Triviality of a surface-link with meridian-based free fundamental group

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ABSTRACT

It is proved that every disconnected surface-link with meridian-based free fundamental group is a trivial (i.e., an unknotted-unlinked) surface-link. This result is a surface-link version of the author's recent announcement result on smooth unknotting of a surface-knot.

Keywords: Boundary surface-link, Stably trivial Surface-link, Trivial surface-link, Unknotted-unlinked surface-link.
Mathematics Subject Classification 2010: Primary 57Q45; Secondary 57M15, 57N13

1 Introduction

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When F is connected, it is also called a *surface-knot*. When the surface F is fixed, it is also called an *F-link*. Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism $f : S^4 \to S^4$. A *trivial* surface-link is a surface-link F which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in S^4, where a handlebody is a 3-manifold which is a 3-ball or a disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted-unlinked* surface-link. For every given closed oriented (possibly disconnected) surface F, a trivial F-link exists uniquely up to equivalences [3]).

In the previous paper [8] following the arguments of [5, 6, 7], the author claimed that a surface-knot F in $\mathbf{R}^4$ is a trivial surface-knot if and only if it has the infinite cyclic fundamental group.

In this paper, this result is generalized to a surface-link with meridian-based free fundamental group. For our argument here, a surface-link in $\mathbf{R}^4$ is considered as a surface-link in the one-point compactification $\mathbf{R}^4 \cup \{\infty\} = S^4$. Then the *exterior* of a surface-link F is the compact 4-manifold $E = \mathrm{cl}(S^4 \setminus N(F))$ for a tubular neighborhood $N(F)$ of F in S^4. Let q_0 be a fixed base point in the interior of E. A surface-link with *meridian-based free fundamental group* is a surface-link F in S^4 such that the fundamental group $\pi_1(E, q_0)$ is a free group with a meridian basis. A trivial surface-link is a surface-link with meridian-based free fundamental group. The following unknotting-unlinking result is our main theorem.

Theorem 1.1. A surface-link is a trivial surface-link if and only if it is a surface-link with meridian-based free fundamental group.

This result answers positively the problem [9, Problem 1.55 (B)] for a cS^2-link (i.e., a 2-link with c components) for any c. In this paper, surface-links are discussed in DIFF-category and the same argument goes well in PL-category. Even in TOP-category, the author believes that the same argument goes well by using a TOP-transversality in [1], so that in the case of a surface-knot, the method of proof is different from the proof given by [2, 4].

A *stabilization* of a surface-link F is a connected sum $\bar{F} = F\#_{k=1}^{s} T_k$ of F and a trivial torus-link $T = \cup_{k=1}^{s} T_k$. By granting $T = \emptyset$, we understand that a surface-link F itself is a stabilization of F. The trivial torus-link T is called the *stabilizer* with $T_k\,(k = 1, 2, ..., s)$ the *stabilizer components* on the stabilization $\bar{F}$.

A *stably trivial* surface-link is a surface-link F such that a stabilization $\bar{F}$ of F is a trivial surface-link. In [8, Corollary 1.2], it is shown that every stably trivial surface-link is a trivial surface-link. Therefore, the proof of Theorem 1.1 is completed by combining [8, Corollary 1.2] with the following lemma, which generalizes the result of [3, Theorem 2.10] to a surface-link.

Lemma 1.2. Every surface-link with meridian-based free fundamental group is a stably trivial surface-link.

2 Proof of Lemma 1.2.

Throughout this section, the proof of Lemma 1.2 will be given.

Let F be a surface-link in S^4 with components $F_i\,(i=1,2,\ldots,r)$. Let $N(F) = \cup_{i=1}^r N(F_i)$ be a tubular neighborhood of $F = \cup_{i=1}^r F_i$ in S^4 which is a trivial disk bundle over F, and $E = \mathrm{cl}(S^4 \setminus N(F))$ the exterior of a surface-link F. Then the boundary $\partial E = \partial N(F) = \cup_{i=1}^r \partial N(F_i)$ is a trivial circle bundle over $F = \cup_{i=1}^r F_i$. Identify $\partial N(F_i) = F_i \times S^1$ such that the composite inclusion

$$F_i \times 1 \to \partial N(F_i) \to \mathrm{cl}(S^4 \setminus N(F_i)$$

induces the zero-map in the integral first homology, where S^1 denotes the set of complex numbers of norm one. Let $q_i \times 1$ be a point in $F_i \times S^1$ for every $i\,(i = 1,2,\ldots,r)$.

Let

$$K = (\cup_{i=1}^r a_i) \bigcup (\cup_{i=1}^r S_i)$$

be a connected graph in E such that

(1) a_i is an edge embedded in E joining q_0 and $q_i \times 1$ such that the interiors of $a_i\,(i=1,2,\ldots,r)$ are mutually disjoint,

(2) $S_i = q_i \times S^1\,(i=1,2,\ldots,r)$, and

(3) the inclusion $K \to E$ induces an isomorphism $\pi_1(K, q_0) \to \pi_1(E, q_0)$ such that the element $t_i = [a_i \cup S_i] \in \pi_1(E, q_0)$ is the ith meridian generator.

By the assumption that $\pi_1(E, q_0)$ is a free group with a meridian basis, there is a graph K with properties (1), (2) and (3). Further, we have the following lemma.

Lemma 2.1. The composite inclusion $F_i \times 1 \to \partial N(F_i) \to E$ is null-homotopic for all i.

Proof of Lemma 2.1. Since $\partial N(F_i) = F_i \times S^1$, the fundamental group elements between the factors $F_i \times 1$ and $q_i \times S^1$ are commutive. On the other hand, since $\pi_1(E, q_0)$ is a free group, the image of the homomorphism $\pi_1(a_i \cup F_i \times 1, q_0) \to \pi_1(E, q_0)$ is in the infinite cyclic group $< t_i >$ generated by t_i. The surface $F_i \times 1$ in $\partial N(F_i) = F_i \times S^1$ is chosen so that the inclusion $F_i \times 1 \to \mathrm{cl}(S^4 \setminus N(F_i))$ induces the zero-map in the integral first homology. This implies that the inclusion $F_i \times 1 \to E$ is null-homotopic. □

Let

$$K(\partial N) = (\cup_{i=1}^r a_i) \cup (\cup_{i=1}^r \partial N(F_i))$$

be a polyhedron in E. Let $p : K(\partial N) \to K$ be the map defined by the projection $F_i \times S^1 \to q_i \times S^1$ sending F_i to q_i for all i.

Lemma 2.2. The map $p : K(\partial N) \to K$ extends to a map $g : E \to K$.

Proof of Lemma 2.2. Since K is a $K(\pi, 1)$-space, there is a map $f : E \to K$ inducing the inverse isomorphism $\pi_1(E, q_0) \to \pi_1(K, p_0)$ of the isomorphism $\pi_1(K, p_0) \to \pi_1(E, q_0)$. Then the composite map $fj : K(\partial N) \to K$ of f with the inclusion $j : K(\partial N) \to E$ and the map $p : K(\partial N) \to K$ induces the same homomorphism

$$(fj)_\# = p_\# : \pi_1(K(\partial N), q_0) \to \pi_1(K, q_0).$$

Since K is a $K(\pi, 1)$-space, the map fj is homotopic to p. By the homotpy extension property in [11], there is a map $g : E \to K$ homotopic to the map f such that $gj = p$. □

Replacing g by a piecewise smooth approximation keeping the map p fixed, we can use a transverse regularity argument to obtain a regular point $q_i \times t_i \in q_i \times S^1$ for each $i\,(i = 1, 2, \ldots, r)$ such that the preimage $V_i = g^{-1}(q_i \times t_i)$ is a compact oriented smooth 3-manifold with boundary $\partial V_i = p^{-1}(q_i \times t_i) = F_i \times t_i$. By discarding a closed component from V_i, we assume that V_i is connected. Let $\alpha_{ik}\,(k = 1, 2, \ldots, n_i)$ be mutually disjoint proper arcs in V_i such that the exterior $V_i^* = \mathrm{cl}(V_i \setminus N_i))$ for a regular neighborhood $N_i = N(\cup_{k=1}^{n_i} \alpha_{ik})$ of the arcs $\alpha_{ik}\,(k = 1, 2, \ldots, n_i)$ in V_i is a handlebody. Let $V(C)_i$ be a compact connected oriented smooth 3-manifold with boundary $\partial V(C)_i = F_i$ obtained from V_i by adding a collar C_i joining $F_i \times t_i$ with F_i in the trivial disk bundle $N(F_i)$ over F_i. Let $\alpha(C)_{ik}\,(k = 1, 2, \ldots, n_i)$ be mutually disjoint proper arcs in $V(C)_i$ obtained from the arcs $\alpha_{ik}\,(k = 1, 2, \ldots, n_i)$ by extending them until they reach F_i through the collar C_i. Then the exterior $V(C)_i^* = \mathrm{cl}(V(C)_i \setminus N(C)_i))$ for a regular neighborhood $N(C)_i = N(\cup_{k=1}^{n_i} \alpha(C)_{ik})$ of the arcs $\alpha(C)_{ik}\,(k = 1, 2, \ldots, n_i)$ in $V(C)_i$ is a handlebody for all i. Let $\alpha(C)_{ik}^0$ be an arc obtained from a simple arc β_{ik} in F_i with $\partial \beta_{ik} = \partial \alpha(C)_{ik}$ by pushing the interior of β_{ik} into the collar C_i. Note that every loop in V_i is null-homotopic in E since the map g induces an isomorphism $g_\# : \pi_1(E, q_0) \cong \pi_1(K, q_0)$. This means that the arc $\alpha(C)_{ik}$ is homotopic to an arc $\alpha(C)_{ik}^0$ by a homotopy in $(S^4 \setminus F) \cup F_i$ relative to F_i. By an argument of HoK, we see that the 1-handle h_{ik} on F thickening the arc $\alpha(C)_{ik}$ is a trivial 1-handle whose surgery of F is the connected sum of F and a trivial torus-knot on the component F_i. Since the surface-link obtained from F by the surgery along the 1-handles h_{ik} for all i and k bounds a system of handlebodies $V(C)^*_i$ $(i = 1, 2, \ldots, r)$, the surface-link F is a stably trivial surface-link. This completes the proof of Lemma 1.2. □

A surface link F with $r(\geq 2)$ components is a *boundary surface-link* if it bounds mutually disjoint r bounded connected oriented smooth 3-manifolds in S^4. The following corollary is contained in the proof of Lemma 1.2 whose technique is known in classical link theory (see [10]).

Corollary 2.3. A disconnected surface link F is a boundary surface-link if there is an epimorphism from the fundamental group $\pi_1(E, q_0)$ to a meridian-based free group.

Acknowledgements. This work was in part supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics).

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