{c_1\}, \cdots, e_s = \{c_s\}$ be a basis for $K_{m-1}(V)$. By the Mayer/Vietoris sequence, we have a basis $\bar{e}_1, \cdots, \bar{e}_s, \bar{e}_{s+1}, \cdots, \bar{e}_n$ of $H_m(\hat{W}^{(2)})$ such that $\bar{e}_i = \{s(c_i)\}$ for $i \leq s$ and, for $i \geq s+1$ $\bar{e}_i$ is represented by a cycle in $\partial \hat{W}^{(2)}$, where $s(c_i) = \bar{c}_i - \bar{c}_i^1$ for m-chains $\bar{c}_i$ in $\hat{W}'_1$ and $\bar{c}_i^1$ in $\hat{W}'_1$ with $\partial \bar{c}_i = \partial \bar{c}_i^1 = c_i \times 0$ in $V \times 0 \subset \bar{U} = \bar{U}_1$. If i or j is $\geq s+1$, then clearly $\mathrm{Int}_{\hat{W}^{(2)}}(\bar{e}_i, \bar{e}_j) = 0$. Let i and j be $\leq s$. Since $\varepsilon(m) = 1$, we have $\mathrm{Int}_{\hat{W}^{(2)}}(\bar{e}_i, \bar{e}_j) = a_{ij} + a_{ji}$ with $a_{ij} = \mathrm{Link}_M(c_i^+, c_j)$ (cf. The proof of Lemma 3.1). Hence for the linking matrix $A = (a_{ij})$ on $K_{m-1}(V)$, $\mathrm{Int}_{\hat{W}^{(2)}}$ is represented by the block sum of $A + A'$ and a zero matrix, so that sign $\hat{W}^{(2)} =$ sign $(A + A')$. The identity sign $(A + A') = \sigma^{\dot{r}}(M)$ was given in the Main Theorem of [K3]. This completes the proof.

Remark 3.4. The method of construction and computation which we used in this section is familiar in knot theory (cf. for example Kauffman [K], Contreras-Caballero [C], Litherland [Li]). Neumann [N2, p. 166] has also used a similar construction in his computation of γ-invariants.

§ 4. Proof of Theorems I, II and the Proposition

4.1. *Proof of Theorem* I. First we assume that $\gamma \in H^1(W; Z)$ has a leaf U in W whose boundary V is a leaf of $\dot{\gamma} \in H^1(M; Z)$. Let (W_1, M_1) be a copy of (W, M). Let (W_c, γ_c) be the pair constructed in Section 3. By Lemma 2.4, we can assume that W is the union $U \times S^1 \cup W_c \cup W_1$ identifying two copies of $V \times S^1$, contained in $U \times S^1$ and W_c, and then identifying $M \times (-1)$ in W_c with M_1 and $\gamma | W_c = \gamma_c$ and $\gamma | U \times S^1$ is a natural extension of $\gamma_c | V \times S^1$ and $\gamma | W_1 = 0$. Then by Lemmas 2.2 and 2.3, $\tau^{\gamma}_{a \pm 0}(W) = \text{sign } W + \tau^{\gamma_c}_{a \pm 0}(W_c)$, for $\tau^{\gamma | U \times S^1}_{a \pm 0}(U \times S^1) = 0$. By Lemma 3.2, $\sigma^{\dot{\gamma}}_a(M) = \tau^{\gamma_c}_{a-0}(W_c) - \tau^{\gamma_c}_{a+0}(W_c) = \tau^{\gamma}_{a-0}(W) - \tau^{\gamma}_{a+0}(W)$ for all $a \in (-1, 1)$ and $\sigma^{\dot{\gamma}}_{\varepsilon(m)}(M) = \varepsilon(m) \tau^{\gamma_c}_{\varepsilon(m) - \varepsilon(m) 0}(W_c) = \varepsilon(m)(\tau^{\gamma}_{\varepsilon(m) - \varepsilon(m) 0}(W) - \text{sign } W)$. If γ has no leaf, then by [K/S] γ_P has a leaf in W_P whose boundary is a leaf of $\dot{\gamma}_P$ in M_P. By Lemma 2.5 and [K3, Lemma 1.2], we have the same conclusion. This completes the proof.

If $W^{(2)}$ is a double covering space of a closed oriented $2m$-manifold W, then it is known that sign $W^{(2)} = 2$ sign W. In fact, it follows, since $\Omega^{\mathrm{Top}}_{2m}(K(Z_2, 1)) \otimes Q \cong \Omega^{\mathrm{Diff}}_{2m}(K(Z_2, 1)) \otimes Q \cong \Omega^{\mathrm{Diff}}_{2m} \otimes Q$ by [W, p. 190] and [C/F].

Hence we obtain from the Novikov addition theorem the following:

Lemma 4.2. *For compact oriented 2m-manifolds W_i, $i=1, 2$, with the same boundary M, assume that a double covering $M^{(2)}\to M$ is extended to coverings $W_i^{(2)}\to W_i$, $i=1, 2$. Then*

$$\text{sign } W_1^{(2)} - 2 \text{ sign } W_1 = \text{sign } W_2^{(2)} - 2 \text{ sign } W_2.$$

The following lemma means that $\sigma^{\dot\gamma}(M)$ in the case $\varepsilon(m)=1$ is the α-invariant of the double covering space of M associated with $\dot\gamma(2) \in H^1(M; Z_2)$ (though their signs are different) (cf. [H/Z]).

Lemma 4.3. *When $\varepsilon(m)=1$, $\sigma^{\dot\gamma}(M)=\text{sign}_\gamma W - 2 \text{ sign } W$.*

Proof. Assume that γ has a leaf in W whose boundary is a leaf of $\dot\gamma$ in M. By Lemma 4.2, we can assume that (W, γ) is the pair constructed in 4.1. Then $\text{sign}_\gamma W = 2 \text{ sign } W + \text{sign } W_c^{(2)} = 2 \text{ sign } W + \sigma^{\dot\gamma}(M)$ by the Novikov addition theorem and Lemma 3.3. If γ has no leaf, then we consider (W_P, M_P, γ_P). By [K/S] and [K3, Lemma 1.2], we have the same conclusion. This completes the proof.

4.4. *Proof of Theorem* II. It follows from Theorem I′ and Lemma 4.3, since $\sigma^{\dot\gamma}(M)=\sum_{a\in[-1,1]}\sigma_a^{\dot\gamma}(M)$.

4.5. *Proof of the Proposition.* Let $h\colon S^1\times D^2\to S^1\times D^2$ be an orientation-preserving homeomorphism such that $(h\,|\,S^1\times\partial D^2)_*[S^1\times q] = -[S^1\times q]+e[p\times\partial D^2]$ and $(h\,|\,S^1\times\partial D^2)_*[p\times\partial D^2]=-[p\times\partial D^2]$ in $H_1(S^1\times \partial D^2; Z)$, where $p\in S^1$, $q\in\partial D^2$ and e is a non-zero integer. Let W be the mapping torus of h and $\gamma\in H^1(W; Z)$ be an element represented by the associated bundle projection $W\to S^1$. Since W is homotopy equivalent to the Klein bottle, we have that $H_2(W)=B_2(\tilde W)=0$. Hence sign $W= \tau^\gamma_{a\pm 0}(W)=0$ for all $a\pm 0\in[-1, 1]$. But $T_1(\tilde M)=H_1(\tilde M)\cong R\langle t\rangle/(t+1)^2$ and the null space of the quadratic form $\tilde b$ on $T^1(\tilde M)$ is easily seen**) to be $(t+1)T^1(\tilde M)$. This means that $\tilde b$ induces a non-singular form on $T^1(\tilde M)/(t+1)T^1(\tilde M)\cong R\langle t\rangle/(t+1)$, so that $\sigma^{\dot\gamma}_{-1}(M)=\pm 1$. Choose an orientation of W so that $\sigma^{\dot\gamma}_{-1}(M)=s/|s|$. Let (W_s, γ_s) be a pair such that W_s is a boundary-disk sum of $|s|$ copies of W and $\gamma_s\in H^1(W_s; Z)$ is determined by $|s|$ copies of γ. Clearly, this pair gives a desired pair in dimension 4. A desired pair in dimension $2m$ is obtained from this pair by taking the product with $(m/2)-1$ copies of CP^2 (cf. [K3, Lemma 1.2] and Lemma 2.5). This completes the proof.

**) Use the Duality Theorem of [K3, § 1].

Notes added in proof: We used Lemma 1.1 of [K3] in this paper, but the proof of [K3] was incorrect. The true proof is found in [K2, pp. 99–100].

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