Splitting criteria for a definite 4-manifold with infinite cyclic fundamental group*

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ABSTRACT

Two criteria for a closed connected definite 4-manifold with infinite cyclic fundamental group to be TOP-split are given. One criterion extends a sufficient condition made in a previous paper. The result is equivalent to a purely algebraic result on the question asking when a positive definite Hermitian form over the ring of integral one-variable Laurent polynomials is represented by an integer matrix. As an application, an infinite family of orthogonally indecomposable unimodular odd definite symmetric Z-forms is produced.

Mathematics Subject Classification 2010: 57M10, 57M35, 57M50, 57N13

Keywords: Definite intersection form, minimal element, 4-manifold, Infinite cyclic fundamental group, TOP-split, Infinite cyclic covering, Splitting criterion.

*This paper is dedicated to the memory of Dr. Tim D. Cochran.

1. Introduction

A closed connected oriented topological 4-manifold M is called a Z^{H_1}*-manifold* if there is a fixed isomorphism from the first homology group $H_1(M)$ onto Z, and a Z^{π_1}*-manifold* if there is a fixed isomorphism from the fundamental group $\pi_1(M)$ onto Z. A Z^{π_1}-manifold M is *TOP-split* if M is homeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for a simply connected closed 4-manifold M_1 obtained from M by a 2-handle surgery killing $\pi_1(M) \cong Z$, and *virtually* TOP-split if a finite connected covering space of M is TOP-split. A Z^{H_1}-manifold M is *definite* if the rank of the Z-intersection form

$$\mathrm{Int}^M : H_2(M;Z) \times H_2(M;Z) \to Z$$

is equal to the absolute value of the signature, and *positive definite* if, furthermore, the signature is positive. A definite Z^{π_1}-manifold with negative signature is changed to be positive definite by reversing the orientation of M.

Before explaining the main theorem of this paper, a history on the TOP-splitting of a Z^{π_1}-manifold is described here.

In [11], every topological Z^{π_1}-manifold was claimed to be TOP-split. However, this is not true, as noted in [12, 13]. Concerning this error, I. Hambleton and P. Teichner in [7] have constructed an example of a Λ-Hermitian matrix L with determinant $+1$ which cannot be Λ-conjugate to an integral matrix, where $\Lambda = Z[Z] = Z[t, t^{-1}]$ denotes the integral Laurent polynomial ring. By a construction of M. H. Freedman and F. Quinn in [5], everyΛ-Hermitian matrix A is realized by a Λ-intersection matrix on a unique (up to Kirby-Siebenmann obstructions Z^{π_1}-manifold M_A. This Z^{π_1}-manifold M_L is referred to as the *Hambleton-Teichner-Freedman-Quinn* Z^{π_1}*-manifold*. Since every Λ-intersection matrix on a TOP-split Z^{π_1}-manifold is Λ-conjugate to an integral matrix, the Z^{π_1}-manifold M_L is not TOP-split, and thus gives a counterexample to the TOP-splitting claim of a topological Z^{π_1}-manifold. Furthermore, the Z^{π_1}-manifold M_L was a positive definite, non-smoothable and virtually non-TOP-split Z^{π_1}-manifold, which is shown by S. Friedl, I. Hambleton, P. Melvin, and P. Teichner in [6]. In [15], it was shown that every Z^{π_1}-manifold is TOP-split if and only if it is virtually TOP-split, which implies that every indefinite Z^{π_1}-manifold is TOP-split. Further, in [16], a positive definite Z^{π_1}-manifold is TOP-split if every finite covering space of it admits an intersection matrix whose diagonal entries are smaller than or equal to 2. As a consequence of these results in [15, 16], it was shown that every *smooth* Z^{π_1}-manifold is TOP-split.

In this paper, the necessary and sufficient conditions on the TOP-splitting for a positive definite Z^{π_1}-manifold generalizing the sufficient condition of [16] are given.

For this purpose, the following two notions are useful:

• One notion is a notion of a *winding degree* on a positive definite Z^{π_1}-manifold M which is a non-negative integer λ measuring a difference between M and the TOP-split Z^{π_1}-manifold $S^1 \times S^3 \# M_1$. The winding degree λ is defined in §2 to take non-unique value for a given M for convenience, but by definition the minimum $\lambda_{\min}$ of all winding degrees λ on M is seen to be an invariant of M. It will be shown in Theorem 1.1 that $\lambda_{\min} = 0$ on M if and only if M is TOP-split.

• The other notion is a notion of a homology class called a *minimal element* in the second homology group $H_2(M; Z)$ of a positive definite Z^{π_1}-manifold M. This notion is a standard notion used for the proof of Eichler's unique orthogonal indecomposable splitting theorem for a positive definite symmetric bilinear form (see Eichler [4], Kneser [17], Milnor-Husemoller [18]).

Definition. For a positive definite Z^{H_1}-manifold M, the definition of a minimal element is given by the following two notions:

• The *square length* of an element $x \in H_2(M; Z)$, denoted by $||x||^2$, is the Z-self-intersection number $\mathrm{Int}^M(x, x)$.
• An element $x \in H_2(M; Z)$ is *minimal* if $x \neq 0$ and x cannot be the sum $y + z$ of any elements $y, z \in H_2(M; Z)$ such that

$$||x||^2 > ||y||^2 \quad \text{and} \quad ||x||^2 > ||z||^2.$$

As a basic observation, every minimal element of $H_2(M; Z)$ belongs to the unique indecomposable orthogonal sum component of $H_2(M; Z)$.

Notation. For a Z^{H_1}-manifold M, the following notations are used.

• The m-fold cyclic connected covering space of M is denoted by $M^{(m)}$.
• The infinite cyclic connected covering space of M with covering transformation group generated by t is denoted by $\widetilde{M}$.

For every Z^{π_1}-manifold M, it is known that the Λ-module $H_2(\widetilde{M}; Z)$ is a free Λ-module

$$H_2(\widetilde{M}; Z) \cong \Lambda^n$$

of rank $n = \beta_2(M)$. This fact was proved in [10, Lemma 2.1] for a more general oriented compact 4-manifold with infinite cyclic fundamental group by using three integral dualities on an infinite cyclic covering of a topological manifold in [9]. Also, see [5] for another proof.

In [15], it is shown (as stated above) that M is TOP-split if and only if $M^{(m)}$ is TOP-split for some m. Let

$$\Lambda^{(m)} = \Lambda/(t^m - 1)\Lambda$$

be the quotient ring of the Laurent polynomial ring Λ by the ideal $(t^m - 1)\Lambda$. For a Z^{π_1}-manifold M, the Λ-module $H_2(M^{(m)}; Z)$ is identical to the quotient Λ-module

$$H_2(\widetilde{M}; Z)/(t^m - 1)H_2(\widetilde{M}; Z),$$

which is a free $\Lambda^{(m)}$-module of rank n. For an element $\tilde{x} \in H_2(\widetilde{M}; Z)$, let $\tilde{x}^{(m)} \in H_2(M^{(m)}; Z)$ denote the projection image of $\tilde{x}$ under the covering projection homomorphism $H_2(\widetilde{M}; Z) \to H_2(M^{(m)}; Z)$.

Definition. For a positive definite Z^{π_1}-manifold M, the infinite cyclic covering space $\widetilde{M}$ of M and $\Lambda = Z[t, t^{-1}]$, the following definitions are set.

• The Λ-*square length* of an element $\tilde{x} \in H_2(\widetilde{M}; Z)$ denoted by $||\tilde{x}||^2_\Lambda$ is the Λ-self-intersection number $\mathrm{Int}^{\widetilde{M}}_\Lambda(\tilde{x}, \tilde{x})$, which is an integral Laurent polynomial $a(t) \in \Lambda$ in t with t-symmetry $a(t) = a(t^{-1})$ (see § 2).
• The *exponent* of $\tilde{x}$, denoted by $e(\tilde{x})$, is the highest degree of $a(t) = ||\tilde{x}||^2_\Lambda$ which is a non-negative integer.
• An element $\tilde{x} \in H_2(\widetilde{M}; Z)$ is *minimal* if $\tilde{x} \neq 0$ and $\tilde{x}$ cannot be the sum $\tilde{y} + \tilde{z}$ of any elements $\tilde{y}, \tilde{z} \in H_2(\widetilde{M}; Z)$ such that

$$||\tilde{x}||^2 > ||\tilde{y}||^2 \quad \text{and} \quad ||\tilde{x}||^2 > ||\tilde{z}||^2.$$

It is noted in § 2 that the notion of a minimal element in $H_2(\widetilde{M}; Z)$ is a natural generalization of the notion of a minimal element in $H_2(M; Z)$, for a positive definite Z^{π_1}-manifold M. By the positivity of a square length shown in § 2, it is seen that every non-zero element $\tilde{x} \in H_2(\widetilde{M}; Z)$ is the sum of finitely many minimal elements and every minimal element of $H_2(\widetilde{M}; Z)$ belongs to a unique indecomposable orthogonal sum component of $H_2(\widetilde{M}; Z)$. The multiplication $t^k\tilde{x}$ for an integer k is called a t-*power shift* of $\tilde{x}$. The double covering projection $p : M^{(2m)} \to M^{(m)}$ is particularly used in the arguments of this paper. We shall show the following theorem:

Theorem 1.1. The following conditions (0)-(5) on a positive definite Z^{π_1}-manifold M are mutually equivalent:

(0) The Z^{π_1}-manifold M is TOP-split.
(1) There are elements $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ in $H_2(\widetilde{M}; Z)$ such that the elements $\tilde{x}^{(1)}_i$ $(i = 1, 2, \ldots, n)$ are Z-generators for $H_2(M; Z)$ and the Λ-intersection numbers $\mathrm{Int}^{\widetilde{M}}_\Lambda(\tilde{x}_i, \tilde{x}_j)$ are integers for all i, j.

(2) Any minimal Λ-generators $\tilde{x}_i\,(i = 1, 2, \ldots, n)$ of $H_2(\widetilde{M}; Z)$ have the property that after suitable t-power shifts of $\tilde{x}_i\,(i = 1, 2, \ldots, n)$, the Λ-intersection numbers $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i, \tilde{x}_j)$ are integers for all i, j.
(3) Every minimal element $\tilde{x}$ of $H_2(\widetilde{M}; Z)$ is sent to a minimal element $\tilde{x}^{(m)} \in H_2(M^{(m)}; Z)$ for every m such that every element $x' \in H_2(M^{(2m)}; Z)$ with $p_*(x') = \tilde{x}^{(m)}$ satisfies the inequality

$$||x'||^2 \geqq ||\tilde{x}^{(m)}||^2.$$

(4) For any previously given winding degree λ on M, there is an $m \geqq \lambda$ for which there are minimal Z-generators $x_i\,(i = 1, 2, \ldots, s)$ of $H_2(M^{(m)}; Z)$ such that for every i, and for every element $x'_i \in H_2(M^{(2m)}; Z)$ with $p_*(x'_i) = x_i$, the inequality

$$||x'_i||^2 > ||x_i||^2 - 2$$

holds.
(5) The minimal winding degree λ_{min} on M is zero.

In Theorem 1.1, (3), (4) and (5) are new results. Note that (0) is equivalent to (1) without assumption of the positive definiteness on M and is equivalent to the following condition:

(1)* There is a Λ-basis $\tilde{x}_i\,(i = 1, 2, \ldots, n)$ of $H_2(\widetilde{M}; Z)$ such that the Λ-intersection numbers $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i, \tilde{x}_j)$ are integers for all i, j.

It is known in [13, 14] that a Z^{π_1}-manifold M with property (1)* is TOP-split (see the proof of (1) $\to$ (0) in the proof of Theorem 1.1). The Λ-intersection form on M with with property (1)* is said to be *Z-extended* in [7], and also said to be *exact* because such a manifold is a special case of a Z^{H_1}-manifold admitting an exact sequence called an exact Z^{H_1}-manifold, which is discussed in [14].

It follows directly from (2) that for a positive definite Z^{π_1}-manifold M, the connected sum $Y\#M$ for any closed simply connected positive definite 4-manifold Y is TOP-split if and only if M is TOP-split.

For a positive definite Z^{π_1}-manifold M, note that, by definition, if there are Z-generators $x_i\,(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)}; Z)$ with $||x_i||^2 \leqq 2$ for all i, then there are minimal Z-generators $y_j\,(j = 1, 2, \ldots, s)$ of $H_2(M^{(m)}; Z)$ with $||y_j||^2 \leqq 2$ for all j. Then for every element $y'_j \in H_2(M^{(2m)}; Z)$ with $p_*(y'_j) = y_j$, the inequality

$$||y'_j||^2 > 0 \geqq ||y_j||^2 - 2$$

holds. It will be explained in §2 that a winding degree λ on M is taken smaller than or equal to a winding index δ on M defined in [16]. With these observations, the following corollary meaning the main theorem of [16] is obtained as a consequence of Theorem 1.1 (4):

Corollary 1.2 ([16, Theorem 1.1]). A positive definite Z^{π_1}-manifold M is TOP-split if for any previously given winding index δ on M, there is an $m \geqq \delta$ for which there is a Z-basis x_i $(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)}; Z)$ such that $||x_i||^2 \leqq 2$ for all i.

By using Corollary 1.2, it was shown in [16] that every Z^{π_1}-manifold M is TOP-split if for every m the intersection form of $M^{(m)}$ is represented by a block sum of copies of (1) and/or E_8. This means that every *smooth* positive definite Z^{π_1}-manifold M is TOP-split, because for every m the Z^{π_1}-manifold $M^{(m)}$ is a positive definite smooth 4-manifold (see Lemma 2.1 later) and hence the intersection form of $M^{(m)}$ is represented by a block sum of copies of (1) by Donaldson's theorem in [2], where note that the intersection form of $M^{(m)}$ is identical to the intersection form of a closed simply connected smooth 4-manifold obtained from $M^{(m)}$ by killing $\pi_1(M^{(m)}) = Z$.

Thus, every *smooth* Z^{π_1}-manifold M is TOP-split, because every indefinite Z^{π_1}-manifold is seen in [15] to be TOP-split.

As another consequence (shown in [16]), every *smooth* S^2-knot in every *smooth* closed simply connected 4-manifold is topologically unknotted if the knot group is an infinite cyclic group.

For a positive definite Z^{π_1}-manifold M, assume that there are Λ-generators $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ of $H_2(\widetilde{M}; Z)$ with $||\tilde{x}_i||^2 \leqq 2$ for all i. Let m be an integer such that

$$m \geqq \max\{e(\tilde{x}_1) + 1, e(\tilde{x}_2) + 1, \ldots, e(\tilde{x}_n) + 1, \lambda\}$$

for some winding degree λ on M. Then the elements $\tilde{x}_i^{(m)} \in H_2(M^{(m)}; Z)$ $(i = 1, 2, \ldots, n)$ induced from $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ by the covering projection homomorphism $H_2(\widetilde{M}; Z) \to H_2(M^{(m)}; Z)$ form $\Lambda^{(m)}$-generators of $H_2(M^{(m)}; Z)$ with identical square length $||\tilde{x}_i^{(m)}||^2 = ||\tilde{x}_i||^2$ for all i, so that there are Z-generators $x'_{i'}$ $(i' = 1, 2, \ldots, n')$ for $H_2(M^{(m)}; Z)$ such that $||x'_{i'}||^2 \leqq 2$ for all i', meaning that M is TOP-split by Corollary 1.2. Thus, the following corollary is obtained from Theorem 1.1 (4) and the observation just before Corollary 1.2 (see Corollary 3.3 later).

Corollary 1.3. A positive definite Z^{π_1}-manifold M is TOP-split if there are Λ-generators $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ of $H_2(\widetilde{M}; Z)$ such that $||\tilde{x}_i||^2 \leqq 2$ for all i.

The following theorem gives another criterion that a positive definite Z^{π_1}-manifold with standard Z-intersection form is TOP-split.

Theorem 1.4. Let M be a positive definite Z^{π_1}-manifold such that there is a Z-basis $e_i\,(i=1,2,\dots,n)$ of $H_2(M;Z)$ with $\mathrm{Int}^M(e_i,e_j)=\delta_{ij}$ for all i,j. Then the following conditions (0), (1), (2) and (3) on M are mutually equivalent:

(0) The Z^{π_1}-manifold M is TOP-split.
(1) For a Λ-basis $\tilde{x}_i\,(i=1,2,\dots,n)$ of $H_2(\widetilde{M};Z)$, there are elements $a_{ij}(t)\in\Lambda\,(i,j=1,2,\dots,n)$ with Λ-intersection number

$$\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i,\tilde{x}_j)=\sum_{k=1}^{n}a_{ik}(t^{-1})a_{jk}(t)$$

for every i and j.
(2) Every minimal element $\tilde{x}\in H_2(\widetilde{M};Z)$ has square length $||\tilde{x}||^2=1$.
(3) The following conditions (3_1) and (3_2) are satisfied:
(3_1) For every element $\tilde{x}\in H_2(\widetilde{M};Z)$, there are elements $a_i(t)\in\Lambda\,(i=1,2,\dots,s)$ such that the Λ-square length

$$||\tilde{x}||_\Lambda^2=\sum_{i=1}^{s}a_i(t)a_i(t^{-1}).$$

(3_2) There are mutually distinct (up to multiplications of the units of Λ) elements $\tilde{y}_i\,(i=1,2,\dots,n-1)$ in $H_2(\widetilde{M};Z)$ such that the square length $||\tilde{y}_i||^2=1$ for all i.

For example, every Z^{π_1}-manifold M with second Betti number $\beta_2(M)=1$ can be converted to a positive definite Z^{π_1}-manifold with standard Z-intersection form by changing the orientation if necessary. Such a manifold is TOP-split by Theorem 1.4 since the Λ-intersection matrix on $H_2(\widetilde{M};Z)\cong\Lambda$ is (±1). It appears to be unknown *whether every positive definite Z^{π_1}-manifold M with $\beta_2(M)=2$ or 3 is TOP-split*, where we note that every positive definite Z-form of rank up to 7 is known to be standard (see [18]).

We come back to the Hambleton-Teichner-Freedman-Quinn Z^{π_1}-manifold M_L. The matrix

$$L=\begin{pmatrix}1+f+f^2 & f+f^2 & 1+f & f\\ f+f^2 & 1+f+f^2 & f & 1+f\\ 1+f & f & 2 & 0\\ f & 1+f & 0 & 2\end{pmatrix}$$

with $f=t+t^{-1}$ is the Hambleton-Teichner matrix given in [7]. Since the size of the matrix L is 4, the Z-intersection form Int^{M_L} on $H_2(M_L;Z)$ is the standard form. Let $\tilde{x}_i\,(i=1,2,3,4)$ be the Λ-basis of $H_2(\widetilde{M_L};Z)$ giving L as the Λ-intersection matrix. It is shown by Theorem 1.4 (3_1) that M_L is not TOP-split since $||\tilde{x}_i||^2=1+f+f^2\,(i=$

$1, 2)$ cannot be written as a sum $\sum_{i=1}^{s} a_i(t)a_i(t^{-1})$. The Z^{π_1}-manifold M_L has further properties which are given below:

Theorem 1.5. For the Hambleton-Teichner matrix L, we have the following properties (1) and (2).

(1) The Λ-basis $\tilde{x}_i$, $(i = 1, 2, 3, 4)$ of $H_2(\widetilde{M_L}; Z)$ are minimal elements such that the square length $||\tilde{x}_i||^2$ is 3 for $i = 1, 2$ and 2 for $i = 3, 4$.
(2) Let V_G be the Z-intersection matrix on any Z-free subgroup G of $H_2(\widetilde{M_L}; Z)$ of finite rank. Then the determinant $\det V_G$ of V_G is greater than 1.

In particular, we see that the criteria of Theorem 1.4 are not satisfied for M_L.

The following corollary produces an infinite family of orthogonally indecomposable unimodular odd symmetric Z-forms as a corollary to Theorem 1.5.

Corollary 1.6. For the Hambleton-Teichner matrix L and every integer $m \geqq 3$, we have the following properties (1) and (2).

(1) The $\Lambda^{(m)}$-basis $\tilde{x}_i^{(m)}$, $(i = 1, 2, 3, 4)$ of $H_2(M_L^{(m)}; Z)$ are minimal elements such that the square length $||\tilde{x}_i^{(m)}||^2$ is 3 for $i = 1, 2$ and 2 for $i = 3, 4$.
(2) The Z-intersection form on $H_2(M_L^{(m)}; Z) \cong Z^{4m}$ is an orthogonally indecomposable unimodular odd definite symmetric Z-form.

It was shown in [6] by a different method that the Z-intersection form on $H_2(M_L^{(m)}; Z)$ in (2) is not standard for every $m \geqq 3$ and orthogonally indecomposable for $m = 3, 4$. The proof of Corollary 1.6 (2) will be done for $m \geqq 5$. The form in (2) is standard for $m \leqq 2$ and isomorphic to Γ_{12} for $m = 3$ by the classification of orthogonally indecomposable unimodular definite symmetric Z-forms of rank $\leqq 16$ (see [18]). On the other hand, the form in (2) must be different from Γ_{4m} for every $m \geqq 4$, because the square length of every minimal element of Γ_{4m} is 2 or m (see [18]).

As another note on Corollary 1.6, the elements $\tilde{x}_i^{(1)} \in H_2(M_L; Z)$ and $\tilde{x}_i^{(2)} \in H_2(M^{(2)}; Z)$ for $i = 1, 2$ have the equalities

$$||\tilde{x}_i^{(2)}||^2 = ||\tilde{x}_i^{(1)}||^2 - 2 = 3,$$

which give a concrete example that the inequality of Theorem 1.1 (3) does not hold in general unless M is TOP-split.

Given a Hermitian Λ-matrix A with $\det A = 1$, we can construct a *smooth* compact connected oriented 4-manifold E from the 4-dimensional solid torus $S^1 \times D^3$ by attaching some 2-handles to the boundary $\partial(S^1 \times D^3) = S^1 \times S^2$ such that $\pi_1(E; Z) \cong Z$

and the matrix A is a Λ-intersection matrix on $H_2(\widetilde{E}; Z)$, where $\widetilde{E}$ denotes the infinite cyclic connected covering space of E (see [5]). The boundary $\partial E = B$ of E has the same homology as $S^1 \times S^2$ and hence is called a *homology handle*. Since $\det A = 1$, the boundary $\widetilde{B} = \partial \widetilde{E}$ of $\widetilde{E}$ has the trivial homology $H_1(\widetilde{B}; Z) = 0$. In case A is the Hambleton-Teichner matrix L, it is observed in [6] that the homology handle B cannot bound a *smooth* rational homology circle W by using [3] instead of [2], which is generalized as follows:[1]

Observation 1.7. For the Hambleton-Teichner matrix L, the disjoint union nB of n copies of the homology handle B for any $n \geqq 1$ cannot bound a smooth compact oriented 4-manifold W with $H_2(W; Q) = 0$.

It is unknown *whether the homology handle B can bound a smooth compact oriented 4-manifold W with an infinite cyclic covering space $\widetilde{W}$ such that*

$$\dim_Q H_2(\widetilde{W}; Q) < +\infty,$$

in other words, *whether the homology handle B represents a trivial element of the $\widetilde{H}$-cobordism group* $\widetilde{\Omega}(S^1 \times S^2)$ in [8]. It is also observed in [6] that some B can be taken as the Dehn surgery manifold with coefficient 0 along a knot K in the 3-sphere S^3 with trivial Alexander polynomial. With an idea of Cochran-Lickorish [1], we obtain:

Observation 1.8. For the Hambleton-Teichner matrix L, the knot K cannot be converted into the unknot by changing positive crossings into negative crossings.

In § 2, several preliminaries on a winding degree are provided. In § 3, the proofs of Theorems 1.1 and 1.4 are given. In § 4, the proofs of Theorem 1.5, Corollary 1.6 and Observations 1.7 and 1.8 are given.

2. Several preliminaries on the winding degree

For a Z^{π_1}-manifold M, let $\tilde{x} \in H_2(\widetilde{M}; Z)$ be a non-zero element. We assert that the square length $||\tilde{x}||^2 > 0$. To see this, the following lemma is proved (though it was the fact used in [16]).

Lemma 2.1 If a Z^{π_1}-manifold M is positive definite, then $M^{(m)}$ is also positive definite for any m.

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[1] It is assumed in [6] that the natural homomorphism: $H_1(B; Z) \to H_1(W; Z)/(\text{torsion})$ is an isomorphism, but this restriction can be removed.

Proof. There are two ways to see that $\beta_2(M^{(m)}) = m\beta_2(M)$. One way is to use the Euler characteristic identity $\chi(M^{(m)}) = m\chi(M)$. By the Betti numbers $\beta_d(M^{(m)}) = \beta_d(M) = 1$ for $d = 0, 1$, we have $\beta_2(M^{(m)}) = m\beta_2(M)$. The other way is to use that $H_2(\widetilde{M}^{(m)}; Z)$ is a $\Lambda^{(m)}$-module of rank $\beta_2(M)$, showing that $\beta_2(M^{(m)}) = m\beta_2(M)$. If $\sigma(M) = \beta_2(M)$, then the signature identity $\sigma(M^{(m)}) = m\sigma(M)$ shows that $\sigma(M^{(m)}) = \beta_2(M^{(m)})$. □

Note that the image $\tilde{x}^{(m)} \in H_2(M^{(m)}; Z)$ of $\tilde{x}$ under the covering projection homomorphism $H_2(\widetilde{M}; Z) \to H_2(M^{(m)}; Z)$ is not zero for a large m. Hence, we have the square length $||\tilde{x}^{(m)}||^2 > 0$ for a large m. The $\Lambda^{(m)}$-intersection number $\mathrm{Int}^{M^{(m)}}_{\Lambda^{(m)}}(\tilde{x}^{(m)}, \tilde{y}^{(m)})$ of the elements $\tilde{x}^{(m)}, \tilde{y}^{(m)} \in H_2(M^{(m)}; Z)$ is calculated as follows:

$$\mathrm{Int}^{M^{(m)}}_{\Lambda^{(m)}}(\tilde{x}^{(m)}, \tilde{y}^{(m)}) = \sum_{s=0}^{m-1} \left(\sum_{i=-\infty}^{+\infty} \mathrm{Int}^{\widetilde{M}}(t^{s+mi}\tilde{x}, \tilde{y}) t^s \right) \in \Lambda^{(m)}.$$

In particular, the Z-intersection number $\mathrm{Int}^{M^{(m)}}(\tilde{x}^{(m)}, \tilde{y}^{(m)})$ is given as follows:

$$\mathrm{Int}^{M^{(m)}}(\tilde{x}^{(m)}, \tilde{y}^{(m)}) = \sum_{i=-\infty}^{+\infty} \mathrm{Int}^{\widetilde{M}}(t^{mi}\tilde{x}, \tilde{y}) \in Z.$$

By definition, for the exponent $e(\tilde{x})$ of $\tilde{x}$, which is the highest degree of the Λ-square length $||\tilde{x}||^2_\Lambda = \mathrm{Int}^{\widetilde{M}}_\Lambda(\tilde{x}, \tilde{x})$, it is seen that the square length $||\tilde{x}||^2$ is identical to the square length $||\tilde{x}^{(m)}||^2$ for any integer $m \geqq e(\tilde{x}) + 1$, so that if $\tilde{x} \neq 0$ and $m \geqq e(\tilde{x}) + 1$, then we have $||\tilde{x}||^2 = ||\tilde{x}^{(m)}||^2 > 0$. This shows that the square length $||\tilde{x}||^2 > 0$ as asserted.

Definition. A 3-*sphere leaf* of a TOP-split Z^{π_1}-manifold X is a 3-sphere submanifold V of X corresponding to the 3-sphere $1 \times S^3$ under a homeomorphism $X \cong S^1 \times S^3 \# X_1$ for some simply connected 4-manifold X_1.

Let P be a 2-sphere (embedded) in X. We assume that the intersection $L = P \cap V$ is a closed oriented possibly disconnected 1-manifold unless it is empty. Let $D_i\,(i = 1, 2, \ldots, r)$ be the connected regions of P divided by L. Let p_i be a fixed interior point of D_i. Let α_{ij} be an oriented arc in P joining the point p_i to the point p_j transversely meeting L. The absolute value $|\mathrm{Int}^P(\alpha_{ij}, L)|$ of the Z-intersection number $\mathrm{Int}^P(\alpha_{ij}, L)$ is independent of any choices of the points $p_i \in D_i$, $p_j \in D_j$ and the oriented arc α_{ij}. The maximal number $|\mathrm{Int}^P(\alpha_{ij}, L)|$ for all i, j is determined only by the 2-sphere P in X and the 3-sphere leaf V and called the *winding index* of the 2-sphere P in X with respect to the 3-sphere leaf V and denoted by $\delta = \delta(P; V, X)$.

A meaning of the winding index δ is as follows. Let X_V be the fundamental region of the infinite cyclic covering space $\widetilde{X}$, obtained from X by splitting along V. Note that $\widetilde{X}$ is the union of the t^i-shifts $t^i(X_V)\,(i = 0, \pm 1, \pm 2, \dots)$ of X_V. Then the winding index δ is the maximal number of the interiors of $t^i(X_V)\,(i = 0, \pm 1, \pm 2, \dots)$ meeting a fixed lifting 2-sphere $\widetilde{P}$ of P to $\widetilde{X}$. This notion was defined in [16].

A homological version of the winding index for a 2-sphere P in a TOP-split Z^{π_1}-manifold X with a 3-sphere leaf V as follows:

The homology $H_2(\widetilde{X}; Z)$ is a free Λ-module with a Λ-basis $v_i\,(i = 1, 2, \dots, n)$ represented by 2-cycles in X_V. For an element $\tilde{x} \in H_2(\widetilde{X}; Z)$, the Λ-intersection number $\mathrm{Int}_\Lambda(\tilde{x}, v_i)$ is an element of Λ, i.e., an integral Laurent polynomial in t. The maximal and minimal degrees of $\mathrm{Int}_\Lambda(\tilde{x}, v_i)$ in t for all i are independent of a choice of such a Λ-basis $v_i\,(i = 1, 2, \dots, n)$ and denoted by $\max\deg(\tilde{x}; V, X)$ and $\min\deg(\tilde{x}; V, X)$, respectively. The difference $\delta^h(\tilde{x}; V, X) = \max\deg(\tilde{x}; V, X) - \min\deg(\tilde{x}; V, X)$ is independent of any covering translations of $\tilde{x}$ and called the *homological winding index* of $\tilde{x}$ in X with respect to the 3-sphere leaf V. The *homological winding index*$\delta^h(P; V, X)$ of the 2-sphere P in X with respect to the 3-sphere leaf V is the number $\delta^h([\widetilde{P}]; V, X)$ for the homology class $[\widetilde{P}] \in H_2(\widetilde{X}; Z)$ of any lifting 2-sphere $\widetilde{P}$ of P to $\widetilde{X}$.

It is direct to see that

$$\delta^h(P; V, X) \leqq \delta(P; V, X).$$

The notion of a winding degree on a positive definite Z^{π_1}-manifold M is defined from the notion of a homological winding index as follows:

For an integer $u \geqq 1$, let $Q(u) = \#_u \mathbf{C}P^2$ and $\overline{Q}(u) = \#_u \overline{\mathbf{C}P}^2$ be the u-fold connected sum of the complex projective planes $Q = \mathbf{C}P^2$ and $\overline{Q} = \overline{\mathbf{C}P}^2$ with signatures $+1$ and -1, respectively. Let uP be the disjoint union of the 2-spheres $P_k = \overline{\mathbf{C}P}^1_k\,(k = 1, 2, \dots, u)$ in the connected summands $\overline{Q}_k = \overline{\mathbf{C}P}^2_k\,(k = 1, 2, \dots, u)$ of $\overline{Q}(u)$. A tubular neighborhood $N \subset X$ of a generator circle of $\pi_1(X) \cong Z$ is called a *solid tube generator* of X which is unique up to ambient isotopies of X. A *circle union* $X' \circ X''$ of two Z^{π_1}-manifolds X', X'' is a Z^{π_1}-manifold obtained from the exteriors $\mathrm{cl}(X' \setminus N')$ and $\mathrm{cl}(X'' \setminus N'')$ for solid tube generators N' and N'' of X' and X'', respectively, by identifying the boundary $\partial N'$ with boundary $\partial N''$ by an orientation-reversing homeomorphism. For a positive definite Z^{π_1}-manifold M, we consider a Z^{π_1}-manifold $X = \overline{Q}(u)\#Q(v)\#M$ for integers $u \geqq 1$ and $v \geqq 0$. Since X is an indefinite Z^{π_1}-manifold, X is TOP-split by [15]. Consider X as a circle union $X' \circ X''$ with following conditions (i)-(iii).

(i) The Z^{π_1}-manifold X' is a TOP-split Z^{π_1}-manifold with a 3-sphere leaf V' with $V' \cap \mathrm{cl}(X' \setminus N') = B'$ a 3-disk.
(ii) For a solid tube generator $N \subset M$, we have an inclusion $\mathrm{cl}(M \setminus N) \subset \mathrm{cl}(X' \setminus N')$ inducing an isomorphism on the infinite cyclic fundamental groups.

(iii) The Z^{π_1}-manifold X'' is a positive definite Z^{π_1}-manifold.

This circle union splitting $X' \circ X''$ of X always exists although it is not unique. For example, let $X' = \overline{Q}(u)\#Q(v')\#M$ (which is TOP-split by [15]) and $X'' = Q(v'')\#S^1 \times S^3$ for any sum $v = v' + v''$ which give a desired circle union $X' \circ X''$ of X for any 3-sphere leaf V' of X'. In fact, for a solid tube generator N of M, there is a 3-sphere leaf V' of X' with $N \cap V'$ a 3-ball. Let $N' = \mathrm{cl}(N \setminus c(\partial N))$ for a boundary collar $c(\partial N)$ of ∂N in N. Then X' is a union of $\mathrm{cl}(M \setminus N) \cup (c(\partial N)\#\overline{Q}(u)\#Q(v')) \cup N'$. In this decomposition, the conditions (i) and (ii) are satisfied. The condition (iii) is clearly satisfied.

Then note that there is a canonical isomorphism

$$H_2(\widetilde{X}; Z) \cong H_2(\widetilde{X}'; Z) \oplus H_2(\widetilde{X}''; Z).$$

For a connected lift $\widetilde{P}_k$ of the 2-sphere P_k to $\widetilde{X}$, let $[\widetilde{P}_k]'$ be the projection image of the homology class $[\widetilde{P}_k] \in H_2(\widetilde{X}; Z)$ into the direct summand $H_2(\widetilde{X}'; Z)$. The maximum of the homological winding index $\delta^h([\widetilde{P}_k]'; V', X')$ for all k $(k = 1, 2, \ldots, u)$ is denoted by $\delta^h([uP]'; V', X')$.

Definition. A *winding degree* on a positive definite Z^{π_1}-manifold M, denoted by λ, is the non-negative integer $\delta^h([uP]'; V', X')$ given by a choice of integers $u \geqq 1$ and $v \geqq 0$ for the connected sum $X = \overline{Q}(u)\#Q(v)\#M$, a choice of any circle union splitting $X' \circ X''$ of X with properties (i)-(iii), and a choice of a 3-sphere leaf V' of X'.

A winding degree $\lambda = \delta^h([uP]'; V', X')$ is an amount that measures the possible range. of the homology classes $[\widetilde{P}_k] \in H_2(\widetilde{X}; Z)$ in the decompositions of $\widetilde{X}'$ into the t-power shifts $t^k(M_{V'})$ $(i = \pm 1, \pm 2, \ldots)$ of the fundamental region $X'_{V'}$. Let $\lambda_{\min}$ be the minimum of all winding degrees λ on a positive definite Z^{π_1}-manifold$M$ which is, by definition, a topological invariant of M.

Let $P^E_k = P_k \cap \mathrm{cl}(X' \setminus N')$ be a proper surface in the compact manifold $\mathrm{cl}(X' \setminus N')$ whose boundary is an oriented link L^P_k with orientation induced from P^E_k such that L^P_k is in a 3-ball $B'' \subset \partial N' \setminus \partial B'$ by an ambient isotopic deformation of P^E_k. Let D_k be a connected Seifert surface for L^P_k in B''. Let $PD_k = P^E_k \cup (-D_k)$ be the closed connected oriented surface in X', and $\widetilde{P}D_k$ a connected lift of PD_k to $\tilde{X}'$. Note that the homology class $[\widetilde{P}_k]' \in H_2(\widetilde{X}'; Z)$ which is the projection image of the homology class $[\widetilde{P}_k] \in H_2(\widetilde{X}; Z)$ into the direct summand $H_2(\widetilde{X}'; Z)$ is written as

$$[\widetilde{P}_k]' = [\widetilde{P}D_k] = t^r(c_0 + c_1 t + \cdots + c_d t^d),$$

where $r = \min\deg([\widetilde{P}_k]'; V', X')$, $r+d = \max\deg([\widetilde{P}_k]'; V', X')$, and c_i $(i = 0, 1, \ldots, d)$ are homology classes in $H_2(\widetilde{X}'; Z)$ represented by 2-cycles in the fundamental region $X'_{V'}$ with $c_0 \neq 0$ and $c_d \neq 0$.

Let $\widetilde{S}^3$ be a fixed 3-sphere lift of V' to $\widetilde{X}'$. For any integer j, let $\widetilde{L}_{k,j} = t^{-j}(\widetilde{P}D_k \cap t^j\widetilde{S}^3)$ be an oriented link (with orientation determined by the orientations of $\widetilde{P}D_k$, $\widetilde{S}^3$ and $\widetilde{X}$) in $\widetilde{S}^3$ unless it is empty. The following lemma is used in our argument.

Lemma 2.2. Let F be a closed oriented surface in $X' \setminus N'$ with $F \cap PD_k = \emptyset$, and $\widetilde{F}$ the preimage of F under the projection $\widetilde{X}' \to X'$. Assume that the surface $\widetilde{F}$ meets the 3-sphere $\widetilde{S}^3$ as a knot $\widetilde{K}$. Then for any integer j with $j \leqq \min\deg([\widetilde{P}_k]'; V', X')$ or $j > \max\deg([\widetilde{P}_k]'; V', X')$ with $\widetilde{L}_{k,j} \neq \emptyset$, the linking number $\mathrm{Link}^{\widetilde{S}^3}(\widetilde{K}, \widetilde{L}_{k,j})$ in the 3-sphere $\widetilde{S}^3$ is 0.

Proof. For any j with $\widetilde{L}_{k,j} \neq \emptyset$, we construct a closed oriented surface $t^j(C'_{k,j}) \cup (-C''_{k,j})$ in $\widetilde{X}'$ where $t^j(C'_{k,j})$ is a Seifert surface of the link $t^j(\widetilde{L}_{k,j})$ in the 3-sphere $t^j(\widetilde{S}^3)$ and $C''_{k,j}$ is a compact surface in $\widetilde{P}D_k$ bounded by $t^j(\widetilde{L}_{k,j})$. If $j \leqq r$ or $j > r + d$, then it is possible to choose $C''_{k,j}$ so that the surface $t^j(C'_{k,j}) \cup (-C''_{k,j})$ is null-homologous in $\widetilde{X}'$. By using the fundamental region $X'_{V'}$, choose a compact 4-submanifold $X'_J = \cup_{i=-J}^{J} t^i(X'_{V'})$ of $\widetilde{X}'$ for a sufficiently large integer J to contain the surface $t^j(C'_{k,j}) \cup (-C''_{k,j})$ and the 3-sphere $t^j(\widetilde{S}^3)$ in the interior. Let $\hat{F}^c$ be a closed oriented surface obtained from the compact surface $F^c = \widetilde{F} \cap X_J$ by adding surfaces in ∂X_J which are translations of a Seifert surface of $\widetilde{K}$ in $\widetilde{S}^3$. Then the Z-intersection number $\mathrm{Int}^{\widetilde{X}}(\hat{F}^c, t^j(C'_{k,j}) \cup (-C''_{k,j}))$ is zero. Since $C''_{k,j} \cap \hat{F}^c = \emptyset$, the Z-intersection number $\mathrm{Int}^{t^j(\widetilde{S}^3)}(t^j(\widetilde{K}), t^j(C'_{k,j}))$ is zero and hence $\mathrm{Link}^{\widetilde{S}^3}(\widetilde{K}, \widetilde{L}_{k,j}) = 0$. □

Throughout the remainder of this section, an estimate of a winding degree from a Λ-intersection matrix for an odd positive definite Z^{π_1}-manifold is explained. Let $A = (a_{ij}(t))$ be a Λ-intersection matrix of size n on an odd positive definite Z^{π_1}-manifold M. It is noted that $a_{ij}(t) = a_{ji}(t^{-1})$ for all i, j. Let $\tilde{x}_i\,(i = 1, 2, \ldots, n)$ be the Λ-basis for $H_2(\widetilde{M}; Z)$ giving the matrix A, and $\tilde{x}'_i\,(i = 1, 2, \ldots, n)$ the dual Λ-basis, i.e., the Λ-basis for $H_2(\widetilde{M}; Z)$ with $\mathrm{Int}^{\widetilde{M}}_{\Lambda}(\tilde{x}_i, \tilde{x}'_j) = \mathrm{Int}^{\widetilde{M}}_{\Lambda}(\tilde{x}'_j, \tilde{x}_i) = \delta_{ij}$ for all i, j, whose Λ-intersection matrix is given by the inverse matrix $A^{-1} = (b_{ij}(t))$. The following identities

$$\begin{aligned} \tilde{x}_j &= \sum_{k=1}^{n} a_{kj}(t)\tilde{x}'_k \quad (j = 1, 2, \ldots, n), \\ \tilde{x}'_j &= \sum_{k=1}^{n} b_{kj}(t)\tilde{x}_k \quad (j = 1, 2, \ldots, n) \end{aligned}$$

are easily established. Consider the following unique splittings of the Laurent polynomials $a_{ii}(t)$ and $b_{ii}(t)$ in t:

$$a_{ii}(t) = \varepsilon_i^a + a'_{ii}(t) + a'_{ii}(t^{-1}), \quad b_{ii}(t) = \varepsilon_i^b + b'_{ii}(t) + b'_{ii}(t^{-1})$$

where ε_i^a and ε_i^b are taken 0 or 1 and $a'_{ii}(t)$ and $b'_{ii}(t)$ are elements in Λ with non-negative constant terms and without any negative powers of t. Let

$$\tilde{a}_{ij}(t) = \begin{cases} a_{ij}(t) & i \neq j \\ a'_{ii}(t) & i = j, \end{cases} \qquad \tilde{b}_{ij}(t) = \begin{cases} b_{ij}(t) & i \neq j \\ b'_{ii}(t) & i = j. \end{cases}$$

Further, for a double index element $f_{ij}(t) \in \Lambda\,(i, j = 1, 2, \ldots, n)$, let

$$f^0_{ij}(t) = \begin{cases} 0 & i > j \\ f_{ij}(t) & i \leqq j. \end{cases} \qquad f^{0*}_{ij}(t) = \begin{cases} 0 & i < j \\ f_{ij}(t) & i \geqq j. \end{cases}$$

Let $\max \lambda(A)$ and $\min \lambda(A)$ be respectively the maximal degree and the minimal degree of the following Laurent polynomials in t:

$$1, \quad a_{ij}(t)\,(i < j), \quad b_{ij}(t)\,(i < j), \quad a'_{ii}(t^{-1}), \quad b'_{ii}(t), \quad c_{ij}(t) = \sum_{k=1}^{\min\{i,j\}} \tilde{a}_{ik}(t) \cdot \tilde{b}_{kj}(t)$$

for all i, j. Let $\lambda(A) = \max \lambda(A) - \min \lambda(A)$. Then an estimate of a winding degree is done as follows:

Lemma 2.3. For an odd positive definite Z^{π_1}-manifold M with Λ-intersection matrix $A = (a_{ij}(t))$, there is a winding degree λ on M such that $\lambda \leqq \lambda(A)$.

Proof. For the orientation-reversed manifold $-M$ of M, it is noted that any circle union $Y = M \circ (-M)$ is TOP-split as $S^1 \times S^3 \# Q(n) \# \overline{Q}(n)$ because odd indefinite forms are diagonal and the connected sum $Y_1 = M_1 \#(-M_1)$ for the simply connected manifold Y_1 obtained from Y by a 2-handle surgery killing $\pi_1(Y) \cong Z$ is homeomorphic to $Q(n)\#\overline{Q}(n)$ by using the vanishing of Kirby-Siebenmann obstruction (see [5]) and the manifold Y is a TOP-split Z^{π_1}-manifold as it is discussed from now. For the Λ-basis $\tilde{x}_i\,(i = 1, 2, \ldots, n)$ and its dual Λ-basis $\tilde{x}'_i\,(i = 1, 2, \ldots, n)$ for $H_2(\widetilde{M}; Z)$ representing A and A^{-1}, respectively, let $\tilde{x}^-$ and $\tilde{x}'^-_i$ correspond to $\tilde{x}$ and $\tilde{x}'_i$ in the direct summand $H_2(-\widetilde{M}; Z)$ of $H_2(\widetilde{Y}; Z)$ for every i, respectively. Let $\tilde{y}_i = \tilde{x}'_i + \tilde{x}'^-_i \in H_2(\widetilde{Y}; Z)\,(i = 1, 2, \ldots, n)$. Note that $\mathrm{Int}^{\widetilde{Y}}_\Lambda(\tilde{y}_i, \tilde{y}_j) = 0$ and $\mathrm{Int}^{\widetilde{Y}}_\Lambda(\tilde{y}_i, \tilde{x}^-_j) = -\delta_{ij}$ for all i, j. The elements $\tilde{z}_i\,(i = 1, 2, \ldots, n)$ in $H_2(\widetilde{Y}; Z)$ are constructed as follows:

$$\begin{aligned} \tilde{z}_1 &= \tilde{x}^-_1 - a'_{11}(t^{-1})\tilde{y}_1, \\ \tilde{z}_2 &= \tilde{x}^-_2 - a_{21}(t^{-1})\tilde{y}_1 - a'_{22}(t^{-1})\tilde{y}_2, \\ &\cdots \\ \tilde{z}_n &= \tilde{x}^-_n - a_{n1}(t^{-1})\tilde{y}_1 - a_{n2}(t^{-1})\tilde{y}_2 - \cdots - a'_{nn}(t^{-1})\tilde{y}_n. \end{aligned}$$

It is checked that

$$\mathrm{Int}_\Lambda^{\widetilde{Y}}(\tilde{y}_i, \tilde{z}_j) = -\delta_{ij}, \quad \mathrm{Int}_\Lambda^{\widetilde{Y}}(\tilde{z}_i, \tilde{z}_j) = -\varepsilon_i^a \delta_{ij}$$

for all i, j, so that $\tilde{y}_i, \tilde{z}_j\,(i, j = 1, 2, \dots, n)$ form a Λ-basis for $H_2(\widetilde{Y}; Z)$ with integral Λ-intersection matrix and thus, the Z^{π_1}-manifold Y is TOP-split. Since Y is odd and has the trivial Kirby-Siebenmann obstruction, there is an identification

$$Y = S^1 \times S^3 \# Q(u) \# \overline{Q}(u)$$

where every component of the 2-spheres nP in $\overline{Q}(n)$ represents a Z-linear combination of the Λ-basis $\tilde{y}_i, \tilde{z}_j\,(i, j = 1, 2, \dots, n)$ of $H_2(\widetilde{Y}; Z)$. Let $X = Y \circ M = M \circ (-M) \circ M = M \circ Y^*$ with $Y^* = (-M) \circ M$. A Λ-basis $\tilde{y}_i^*, \tilde{z}_j^*\,(i, j = 1, 2, \dots, n)$ for $H_2(\widetilde{Y}^*; Z)$ is given as follows:

Let $\tilde{y}_i^* = \tilde{x}_i^- + \tilde{x}_i^*$ where $\tilde{x}_i^-$ and $\tilde{x}_i^*$ correspond to $\tilde{x}_i^- \in H_2(-\widetilde{M}; Z)$ and $\tilde{x}_i \in H_2(\widetilde{M}; Z)$, respectively. Then we have $\mathrm{Int}_\Lambda^{\widetilde{Y}^*}(\tilde{y}_i^*, \tilde{y}_j^*) = 0$ for all i, j. For the dual Λ-basis $\tilde{x}_i'^*\,(i = 1, 2, \dots, n)$ of $\tilde{x}_i^*\,(i = 1, 2, \dots, n)$, let

$$\begin{aligned}
\tilde{z}_1^* &= \tilde{x}_1'^* - b_{11}'(t)\tilde{y}_1^*, \\
\tilde{z}_2^* &= \tilde{x}_2'^* - b_{12}(t)\tilde{y}_1^* - b_{22}'(t)\tilde{y}_2^*, \\
&\dots \\
\tilde{z}_n^* &= \tilde{x}_n'^* - b_{1n}(t)\tilde{y}_1^* - b_{2n}(t)\tilde{y}_2^* - \dots - b_{nn}'(t)\tilde{y}_n^*.
\end{aligned}$$

Then we have

$$\mathrm{Int}_\Lambda^{\widetilde{Y}}(\tilde{y}_i^*, \tilde{y}_j^*) = 0, \quad \mathrm{Int}_\Lambda^{\widetilde{Y}}(\tilde{y}_i^*, \tilde{z}_j^*) = \delta_{ij}, \quad \mathrm{Int}_\Lambda^{\widetilde{Y}}(\tilde{z}_i^*, \tilde{z}_j^*) = \varepsilon_i^b \delta_{ij}$$

for all i, j. This Λ-basis creates a 3-sphere leaf V^* for Y^* and defines a winding degree λ on M.

The Λ-intersection numbers $\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{y}_i, \tilde{y}_j^*)$ and $\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{y}_i, \tilde{z}_j^*)$ are calculated as follows.

$$\begin{aligned}
\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{y}_i, \tilde{y}_j^*) &= \mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_i' + \tilde{x}_i'^-, \tilde{x}_j^- + \tilde{x}_j^*) = \mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_i'^-, \tilde{x}_j^-) = -\delta_{ij}. \\
\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{y}_i, \tilde{z}_j^*) &= \mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_i' + \tilde{x}_i'^-, \tilde{x}_j'^* - \sum_{k'=1}^{j} \tilde{b}_{k'j}(t)(\tilde{x}_{k'}^- + \tilde{x}_{k'}^*) \\
&= \mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_i'^-, -\sum_{k'=1}^{j} \tilde{b}_{k'j}(t)\tilde{x}_{k'}^-) = \tilde{b}_{ij}^0(t).
\end{aligned}$$

The Λ-intersection numbers $\mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{z}_i, \tilde{y}_j^*)$ and $\mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{z}_i, \tilde{z}_j^*)$ are calculated as follows.

$$\begin{aligned}
\mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{z}_i, \tilde{y}_j^*) &= \mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{x}_i^- - \sum_{k=1}^{i} \tilde{a}_{ik}(t^{-1})(\tilde{x}'_k + \tilde{x}'^-_k), \tilde{x}_j^- + \tilde{x}_j^*) \\
&= \mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{x}_i^- - \sum_{k=1}^{i} \tilde{a}_{ik}(t^{-1})\tilde{x}'^-_k, \tilde{x}_j^-) = -a_{ij}(t) + \tilde{a}^{0*}_{ij}(t). \\
\mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{z}_i, \tilde{z}_j^*) &= \mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{x}_i^- - \sum_{k=1}^{i} \tilde{a}_{ik}(t^{-1})(\tilde{x}'_k + \tilde{x}'^-_k), \tilde{x}'^*_j - \sum_{k'=1}^{j} \tilde{b}_{k'j}(t)(\tilde{x}^-_{k'} + \tilde{x}^*_{k'}) \\
&= \mathrm{Int}_\Lambda^{\tilde{X}}(\tilde{x}_i^- - \sum_{k=1}^{i} \tilde{a}_{ik}(t^{-1})\tilde{x}'^-_k, -\sum_{k'=1}^{j} \tilde{b}_{k'j}(t)\tilde{x}^-_{k'}) \\
&= \tilde{b}^0_{ij}(t) - \sum_{k=1}^{\min\{i,j\}} \tilde{a}_{ik}(t) \cdot \tilde{b}_{kj}(t) \\
&= \tilde{b}^0_{ij}(t) - \sum_{k=1}^{\min\{i,j\}} \tilde{a}_{ik}(t) \cdot \tilde{b}_{kj}(t) = \tilde{b}^0_{ij}(t) - c_{ij}(t).
\end{aligned}$$

By examining these calculations (particularly, by noting that $-a_{ij}(t) + \tilde{a}^{0*}_{ij}(t)$ means $-a_{ij}(t)\,(i < j)$, $-\varepsilon_i^a - a'_{ii}(t^{-1})$ or $0\,(i > j)$, and $\tilde{b}^0_{ij}(t)$ means $b_{ij}(t)\,(i < j)$, $b'_{ii}(t)$ or $0\,(i > j)$, it is seen that $\lambda \leqq \lambda(A)$. $\square$

3. Proofs of Theorems 1.1 and 1.4

It is not always assumed that a closed 4-manifold is a smooth or piecewise-linear manifold, but smooth and piecewise-linear techniques can be used for it because a punctured manifold of it is smoothable (see [5]).

The proof of Theorem 1.1 will be basically analogous to the proof of [16, Theorem 1.1]. For the use of a minimal element, we provide the following lemma.

Lemma 3.1. For a TOP-split positive definite Z^{π_1}-manifold M, let $x \in H_2(M; Z)$ be a minimal element. For any element $x' \in H_2(M^{(2)}; Z)$ with $p_*(x') = x$, the inequality $||x'||^2 \geqq ||x||^2$ holds.

Proof. Let $M = S^1 \times S^3 \# M_1$ for a simply connected closed 4-manifold M_1, and $M^{(2)} = S^1 \times S^3 \# M_{1,1} \# M_{1,2}$ for the two copies $M_{1,1}, M_{1,2}$ of M_1. The element x is represented as a 2-cycle in M_1. Let x_1 be an element of $H_2(M^{(2)}; Z)$ represented as a 2-cycle of $M_{1,1}$ such that $p_*(x_1) = x$ and $||x_1||^2 = ||x||^2$. Then there are elements $y_i\,(i = 1, 2)$ of $H_2(M^{(2)}; Z)$ such that y_i is represented as 2-cycles in $M_{1,i}$ and $x' = x_1 + y_1 + y_2$. The identity $-p_*(y_1) = p_*(y_2) \in H_2(M; Z)$ is obtained from $p_*(x') = x$, meaning that y_2 is nothing but the covering translation element of $-y_1$ in $H_2(M^{(2)};Z)$. Denoting $-p_*(y_1)=p_*(y_2)$ by y, we have

$$||y_1||^2 = ||y_2||^2 = ||y||^2.$$

Since $p_*(x_1+y_1)=x-y$ and $||x_1+y_1||^2=||x-y||^2$, we have $||x'||^2=||x-y||^2+||y||^2$. It is noted that this equality holds without the positive definiteness. Using that M is positive definite and x is a minimal element with $x=(x-y)+y$, we must have

$$||x-y||^2 \geqq ||x||^2 \quad \text{or} \quad ||y||^2 \geqq ||x||^2.$$

Thus, the inequality $||x'||^2 \geqq ||x||^2$ holds. □

The proof of Theorem 1.1 will be done as follows:

Proof of Theorem 1.1.
Proof of (0) → (2). Let $M=S^1\times S^3\#M_1$ for a simply connected closed 4-manifold M_1. Then $\widetilde{M}$ is the connected sum of $R^1\times S^3$ and the infinite copies $M_{1j}\,(j=0,\pm1,\dots)$ of M_1. Then every minimal element $\tilde{x}_i$ is represented by a 2-cycle in one copy M_{1j} after a t-power shift, so that the elements $\tilde{x}_i\,(i=1,2,\dots,n)$ are represented by the same copy M_{10} after suitable t-power shifts of $\tilde{x}_i\,(i=1,2,\dots,n)$, showing (0) → (2). □

Proof of (2) → (1). This assertion is obvious since after suitable t-shifts of $\tilde{x}_i\,(i=1,2,\dots,n)$, the Λ-intersection numbers $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i,\tilde{x}_j)$ belong to Z and the elements $\tilde{x}_i^{(1)}\,(i=1,2,\dots,n)$ are Z-generators for $H_2(M;Z)$. □

Proof of (1) → (0). This assertion will mean that the conditions (0), (1) and (2) are mutually equivalent. Assume that the elements $\tilde{x}_i\,(i=1,2,\dots,n)$ in $H_2(\widetilde{M};Z)$ with $a_{ij}=\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i,\tilde{x}_j)\in Z$ for all i,j induce Z-generators $\tilde{x}_i^{(1)}\,(i=1,2,\dots,n)$ for $H_2(M;Z)$. Then we have the Z-intersection number $\mathrm{Int}^M(\tilde{x}_i^{(1)},\tilde{x}_j^{(1)})=a_{ij}$ for all i,j. Let $H_2(M;Z)$ be a Z-free group of rank s. For a Z-basis $y_j\,(j=1,2,\dots,s)$ of $H_2(M;Z)$, every basis element y_j is a Z-linear combination of the elements $\tilde{x}_i^{(1)}\,(i=1,2,\dots,n)$. Let $\tilde{y}_j$ be the element in $H_2(\widetilde{M};Z)$ to be the Z-linear combination on $\tilde{x}_i\,(i=1,2,\dots,n)$ whose Z-coefficients are the same as the Z-coefficients in the Z-linear combination of y_j on $\tilde{x}_i^{(1)}\,(i=1,2,\dots,n)$. Then the Λ-intersection matrix $\left(\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{y}_j,\tilde{y}_{j'})\right)$ is equal to the Z-intersection matrix $(\mathrm{Int}^M(y_j,y_{j'}))$, which is a unimodular matrix of size s. Using that $H_2(\widetilde{M};Z)$ is a Λ-free module of rank s, we see that the elements $\tilde{y}_j\,(j=1,2,\dots,s)$ form a Λ-basis for $H_2(\widetilde{M};Z)$. Then it is shown in [13, Corollary 3.4] that M is TOP-split, but reproved here for convenience. In fact, for Z^{π_1}-manifold $M_0 = S^1 \times S^3 \# M_1$ with M_1 the simply connected 4-manifold obtained from M by a 2-handle surgery killing $\pi_1(M) \cong Z$, the Λ-intersection forms on $\widetilde{M}$ and $\widetilde{M}_0$ are Λ-isomorphic. Since M and M_0 have the same Kirby-Siebenmann obstruction, M and $S^1 \times S^3 \# M_1$ are homeomorphic by [5]. Thus, M is TOP-split, showing (1)→ (0). □

Proof of (0) → (3). For $M = S^1 \times S^3 \# M_1$ and $\widetilde{M} = R^1 \times S^3 \#_{j=-\infty}^{+\infty} M_{1j}$ (already established), every minimal element $\widetilde{x} \in H_2(\widetilde{M}; Z)$ is represented by a 2-cycle c_x in the copy M_{10} after a t-power shift, which is not null-homologous in M_{10}. Assume that the element $\widetilde{x}^{(m)} \in H_2(M^{(m)}; Z)$ is the sum $y+z$ for non-zero elements $y, z \in H_2(M^{(m)}; Z)$. Let $y = [c'_y + c''_y], z = [c'_z + c''_z] \in H_2(M^{(m)}; Z)$ where c'_y, c'_z are 2-cycles in M_{10} and c''_y, c''_z are 2-cycles in the complement $\mathrm{cl}(M^{(m)} \setminus M_{10})$. Since $\widetilde{x} = [c_x] = [c_y] + [c_z] \in H_2(\widetilde{M}; Z)$, we have $||\widetilde{x}||^2 \leqq ||[c_y]||^2$ or $||\widetilde{x}||^2 \leqq ||[c_z]||^2$ by the definition of minimal elements. This means that

$$||\widetilde{x}^{(m)}||^2 \leqq ||[c_y]||^2 \leqq ||y||^2 \quad \text{or} \quad ||\widetilde{x}^{(m)}||^2 \leqq ||[c_z]||^2 \leqq ||z||^2.$$

Thus, the element $\widetilde{x}^{(m)} \in H_2(M^{(m)}; Z)$ is a minimal element for every m. Let $x' \in H_2(M^{(2m)}; Z)$ be an element with $p_*(x') = \widetilde{x}^{(m)}$. Since $M^{(m)}$ is TOP-split, we see from Lemma 3.1 that $||x'||^2 \geqq ||\widetilde{x}^{(m)}||^2$, showing (0) → (3). □

Proof of (3) → (4). Let $\widetilde{x}_i \, (i = 1, 2, \ldots, n)$ be minimal Λ-generators of $H_2(\widetilde{M}; Z)$. By (3), the elements

$$t^j(\widetilde{x}_i^{(m)}) \in H_2(M^{(m)}; Z) \quad (i = 1, 2, \ldots, n; j = 0, 1, \ldots, m-1)$$

give desired minimal Z-generators of $H_2(M^{(m)}; Z)$ without assumption that $m \geqq \lambda$, showing (3) → (4). □

It will be directly shown in Proposition 3.2 that the inequality $||x'_i||^2 > ||x_i||^2 - 2$ in (4) is always equivalent to the inequality $||x'_i||^2 \geqq ||x_i||^2$.

Proof of (4) → (0). By [15], M is TOP-split if and only if $M^{(m)}$ is TOP-split. The proof will be done by replacing $M^{(m)}$ with $m \geqq \lambda$ for a previously given winding degree λ on M by M with $1 \geqq \lambda$. Let $x_i \, (i = 1, 2, \ldots, s)$ be minimal Z-generators of $H_2(M; Z)$ such that $||x'_i||^2 > ||x_i||^2 - 2$ for every element $x'_i \in H_2(M^{(2)}; Z)$ with $p_*(x'_i) = x_i$ and every i. The winding degree λ on M is given by the number $d([uP]'; V', X')$ on $X = \overline{Q}(u) \# Q(v) \# M = X' \circ X''$ as defined in § 2. Let P_k be the kth 2-sphere in the 2-sphere union uP in $\overline{Q}(u)$. Let $\widetilde{P}_k$ be a connected lift of P_k

to $\widetilde{X}$. After a t-power shift, the homology class $[\widetilde{P}_k]' \in H_2(\widetilde{X}'; Z)$ is written as

$$[\widetilde{P}_k]' = [\widetilde{P}D_k] = c_{k,0} + c_{k,1}t,$$

where $\widetilde{P}D_k$ is a connected lift of a closed connected oriented surface $PD_k = P_k^E \cup (-D_k) \subset X'$ constructed in § 2 so that $P_k^E \subset \mathrm{cl}(X' \setminus N')$ and $D_k \subset \partial N' \setminus \partial B'$, and $c_{k,i}\,(i = 0, 1)$ are homology classes in $H_2(\widetilde{X}'; Z)$ represented by 2-cycles in the fundamental region $X'_{V'}$.

Let $\widetilde{S}^3$ be a fixed connected lift of the 3-sphere leaf $S^3 = V'$ of X' to the infinite cyclic covering space $\widetilde{X}'$. Let $\widetilde{L}_{k,j} = t^{-j}(\widetilde{P}D_k \cap t^j \widetilde{S}^3)$ be an oriented link (possibly the empty link) in $\widetilde{S}^3$, and $L_{k,j}$ the projection of $\widetilde{L}_{k,j}$ into S^3. Represent the element x_i by a closed connected oriented surface F_i (embedded) in $M \setminus N$, which are disjoint from the surfaces PD_k. By modifying F_i in a collar $S^3 \times [-1, 1]$ of the 3-sphere leaf S^3 in X', assume that the intersection $K_i = F_i \cap S^3$ is an oriented trivial knot bounding a disk Δ_i in S^3 such that the system $\Delta_i\,(i = 1, 2, \ldots, s)$ are mutually disjoint and meet PD_k transversely in a finite number of points.

For the preimage $\widetilde{F}_i$ of F_i under the covering $\widetilde{X}' \to X'$, let $\widetilde{K}_i = \widetilde{F}_i \cap \widetilde{S}^3$, and $\widetilde{\Delta}_i$ the lift of Δ_i to $\widetilde{S}^3$ with $\partial\widetilde{\Delta}_i = \widetilde{K}_i$. We show that

$$\ell_{i,k,j} = \mathrm{Link}^{\widetilde{S}^3}(\widetilde{K}_i, \widetilde{L}_{k,j}) = 0$$

for all i, j, k. By Lemma 2.2, $\ell_{i,k,j} = 0$ for $j \leqq 0$ or $j > 1$ and all i and k. Suppose $\ell_{i,k,1}$ is not zero for some i and k.

Construct an immersed disk Δ_i^* in X with $\partial\Delta_i^* = K_i$ from Δ_i by replacing a meridian disk of every point $L_{k,j} \cap \Delta_i$ for all j in Δ_i with a disk disjointedly parallel to P_k for every k.

Let G_i be a singular closed connected oriented surface in X obtained from F_i by cutting along K_i and attaching two anti-oriented copies $\pm\Delta_i^*$ of Δ_i^*. As an important note, by taking

$$a_i = \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{u} \ell_{i,,k,j}^2 = \sum_{k=1}^{u} \ell_{i,k,1}^2 > 0,$$

the surface Δ_i^* has the Z-self-intersection number $-a_i$ in X with respect to the Seifert framing of K_i in S^3. Let $x_{i,X}$ be the image of x_i by the monomorphism $H_2(M; Z) \to H_2(X; Z)$. Then we have

$$x_{i,X} = [F_i] = [G_i] \in H_2(X; Z).$$

For a connected lift G'_i of G_i to the double covering space $X^{(2)}$ of X, let

$$x'_{i,X} = [G'_i] \in H_2(X^{(2)}; Z).$$

The square length of the element $x'_{i,X}$ is estimated as follows:

$$(*) \qquad ||x'_{i,X}||^2 = ||x_{i,X}||^2 - 2a_i \leqq ||x_{i,X}||^2 - 2 = ||x_i||^2 - 2.$$

Since $\Delta_i^* \cap uP = \emptyset$, the surface Δ_i^* is regarded as a surface in the positive definite Z^{π_1}-manifold $M_Q = Q(v)\#M$ obtained from X by blowing down on $\overline{Q}(u)$. Let $x_{i,Q}$ be the image of x_i by the monomorphism $H_2(M;Z) \to H_2(M_Q;Z)$. Then we have

$$x_{i,Q} = [F_i] = [G_i] \in H_2(M_Q;Z).$$

Since the surface G'_i is in the connected summand $M_Q^{(2)}$ of $X^{(2)}$, let

$$x'_{i,Q} = [G'_i] \in H_2(M_Q^{(2)};Z),$$

which is sent to $x_{i,Q}$ under the double covering projection homomorphism $H_2(M_Q^{(2)};Z) \to H_2(M_Q;Z)$. Then the inequality $(*)$ is equivalent to the inequality

$$(**) \qquad ||[x'_{i,Q}||^2 \leqq ||x_{i,Q}||^2 - 2 = ||x_i||^2 - 2.$$

By using that $H_2(M^{(2)};Z)$ is an orthogonal summand of $H_2(M_Q^{(2)};Z)$, let x'_i be the image of $x'_{i,Q}$ by the orthogonal summand projection $H_2(M_Q^{(2)};Z) \to H_2(M^{(2)};Z)$. Since $M_Q^{(2)}$ is positive definite, we have

$$||[x'||^2 \leqq ||x'_{i,Q}||^2, \quad \text{so that} \quad ||x'_i||^2 \leqq ||x_i||^2 - 2.$$

This contradicts the inequality $||x'_i||^2 > ||x_i||^2 - 2$ given by the assumption of (4), because the double covering projection homomorphism $p_* : H_2(M^{(2)};Z) \to H_2(M;Z)$ sends x'_i to x_i. Thus, we have

$$\ell_{i,k,j} = \mathrm{Link}^{\widetilde{S}^3}(\widetilde{K}_i, \widetilde{L}_{j,k}) = 0$$

in the 3-sphere $\widetilde{S}^3$ for all i, j, k. This means that all the Z-intersection numbers (containing the Z-self-intersection numbers) on any connected surfaces lifting the surfaces $\Delta_i^* (i = 1, 2, \ldots, n)$ to the infinite cyclic covering space $\widetilde{X}$ have 0 with respect to the lifted framings of the Seifert framings of $K_i (i = 1, 2, \ldots, n)$ in S^3. Let $\widetilde{G}_i$ be a connected lift of the singular closed oriented surface G_i in X (already constructed above) to $\widetilde{X}$. Let

$$\tilde{x}_{i,X} = [\widetilde{G}_i] \in H_2(\widetilde{X};Z)\,(i = 1, 2, \ldots, n).$$

Then, from construction, we see that the Λ-intersection numbers $\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_{i,X}, \tilde{x}_{i',X})$ are integers for all i, i'. Since the surfaces $\widetilde{G}_i\,(i = 1, 2, \ldots, n)$ belong to the infinite cyclic covering space $\widetilde{M}_Q$ of M_Q, let

$$\tilde{x}_{i,Q} = [\widetilde{G}_i] \in H_2(\widetilde{M}_Q;Z)\,(i = 1, 2, \ldots, n).$$

Note that the element $\tilde{x}_{i,Q}$ is sent to $x_{i,Q}$ under the covering projection homomorphism $H_2(\widetilde{M}_Q; Z) \to H_2(M_Q; Z)$ and

$$\mathrm{Int}_\Lambda^{\widetilde{X}}(\tilde{x}_{i,X}, \tilde{x}_{i',X}) = \mathrm{Int}_\Lambda^{\widetilde{M}_Q}(\tilde{x}_{i,,Q}, \tilde{x}_{i',Q}) = \mathrm{Int}^{M_Q}(x_{i,Q}, x_{i',Q}) = \mathrm{Int}^M(x_i, x_{i'}) \in Z$$

for all i, i'. Then, by an argument of (1) $\to$ (0), there is a free Λ-submodule $\widetilde{H}$ of $H_2(\widetilde{M}_Q; Z)$ with a Λ-basis $\tilde{y}_i\,(i = 1, 2, \ldots, s^*)$ such that

(i) every basis element $\tilde{y}_i$ is a Z-linear combination of $\tilde{x}_{i,,Q}\,(i = 1, 2, \ldots, s)$
(ii) every element $\tilde{x}_{i,,Q}$ is written as a Z-linear combination of $\tilde{y}_i\,(i = 1, 2, \ldots, s^*)$.
(iii) the Λ-intersection matrix with respect to $\tilde{y}_i\,(i = 1, 2, \ldots, s^*)$ is an integral matrix with determinant $+1$.

We claim that there are minimal elements $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ in $\widetilde{H}$ such that $\mathrm{Int}_\Lambda^{\widetilde{M}_Q}(\tilde{z}_i, \tilde{z}_{i'}) \in Z$ for all i, i' and, by the covering projection homomorphism $H_2(\widetilde{M}_Q; Z) \to H_2(M_Q; Z)$, the elements $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ are sent to the minimal elements $x_{i,Q}\,(i = 1, 2, \ldots, s)$, respectively. In fact, by an argument on an orthogonal complement of the Λ-intersection form, there is an orthogonal splitting

$$H_2(\widetilde{M}_Q; Z) = \widetilde{H} \oplus \widetilde{H}'$$

with respect to the Λ-intersection form Int_Λ on $H_2(\widetilde{M}_Q; Z)$. By Freedman-Quinn construction in [5], there is a circle union splitting $Y \circ Y'$ of M_Q for some positive definite Z^{π_1}-manifolds Y, Y' such that the Λ-intersection form on $H_2(\widetilde{Y}; Z)$ is isomorphic to the restriction of the Λ-intersection form on $H_2(\widetilde{M}_Q; Z)$ to $\widetilde{H}$. By (1) $\to$ (0), Y is a TOP-split. Let Y_V be a fundamental region of Y splitting along a 3-sphere leaf V. Let $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ be Λ-generators of $H_2(\widetilde{Y}; Z) = \widetilde{H}$ represented by 2-cycles in Y_V and sent respectively to the minimal elements $x_{i,Q}\,(i = 1, 2, \ldots, s)$. Because $H_2(Y_V; Z)$ is isomorphic to the orthogonal summand $H_2(M; Z)$ of $H_2(\widetilde{M}_Q; Z)$ by the covering projection homomorphism $H_2(\widetilde{M}_Q; Z) \to H_2(M_Q; Z)$ and $H_2(Y_V; Z)$ is an orthogonal summand of $H_2(\widetilde{Y}; Z) = \widetilde{H}$, it follows that the elements $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ are minimal elements in $\widetilde{H}$, as desired.

Using that $\widetilde{H}$ is an orthogonal summand of $H_2(\widetilde{M}_Q; Z)$, we see that the elements $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ are minimal elements in $H_2(\widetilde{M}_Q; Z)$. Every minimal element of $H_2(CP^2; Z)$ must be a generator, so that the minimal element $\tilde{z}_i \in H_2(\widetilde{M}_Q; Z)$ is represented by a 2-cycle in $\widetilde{M}$ because $\tilde{z}_i$ is sent to $x_{i,Q}$. This implies that there are elements $\tilde{x}_i\,(i = 1, 2, \ldots, s)$ in $H_2(\widetilde{M}; Z)$ coming from $\tilde{z}_i\,(i = 1, 2, \ldots, s)$ such that the elements $\tilde{x}_i^{(1)}, (i = 1, 2, \ldots, s)$ are equal to the Z-generators $x_i\,(i =$

$1, 2, \ldots, s$) of $H_2(M; Z)$ coming from $x_{i,Q}$ $(i = 1, 2, \ldots, s)$ and the Λ-intersection numbers $\mathrm{Int}_{\Lambda}^{\widetilde{M}}(\tilde{x}_i, \tilde{x}_{i'})$ are integers for all i, i'. By (1) $\to$ (0), M is TOP-split, showing (4) $\to$ (0). $\square$

Proof of (0) $\to$ (5). If M is TOP-split, then there is a winding degree λ on M with $\lambda = 0$ by definition. $\square$

Proof of (5) $\to$ (0). Assume that $\lambda_{\min} = \lambda = d([uP]'; V', X') = 0$ on $X = \overline{Q}(u)\#Q(v)\#M = X' \circ X''$ as defined in § 2. The proof is almost similar to the proof of the assertion (4) $\to$ (0). Let P_k be the kth 2-sphere in the 2-sphere union uP in $\overline{Q}(u)$. Let $\widetilde{P}_k$ be a connected lift of P_k to $\widetilde{X}$. After a t-power shift, the homology class $[\widetilde{P}_k]' \in H_2(\widetilde{X}'; Z)$ is written as

$$[\widetilde{P}_k]' = [\widetilde{P}D_k] = c_{k,0},$$

where $\widetilde{P}D_k$ is a connected lift of a closed connected oriented surface $PD_k = P_k^E \cup (-D_k) \subset X'$ constructed in § 2 so that $P_k^E \subset \mathrm{cl}(X' \setminus N')$ and $D_k \subset \partial N' \setminus \partial B'$, and $c_{k,0}$ is a homology class in $H_2(\widetilde{X}'; Z)$ represented by a 2-cycle in the fundamental region $X'_{V'}$.

Let $\widetilde{S}^3$ be a fixed connected lift of the 3-sphere leaf $S^3 = V'$ of X' to the infinite cyclic covering space $\widetilde{X}'$. Let $\widetilde{L}_{k,j} = t^{-j}(\widetilde{P}D_k \cap t^j \widetilde{S}^3)$ be an oriented link (possibly the empty link) in $\widetilde{S}^3$, and $L_{k,j}$ the projection of $\widetilde{L}_{k,j}$ into S^3.

Let x_i $(i = 1, 2, \ldots, s)$ be any minimal Z-generators of $H_2(M; Z)$. Represent the element x_i by a closed connected oriented surface F_i (embedded) in $M \setminus N$, which are disjoint from the surfaces PD_k. By modifying F_i in a collar $S^3 \times [-1, 1]$ of the 3-sphere leaf S^3 in X', assume that the intersection $K_i = F_i \cap S^3$ is an oriented trivial knot bounding a disk Δ_i in S^3 such that the system Δ_i $(i = 1, 2, \ldots, s)$ are mutually disjoint and meet PD_k transversely in a finite number of points.

For the preimage $\widetilde{F}_i$ of F_i under the covering $\widetilde{X}' \to X'$, let $\widetilde{K}_i = \widetilde{F}_i \cap \widetilde{S}^3$, and $\widetilde{\Delta}_i$ the lift of Δ_i to $\widetilde{S}^3$ with $\partial\widetilde{\Delta}_i = \widetilde{K}_i$. Then, by Lemma 2.2, we have

$$\ell_{i,k,j} = \mathrm{Link}^{\widetilde{S}^3}(\widetilde{K}_i, \widetilde{L}_{k,j}) = 0$$

for all i, j, k.

The rest of the proof is completely the same as the proof of the assertion (4) $\to$ (0). This shows the assertion (5) $\to$ (0). $\square$

This completes the proof of Theorem 1.1. $\square$

As an additional note to Theorem 1.1, the following proposition clarifies a reason why the square length $||x||^2$ in (2) may be replaced by $||x_i||^2 - 2$ in (3).

Proposition 3.2. For every Z^{π_1}-manifold M, any elements $x \in H_2(M;Z)$ and $x' \in H_2(M^{(2)};Z)$ with $p_*(x') = x$ have the congruence $||x'||^2 \equiv ||x||^2 \pmod 2$.

Proof. For a *smooth* Z^{π_1}-manifold M, this property follows from the fact that every covering preserves the second Wu class. Let M_0 be a TOP-split Z^{π_1}-manifold. Then this property on M_0 holds as it is seen from a calculation of the Z-self-intersection numbers in the proof of Lemma 3.1. For a general Z^{π_1}-manifold M, there is a Z-homology cobordism W from M to a TOP-split Z^{π_1}-manifold M_0, which is seen from the proof of [11, Theorem 1.1] although this theorem itself contains a serious error (cf. [12, 13, 14]). Then the double covering space $W^{(2)}$ gives a $Z/2Z$-homology cobordism from $M^{(2)}$ to $M_0^{(2)}$. The pair $(W^{(2)}, W)$ sends the pair of elements x', x with $p_*(x') = x$ to a pair of elements $x'_0 \in H_2(M_0^{(2)};Z), x_0 \in H_2(M_0^{(2)};Z)$ with $p_*(x'_0) = x_0$ up to 2 times elements. Thus, the congruence $||x'||^2 \equiv ||x||^2 \pmod 2$ is obtained. □

The following corollary is slightly stronger than Corollary 1.2 and obtained from Theorem 1.1 (4) since, as noted in § 1, if there are Z-generators $x_i\,(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)};Z)$ with $||x_i||^2 \leqq 2$ for all i, then there are minimal Z-generators $y_j\,(j = 1, 2, \ldots, s)$ of $H_2(M^{(m)};Z)$ with $||y_j||^2 \leqq 2$ for all j and, for every element $y'_j \in H_2(M^{(2m)};Z)$ with $p_*(y'_j) = y_j$, the inequality $||y'_j||^2 > ||y_j||^2 - 2$ holds.

Corollary 3.3. A positive definite Z^{π_1}-manifold M is TOP-split if for any previously given winding degree λ on M, there is an $m \geqq \lambda$ for which there are Z-generators $x_i\,(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)};Z)$ such that $||x_i||^2 \leqq 2$ for all i.

The proof of Theorem 1.4 will be done as follows:

Proof of Theorem 1.4. Assume that M is TOP-split. Then there is a Λ-basis $\tilde{e}_i \in H_2(\widetilde{M};Z)\,(i = 1, 2, \ldots, n)$ with $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{e}_i, \tilde{e}_j) = \delta_{ij}$ for all i, j. For a Λ-basis $\tilde{x}_i \in H_2(\widetilde{M};Z)\,(i = 1, 2, \ldots, n)$, let $\tilde{x}_i = \sum_{k=1}^{s} a_{ik}(t)\tilde{e}_k$. Then we have

$$\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i, \tilde{x}_j) = \sum_{k=1}^{n} a_{ik}(t^{-1})a_{jk}(t),$$

showing (0) → (1). Assume (1). Let P be the matrix of size n whose (i, j) entry is $a_{ij}(t)$. Then the Λ-intersection matrix S whose (i, j) entry is $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i, \tilde{x}_j)$ is given by $P\overline{P}^T$. Since the determinant $\det S = 1$, the matrix P is non-singular in Λ, so that $P^{-1}S(\overline{P}^T)^{-1}$ is the identity matrix and M is TOP-split by for example Corollary 1.3, showing (1) → (0).

The assertion (0) $\to$ (2) is direct. The converse (2) $\to$ (0) is obtained from Corollary 1.3 because there are finitely many minimal Λ-generators of $H_2(\widetilde{M};Z)$.

(1) implies (3_1) since every element of $H_2(\widetilde{M};Z)$ is a Λ-linear combination of any given Λ-basis of $H_2(\widetilde{M};Z)$. Also, (2) implies (3_2) since there are Λ-linearly independent minimal elements $\tilde{y}_i\,(i=1,2,\dots,n)$. Thus, we have (0) $\to$ (3). To show (3) $\to$ (0), assume (3). Let $\tilde{x}_i\,(i=1,2,\dots,n-1)$ be mutually distinct elements of $H_2(\widetilde{M};Z)$ up to multiplications of the units of Λ such that the square length $||\tilde{x}_i||^2=1$ for all i. It is noted that if an element $f(t)\in\Lambda$ is written as $\sum_{i=p}^q c_i t^i\,(c_i\in Z)$, then the constant term c of the product $f(t)f(t^{-1})$ is written as the sum $\sum_{i=p}^q c_i^2$, so that according to whether $c=0$ or 1, we have $f(t)f(t^{-1})=0$ or 1, respectively. This means that the Λ-square length $||\tilde{x}_i||^2_\Lambda=1$ for all i. For the Λ-submodule $\Lambda[\tilde{x}_1]$ of $H_2(\widetilde{M};Z)$ generated by $\tilde{x}_1$, let $H_2(\widetilde{M})=\Lambda[\tilde{x}_1]\oplus H'$ be the orthogonal splitting with respect to the Λ-intersection form $\mathrm{Int}_\Lambda^{\widetilde{M}}$ on $H_2(\widetilde{M};Z)$. Then $\tilde{x}_2$ is written as $a_1(t)\tilde{x}_1+\tilde{x}_2'$ for an element $a_1(t)\in\Lambda$ and an element $\tilde{x}_2'$ in H'. The square length $||\tilde{x}_2||^2$ is computed from the constant terms of the following identities:

$$||\tilde{x}_2||^2_\Lambda=a_1(t)a_1(t^{-1})+||\tilde{x}_2'||^2_\Lambda=1.$$

If $\tilde{x}_2'\neq 0$, then the square length $||\tilde{x}_2'||^2>0$, so that $a_1(t)=0$ and $\tilde{x}_2=\tilde{x}_2'$. If $\tilde{x}_2'=0$, then $a_1(t)=\pm t^k$ for some integer k, so that $\tilde{x}_1$ and $\tilde{x}_2$ are equal up to the multiplication of a unit of Λ, which is a contradiction. Thus, the identity $\tilde{x}_2=\tilde{x}_2'\in H'$ is obtained. By a similar argument, the elements $\tilde{x}_i\,(i=2,3,\dots,n-1)$ are seen to be in H'. By an inductive argument, it is shown that the elements $\tilde{x}_i\,(i=1,2,\dots,n-1)$ have the identities $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i,\tilde{x}_j)=\delta_{ij}$ for all i,j and hence form a Λ-basis of the Λ-submodule $\Lambda[\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_{n-1}]$ of $H_2(\widetilde{M};Z)$ generated by $\tilde{x}_i\,(i=1,2,\dots,n-1)$. Let H_1' be the orthogonal complement of $\Lambda[\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_{n-1}]$ in $H_2(\widetilde{M};Z)$, which is Λ-isomorphic to Λ with Λ-intersection matrix (1). Thus, there is a Λ-basis $\tilde{x}_i\,(i=1,2,\dots,n)$ of $H_2(\widetilde{M})$ with identities $\mathrm{Int}_\Lambda^{\widetilde{M}}(\tilde{x}_i,\tilde{x}_j)=\delta_{ij}$ for all i,j. For example by Corollary 1.3, M is TOP-split, showing (3) $\to$ (0). □

4. Proofs of Theorem 1.5, Corollary 1.6 and Observations 1.7 and 1.8

The proof of Theorem 1.5 will be done as follows:

Proof of Theorem 1.5. The square length $||\tilde{x}_i||^2$ is 3 for $i=1,2$ and 2 for $i=3,4$. If $\tilde{x}_1$ is not minimal, then there are Λ-generators $\tilde{x}_{i'}'\,(i'=1,2,\dots,k)$ of $H_2(\widetilde{M}_L;Z)$ with $||\tilde{x}_{i'}'||^2\leqq 2$ for all i' because $||\tilde{x}_1-\tilde{x}_2||^2=2$. Then by Corollary 1.3, the Z^{π_1}-manifold M_L must be TOP-split, which contradicts that M_L is not TOP-split. Hence $\tilde{x}_1$ is a minimal element. By a similar argument, $\tilde{x}_2$ is also a minimal element. If $\tilde{x}_3$

is not minimal, then $\tilde{x}_3$ is the sum of two elements $\tilde{x}_3', \tilde{x}_3''$ with square lengths

$$||\tilde{x}_3'||^2 = ||\tilde{x}_3''||^2 = 1.$$

The elements $t^i\tilde{x}_1$ ($i \in Z$) must belong to the Z-orthogonal complement of the infinite cyclic group generated by $\tilde{x}_3'$ since they are minimal elements with square length 3. Hence $\mathrm{Int}_\Lambda^{\widetilde{M}_L}(\tilde{x}_1, \tilde{x}_3') = 0$. Similarly, $\mathrm{Int}_\Lambda^{\widetilde{M}_L}(\tilde{x}_1, \tilde{x}_3'') = 0$, so that $\mathrm{Int}_\Lambda^{\widetilde{M}_L}(\tilde{x}_1, \tilde{x}_3) = 0$ contradicting that $\mathrm{Int}_\Lambda^{\widetilde{M}_L}(\tilde{x}_1, \tilde{x}_3) \neq 0$. Hence $\tilde{x}_3$ is a minimal element. By a similar method, $\tilde{x}_4$ is also a minimal element, showing (1). To see (2), suppose $\det V_G = 1$. The set $B = \{t^i\tilde{x}_k |\, i \in Z, k = 1, 2, 3, 4\}$ forms a Z-basis for $H_2(\widetilde{M}_L; Z)$ consisting of minimal elements. By a property of a minimal element, every element of B belongs to either G or the Z-orthogonal complement $G^\perp$ of G constructed by using $\det V_G = 1$. However, it is impossible because any two elements of B are connected by a sequence $v_j \in B\,(j = 1, 2, \ldots, s)$ with $\mathrm{Int}^{\widetilde{M}_L}(v_j, v_{j+1}) \neq 0$ for every j. Thus, $\det V_G > 1$, showing (2). □

The proof of Corollary 1.6 will be done as follows:

Proof of Corollary 1.6. Since $m \geqq 3 \geqq e(\tilde{x}_i) + 1$ $(i = 1, 2, 3, 4)$, we have the square length

$$||\tilde{x}_i^{(m)}||^2 = ||\tilde{x}_i||^2 \quad (i = 1, 2, 3, 4)$$

which is 3 for $i = 1, 2$ and 2 for $i = 3, 4$. A winding degree λ on M_L with $\lambda \leqq 5$ is found from Lemma 2.3 by considering the following matrices:

$$A = \begin{pmatrix} 1+f+f^2 & 1 & 1+f & f \\ 1 & 2 & 1 & -1 \\ 1+f & 1 & 2 & 0 \\ f & -1 & 0 & 2 \end{pmatrix},$$

$$A^{-1} = \begin{pmatrix} 4 & -2 & -1-2f & -1-2f \\ -2 & 2 & f & 1+f \\ -1-2f & f & 1+f+f^2 & f+f^2 \\ -1-2f & 1+f & f+f^2 & 1+f+f^2 \end{pmatrix},$$

where the matrix A is obtained as the Λ-intersection matrix of $H_2(\widetilde{M}_L; Z)$ given by the Λ-basis $\tilde{x}_1, \tilde{x}_1 - \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$, and A^{-1} is obtained as the inverse matrix of A.

The numbers $\max\lambda(A) = 3$ and $\min\lambda(A) = -2$ are determined by actual calculations, where $\max\lambda(A) = 3$ is attained only by the Laurent polynomials

$$c_{13}(t) = c_{14}(t) = (1 + t + t^2)(1 + 2f),$$

and $\min \lambda(A) = -2$ is attained only by the following Laurent polynomials:

$$\begin{aligned} b_{34}(t) &= f + f^2, \quad a'_{11}(t^{-1}) = 1 + t^{-1} + t^{-2}, \\ c_{33}(t) &= -(1 + 2f + 2f^2) + (1 + t + t^2), \quad c_{34}(t) = -f - f^2, \\ c_{43}(t) &= -2f - 2f^2, \quad c_{44}(t) = -1 - 2f - 2f^2. \end{aligned}$$

Thus,

$$\lambda(A) = \max \lambda(A) - \min \lambda(A) = 3 - (-2) = 5.$$

For $m \geqq 5 \geqq \lambda$, it is seen that $\tilde{x}_i^{(m)}$ is minimal for $i = 1, 2$ by a similar consideration of the proof of Theorem 1.5 (1) using Corollary 3.3 instead of Corollary 1.3 since

$$||\tilde{x}_1^{(m)} - \tilde{x}_2^{(m)}||^2 = ||\tilde{x}_1 - \tilde{x}_2||^2 = 2.$$

Also, by a similar consideration of the proof of Theorem 1.5 (1), $\tilde{x}_i^{(m)}$ is minimal for $i = 3, 4$, showing (1) for $m \geqq 5$. To see (2) for $m \geqq 5$, let G be a proper Z-free subgroup of $H_2(M_L^{(m)}; Z)$. Since the Z-basis $t^j \tilde{x}_i^{(m)}$ $(i = 1, 2, 3, 4; j = 0, 1, 2, \ldots, m-1)$ of $H_2(M_L^{(m)}; Z)$ are minimal, a similar consideration of the proof of Theorem 1.5 (2) shows that the Z-intersection form on G has the determinant greater than 1, showing (2) for $m \geqq 5$. The assertion (2) for $m = 3, 4$ was given in [6]. By assuming it, the assertion (1) for $m = 3, 4$ is shown as follows:

The elements $\tilde{x}_i^{(m)}$ for $i = 3, 4$ must be minimal because $||\tilde{x}_i^{(m)}||^2 = 2$ and every element of the square length 1 generates an orthogonal summand Z. Suppose $\tilde{x}_1^{(m)}$ is written as a sum $x'_1 + x''_1$ with $||x'_1||^2 < 3$ and $||x''_1||^2 < 3$. Then $||x'_1||^2 = ||x''_1||^2 = 2$ which contradicts $||\tilde{x}_1^{(m)}|| = 3$. Hence $\tilde{x}_1^{(m)}$ is minimal. Similarly, $\tilde{x}_2^{(m)}$ is shown to be minimal. □

The proof of Observation 1.7 will be done as follows:

Proof of Observation 1.7. Suppose the 3-manifold nB bounds a smooth compact oriented 4-manifold W with $H_2(W; Q) = 0$. Since $H_2(W, nB; Q) = H^2(W; Q) = 0$ by Poincaré duality, the natural map $H_1(nB; Z) \to H_1(W; Z)/(\text{torsion})$ is injective. Then there is an epimorphism $\gamma : H_1(W; Z) \to Z$ such that the infinite cyclic covering $\widetilde{W} \to W$ belonging to γ lifts at least one component of nB non-trivially. Let $\Lambda_Q = Q[t, t^{-1}]$. Since $H_2(W; Q) = 0$, the Λ_Q-module $H_2(\widetilde{W}; Q)$ is a finitely generated torsion Λ_Q-module for whose rational Alexander polynomial $A(t) \in \Lambda_Q$ has $A(1) \neq 0$. Since there are infinitely many positive integer m such that $t^m - 1$ is coprime with $A(t)$ in Λ_Q, the m-fold cyclic covering space $W^{(m)}$ of W belonging to the epimorphism $H_1(W; Z) \stackrel{\gamma}{\to} Z \to Z/mZ$ has

$$H_2(W^{(m)}; Q) = H_2(\widetilde{W}; Q)/(t^m - 1)H_2(\widetilde{W}; Q) = 0$$

for any such m. The boundary $\partial W^{(m)}$ of the 4-manifold $W^{(m)}$ is a disjoint union of m_i-fold cyclic covering spaces $B^{(m_i)}\,(i = 1, 2, \ldots, s)$ of B which are still homology handles. It is noted that some m_i may be taken sufficiently large when m is taken sufficiently large. Then a smooth compact oriented 4-manifold Y with $H_1(Y;Q) = H_2(Y;Q) = 0$ is constructed from $W^{(m)}$ by 2-handle surgeries on $W^{(m)}$ killing $H_1(W^{(m)};Q)$. Let Y^* be the smooth closed oriented 4-manifold obtained from $-Y$ by attaching $E^{(m_i)}\,(i = 1, 2, \ldots, s)$ along $B^{(m_i)}\,(i = 1, 2, \ldots, s)$, where $E^{(m_i)}$ denotes the m_i-fold cyclic covering space of a smooth compact connected oriented 4-manifold E with boundary B explained in the introduction. By Corollary 1.6, for a large m the smooth 4-manifold Y^* with $H_1(Y^*;Q) = 0$ must have a positive definite non-standard Z-intersection form on $H_2(Y;Z)/(\text{torsion})$, which contradicts [3]. Thus, the 3-manifold nB cannot bound any smooth compact oriented 4-manifold W with $H_2(W;Q) = 0$. □

The proof of Observation 1.8 will be done as follows:

Proof of Observation 1.8. Suppose that the knot K in S^3 is made a trivial knot by changing r positive crossings into negative crossings. Then the $(3,1)$-manifold pair $(-S^3, K)$, where $-S^3$ denotes S^3 with orientation reversed, bounds a $(4,2)$-manifold pair (Y, D), where $Y = D^4 \# Q(r)$ for the 4-disk D^4 and D is a smooth disk with $[D] = 0$ in $H_2(Y, -S^3; Z)$. In this construction, the inclusion $-S^3 \setminus k \subset Y \setminus D$ is assumed to induce an isomorphism $H_1(-S^3 \setminus k; Z) \to H_1(Y \setminus D; Z)$ and an epimorphism $\pi_1(-S^3 \setminus k) \to \pi_1(Y \setminus D)$. Let $W = \mathrm{cl}(Y \setminus N)$ for a tubular neighborhood N of D in Y. Then W is a smooth positive definite compact 4-manifold with boundary $-B$ such that the inclusion $-B \subset W$ induces an isomorphism $H_1(-B; Z) \to H_1(W; Z)$ and an epimorphism $\pi_1(-B) \to \pi_1(W)$. Let X be a smooth positive definite Z^{π_1}-manifold obtained E by attaching W along B. By Corollary 1.6, the Z-intersection form on $H_2(X^{(m)}; Z)$ for the m-fold cyclic covering space $X^{(m)}$ of X for $m \geqq 3$ which is a smooth Z^{π_1}-manifold must have a positive definite non-standard form as an orthogonal summand, which contradicts [3]. Thus, the knot k cannot be made a trivial knot by changing positive crossings into negative crossings. □

Acknowledgement. This work was supported by JSPS KAKENHI Grant Number 26287013. The author thanks to a referee for a very careful reading.

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