Topology of a 4D universe for every 3-manifold

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Abstract

A 4D universe is a 4-dimensional boundary-less connected oriented manifold with every closed 3-manifold (i.e., a 3-dimensional closed connected oriented manifold) embedded. A 4D punctured universe is a 4-dimensional boundary-less connected oriented manifold with the punctured manifold of every closed 3-manifold embedded. Every 4D universe and every 4D punctured universe are open 4-dimensional manifolds. If a closed 3-manifold is considered as a 3D universe, then every 4D spacetime is embedded in every 4D universe and hence every 4D universe is a classifying space for every spacetime. In this paper, it is observed that a full 4D universe is produced by collision modifications between 3-sphere fibers in the 4D spherical shell (i.e., the 3-sphere bundle over the real line) embedded properly in any 5-dimensional open manifold. As a previous result, it was shown that any 4D universe and 4D punctured universe must have infinity on some homological indexes. It is shown in this paper that the second rational homology groups of every 4D universe and every 4D punctured universe are always infinitely generated.

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1. Introduction

Throughout this paper, by a *closed 3-manifold* we mean a closed connected oriented 3-manifold M, and by a *punctured 3-manifold* the punctured

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manifold M^0 of a closed 3-manifold M obtained from M by removing the interior of a 3-ball. Let $\mathbb{M}$ be the set of closed 3-manifolds M, and $\mathbb{M}^0$ the set of punctured 3-manifolds M^0. By a *4D universe* or simply a *universe*, we mean a boundary-less connected oriented 4-manifold with every closed 3-manifold M embedded, and by a *4D punctured universe* or simply a *punctured universe* a boundary-less connected oriented 4-manifold with every punctured 3-manifold M^0 embedded. Every universe and every punctured universe are open 4-manifolds (see [6, 10]). If a closed 3-manifold M is 3D universe[1] and a smooth map $t : M \to R$ for the the real line R is a time function, then there is a smooth embedding $M \to M \times R$ sending every point $x \in M$ to the point $(x, t(x)) \in M \times R$. The product $M \times R$, regarded as the M-bundle over R, is called the *spacetime* of M. Since every closed 3-manifold M embedded in U admits a trivial normal line bundle $M \times R$ in U, every universe is considered as a "classifying space" for the spacetime of every 3D universe M.

A standard physical spacetime model called the *hyersphere world-universe model* (see for example [13]) is topologically the product $S^3 \times R$, called the *4D spherical shell* or simply the *spherical shell*. In Section 2, the spherical shell $S^3 \times R$ is assumed to be properly and smoothly embedded in an open 5-manifold W. Then we define a collision modification on two distinct 3-sphere fibers $S^3_t, S^3_{t'}\,(t, t' \in R, t \neq t')$ of the spherical shell $S^3 \times R$ and show in Theorem 2.1 that *a universe U is constructed in W from the spherical shell $S^3 \times R$ by infinitely many collision modifications on 3-sphere fibers of $M \times R$.* It may be something useful to mention that there are 5-dimensional physical universe models such as Kaluza-Klein model (see [2, 12]) and Randall-Sundrum model [14, 15] and an argument on the physical collision of a brane in the bulk space such as [11].

For a boundary-less connected oriented 4-manifold X, we note that there are two types of embeddings $k : M \to X$. An embedding $k : M \to X$ is *of type 1* if the complement $X \backslash k(M)$ is connected, and *of type 2* if the complement $X \backslash k(M)$ is disconnected. For example, the smooth embedding $M \to M \times R$ given by a time function $t : M \to R$ is of type 2 (see [6]). If there is a type 1 embedding $k : M \to X$, then there is an element $x \in H_1(X; Z)$ with the intersection number $\mathrm{Int}_U(x, k(M)) = +1$, so that the intersection form $\mathrm{Int}_X : H_1(X; Z) \times H_3(X; Z) \to Z$ induces an epimorphism

$$I_d : H_d(X; Z) \to Z$$

[1] A model of our living 3-space.

for $d = 1, 3$ such that the composite $I_3 k_* : H_3(M; Z) \to H_3(X; Z) \to Z$ is an isomorphism and the composite $I_1 k_* : H_1(M; Z) \to H_1(X; Z) \to Z$ is the 0-map (see [6, 10]). By using of embeddings of types 1 and 2, the special universes are considered in [10]: Namely, a universe U is a *type 1 universe* if every $M \in \mathbb{M}$ is type 1 embeddable in U, and a *type 2 universe* if every $M \in \mathbb{M}$ is type 2 embedded in U. A universe U is a *full universe* if U is a type 1 universe and a type 2 universe. In Theorem 2.1, a full universe U will be constructed in every open 5-manifold W from the spherical shell $S^3 \times R$ by infinitely many collision modifications on 3-sphere fibers of $S^3 \times R$.

Actually, there exist quite many 4D universes and 4D punctured universes. The following comparison theorem between them is established in [10, Theorem 2.1]:

Comparison Theorem.

$$\begin{array}{ccccc} & & \text{Type 1 universe} & & \\ & \nearrow & & \searrow & \\ \star\ \text{Full universe} & & & & \text{Universe} \to \text{Punctured universe.} \\ & \searrow & & \nearrow & \\ & & \text{Type 2 universe} & & \end{array}$$

- $\star$ Type 1 universe $\not\to$ Full universe.
- $\star$ Type 2 universe $\not\to$ Full universe.
- $\star$ Universe $\not\to$ Type 1 universe.
- $\star$ Universe $\not\to$ Type 2 universe.
- $\star$ Punctured universe $\not\to$ Universe.

For a universe or punctured universe U, the following topological invariants called the *topological indexes*

$$\hat{\beta}_d(U)(d = 1, 2),\ \delta(U),\ \delta_i(U)\,(i = 0, 1, 2),\ \rho(U),\ \rho_i(U)\,(i = 0, 1, 2)$$

taking values in the set $\{0, 1, 2, \ldots, +\infty\}$ are defined in [10] to investigate the topological shape of U. The definitions are explained as follows:

For a non-compact oriented 4-manifold X and the intersection form

$$\text{Int} : H_d(X; Z) \times H_{4-d}(X; Z) \to Z,$$

we define the *dth null homology* of X to be the subgroup

$$O_d(X;Z) = \{x \in H_d(X;Z) | \operatorname{Int}(x, H_{4-d}(X;Z)) = 0\}$$

of the dth homology group $H_d(X;Z)$ and the *dth non-degenerate homology* of X to be the quotient group

$$\hat{H}_d(X;Z) = H_d(X;Z)/O_d(X;Z),$$

which is a free abelian group by [10, Lemma 3.1]. Let $\hat{\beta}_d(X)$ be the Z-rank of $\hat{H}_d(X;Z)$.

For $M^0 \in \mathbb{M}^0$, let $\delta(M^0 \subset X)$ be the minimal Z-rank of the image of the homomorphism

$$k^0_* : H_2(M^0;Z) \longrightarrow H_2(X;Z)$$

for all embeddings $k^0 : M^0 \to X$.

For an abelian group G, let $G^{(2)} = \{x \in G | 2x = 0\}$, which is a direct sum of some copies of Z_2. Let $\rho(M^0 \subset X)$ be the minimal Z_2-rank of the homomorphism image group

$$\operatorname{Im}[k^0_* : H_2(M^0;Z) \longrightarrow H_2(X;Z)]^{(2)}$$

for all embeddings $k^0 : M^0 \to X$ with Z-rank $\delta(M^0 \subset X)$. By taking the value 0 for the non-embeddable case, we define the following topological invariants of X:

$$\begin{aligned}
\delta_0(X) &= \sup\{\delta(M^0 \subset X) | M^0 \in \mathbb{M}^0\}, \\
\rho_0(X) &= \sup\{\rho(M^0 \subset X) | M^0 \in \mathbb{M}^0\}.
\end{aligned}$$

For $M \in \mathbb{M}$, let $\delta(M \subset X)$ be the minimal Z-rank of the image of the homomorphism

$$k_* : H_2(M;Z) \longrightarrow H_2(X;Z)$$

for all embeddings $k : M \to X$. Let $\rho(M \subset X)$ be the minimal Z_2-rank of the homomorphism image group

$$\operatorname{Im}[k_* : H_2(M;Z) \longrightarrow H_2(X;Z)]^{(2)}$$

for all embeddings $k : M \to X$ with Z-rank $\delta(M \subset X)$.

Note that in [10], the Z-rank condition in the definitions of $\rho(M^0 \subset X)$ and $\rho(M \subset X)$ was erroneously omitted.

By taking the value 0 for the non-embeddable case, we define the following invariants of X:

$$\begin{aligned}\delta(X) &= \sup\{\delta(M \subset X) \mid M \in \mathbb{M}\},\\ \rho(X) &= \sup\{\rho(M \subset X) \mid M \in \mathbb{M}\}.\end{aligned}$$

Restricting all embeddings $k : M \to X$ to all embeddings $k : M \to X$ of type i for $i = 1, 2$, we obtain the topological indexes $\delta_i(X)$ and $\rho_i(X)$ $(i = 1, 2)$ of X in place of $\delta(X)$ and $\rho(X)$.

The results given in [10] are explained as follows:

• For a punctures universe U, one of the topological indexes $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$. Further, in every case, there is a punctured spin universe U with the other topological indexes taken 0.

• For a type 1 universe U, one of the topological indexes $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$. The condition $\hat{\beta}_1(U) \geq 1$ always holds, but in the case of $\rho_1(U) = +\infty$, the condition $\hat{\beta}_1(U) = +\infty$ holds. Further, in every case, there is a type 1 spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ taken 0.

• For a type 2 universe U, one of the topological indexes $\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. Further, in every case, there is a type 2 spin universe U with the other topological index taken 0.

• For any universe U, one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$. In the case of $\rho(U) = +\infty$, the condition $\hat{\beta}_1(U) = +\infty$ is added. Further, in every case, there is a spin universe U with the other topological indexes on $\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.

• For a full universe U, one of the topological indexes $\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$. The condition $\hat{\beta}_1(U) \geq 1$ always holds. Further, in every case, there is a full spin universe U with the other topological index on $\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In Section 3, we explain the signature theorem of an infinite cyclic covering space over a non-compact 4-manifold with closed 3-manifolds as the boundary, which is given in [10]. We also introduce a new class of closed 3-manifolds called *connected sums of homology 3-tori* and study some homological properties of them. By using the results in Section 3, we show in Section 4 the following main result:

Theorem 4.1. Let X be a non-compact oriented 4-manifold with the second Betti number $\beta_2(X) < +\infty$. Then there is a punctured 3-manifold $M^0 \in \mathbb{M}^0$ which is not embeddable in X.

Thus, we can add the following crucial property to the properties stated above on the universes and punctured universes:

Corollary 4.2. The second rational homology group $H_2(U; Q)$ of every universe or punctured universe U is always infinitely generated over Q. Namely, $\beta_2(U) = +\infty$.

This result could not be shown by the techniques in [10] because there are a punctures universe U with the topological indexes $\rho_0(U) = +\infty$ and $\hat{\beta}_2(U) = \delta_0(U) = 0$ and a type 1 universe U with the topological indexes $\rho_1(U) = +\infty$ and $\hat{\beta}_2(U) = \delta_1(U) = 0$, which use Samsara 4-manifolds on 3-manifolds constructed in [9]. By noting that the Samsara 4-manifolds on most 3-manifolds have 3-torus boundary components, there was a pretty discovery that every oriented 4-manifold X whose boundary ∂X has a 3-torus component must have $H_2(X; Q) \neq 0$ (see Corollary 3.5). For this proof of the main result, we use a generalization of this fact besides the arguments of [10].

As the final note in the introduction, it would be interesting to observe that the infinity in every case of a 4D universe comes from the existence of the connected sums of copies of the trefoil knot, which occurs frequently next to the trivial knot (see [1, 16, 17]). In fact, the closed 3-manifolds contributing to the infinities in [10] are called c-efficient 3-manifolds which are the connected sums of the homology handles obtained from the 3-sphere S^3 by the 0-surgery along the connected sums of certain copies of the trefoil knot. The closed 3-manifolds contributing to the present infinity $\beta_2(U) = +\infty$ are the connected sums of homology 3-tori constructed from the 3-torus T^3 by replacing the standard solid torus generators with the exteriors of the connected sums of certain copies of the trefoil knot (see Section 3, especially Example 3.2).

2. A collision modification of the spherical shell

Let W be an open connected oriented 5-manifold. Let X and X' be two disjoint compact oriented connected 4-manifolds smoothly embedded in W.

By isotopic deformations $\tilde{i} : X \to W$ and $\tilde{i}' : X' \to W$ of the inclusion maps $i : X \subset W$ and $i' : X' \subset W$, we consider that the images $\tilde{i}X$ and $\tilde{i}'X'$ meet tangently and opposite-orientedly in W with a compact 4-submanifold V, where V is assumed to be in the interiors of the 4-manifolds X and X'. We call such a V a *collision field* of the 4-manifolds X and X' in the 5-manifold W. A *collision modification of* X *and* X' *in* W *with a collision field* V is the 4-manifold

$$X'' = \mathrm{cl}(\tilde{i}X \setminus V) \bigcup \mathrm{cl}(\tilde{i}'X' \setminus V).$$

This collision modification gives a standard procedure to construct a new 4-manifold X'' from X and X' through a regular neighborhood of V in W. In the spherical shell $S^3 \times R$ embedded properly and smoothly in an open 5-manifold W, we understand that a *collision modification on distinct 3-sphere fibers* S^3_t and $S^3_{t'}$) of $S^3 \times R$ in W is a collision modification of the disjoint compact spherical shells $S^3 \times I$ and $S^3 \times I'$ in W with a collision field V for any disjoint closed interval neighborhoods I and I' of the points t and t' in R, respectively. In the following theorem, it is explained how a full universe is constructed from the spherical shell $M \times R$ by infinitely many collision modifications on distinct 3-sphere fibers of $S^3 \times R$.

Theorem 2.1. Assume that the spherical shell $M \times R$ is embedded properly in a 5-dimensional open manifold W. Then a full universe U is produced in W by infinitely many collision modifications on distinct 3-sphere fibers of the sphere shell $S^3 \times R$.

Proof. Let $S^3 \times I$ and $S^3 \times I'$ be any disjoint compact spherical shells in $S^3 \times R$. Let ℓ be an arc in W joining an interior point p of $S^3 \times I$ with an interior point p' of $S^3 \times I'$ such that $\ell \cap S^3 \times R = \partial\ell = \{p, p'\}$. Let $h(\ell) = \ell \times D^4$ be a 1-handle with core ℓ on $S^3 \times I$ and $S^3 \times I'$ in W joining a 4-ball neighborhood $p \times D^4$ of p in $S^3 \times I$ and a 4-ball neighborhood $p' \times D^4$ of p in $S^3 \times I'$. By taking a 4D solid torus $x \times S^1 \times D^3$ in the interior of the 4-disk $x \times D^4$ for an interior point x of ℓ, a collision modification X'' of $S^3 \times I$ and $S^3 \times I'$ in W with a collision field $V = x \times S^1 \times D^3$ in the 4-disk $x \times D^4$ is produced so that X'' is homeomorphic to the connected sum $S^3 \times I \# S^3 \times I' \# S^2 \times S^2$. This can be seen by a topological argument as follows: Namely, X'' is homeomorphic to the union

$$\mathrm{cl}(S^3 \times I \setminus p \times S^1 \times D^3) \bigcup \ell \times S^1 \times \partial D^3 \bigcup \mathrm{cl}(S^3 \times I' \setminus p' \times S^1 \times D^3)$$

and the complement $\mathrm{cl}(D^4 \backslash S^1 \times D^3)$ is homeomorphic to the product $S^2 \times D^2$ with an open 4-ball removed and the double of $S^2 \times D^2$ is $S^2 \times S^2$. By this collision modification, the spherical shell $S^3 \times R$ changes into an open 4-manifold $S^3 \times R \# S^2 \times S^2 \# S^1 \times S^3$. Continuing this collision modification, we have an open 4-manifold U which is the connected sum of $S^3 \times R$ and infinitely many copies of $S^2 \times S^2$ and $S^1 \times S^3$. This open 4-manifold U is a full universe. In fact, U is a type 2 universe because U contains the stable 4-space SR^4, that is, the connected sum of R^4 and infinitely many copies of $S^2 \times S^2$, which is known as a type 2 universe (see [7, 10]). Further, U is a type 1 universe because U contains the connected sum of SR^4 and $S^1 \times S^3$ which is known as type 1 universe (see [10, Figure 2]). □

3. Homological algebra on an infinite cyclic covering of a non-compact 4-manifold

Let X be a non-compact oriented 4-manifold. A homomorphism $\gamma : H_1(X; Z) \to Z$ is *end-trivial* if the restriction $\gamma|_{\mathrm{cl}(X \backslash X')} = 0$ for a compact 4-submanifold X' of X. Assume that a closed 3-manifold B is the boundary ∂X of X. Let $(\tilde{X}, \tilde{B})$ be the infinite cyclic covering of the pair (X, B) on an end-trivial homomorphism γ, where note that $\tilde{B}$ is the infinite cyclic covering space of B associated with the restriction $\dot{\gamma} = \gamma|_B : H_1(B; Z) \to Z$. Let $\Gamma = Q[t, t^{-1}]$ be the Laurent polynomial ring over Q. Consider the Γ-intersection form

$$\mathrm{Int}_\Gamma : H_2(\tilde{X}; Q) \times H_2(\tilde{X}; Q) \to \Gamma.$$

Let

$$O_2(\tilde{X}; Q)_\Gamma = \{x \in H_2(\tilde{X}; Q) \,|\, \mathrm{Int}_\Gamma(x, H_2(\tilde{X}; Q)) = 0\}.$$

Then the quotient Γ-module $\hat{H}_2(\tilde{X}; Q)_\Gamma = H_2(\tilde{X}; Q)/O_2(\tilde{X}; Q)_\Gamma$ is a free Γ -module of finite rank for an end-trivial homomorphism $\gamma : H_1(X; Z) \to Z$ (see [10]).

Let $A(t)$ be a Γ-Hermitian matrix representing the Γ-intersection form Int_Γ on $\hat{H}_2(\tilde{X}; Q)_\Gamma$. For $x \in (-1, 1)$ let $u(x) = x + \sqrt{1 - x^2}i$, which is a complex number of norm one. For $a \in (-1, 1)$ we define the signature invariant of $\tilde{Y}$ by

$$\tau_{a\pm 0}(\tilde{X}) = \lim_{x \to a\pm 0} \mathrm{sign} A(u(x)).$$

The signature invariants $\sigma_a(\tilde{B})$ ($a \in [-1,1]$) of $\tilde{B}$ are also defined in [3, 4] by using the quadratic form

$$b : \mathrm{Tor}_\Gamma H_1(\tilde{B};Q) \times \mathrm{Tor}_\Gamma H_1(\tilde{B};Q) \to Q$$

on the Γ-torsion part $\mathrm{Tor}_\Gamma H_1(\tilde{B};Q)$ of $H_1(\tilde{B};Q)$. For $a \in [-1,1]$, let

$$\sigma_{[a,1]}(\tilde{B}) = \sum_{a \le x \le 1} \sigma_x(\tilde{B}),$$
$$\sigma_{(a,1]}(\tilde{B}) = \sum_{a < x \le 1} \sigma_x(\tilde{B}).$$

Let $M = M(k)$ be the homology handle obtained from the 3-sphere S^3 by the 0-surgery along an oriented knot k, and $\tilde{M}$ the infinite cyclic connected covering of M associated with a generator $\dot{\gamma}_M \in H^1(M;Z)$. Let $\sigma_{[a,1]}(k) = \sigma_{[a,1]}(\tilde{M})$ and $\sigma_{(a,1]}(k) = \sigma_{(a,1]}(\tilde{M})$ for every $a \in (-1,1)$ (see [8]). The signature invariant $\sigma_{[a,1]}(k)$ of a knot k is *critical* if $\sigma_{[a,1]}(k) \neq 0$ and $\sigma_{[x,1]}(k) = 0$ for every $x \in (a,1)$.

The following identities are given as the non-compact version signature theorem in [10] (although the compact version signature theorem is given in [5, 6]).

$$\tau_{a-0}(\tilde{X}) - \mathrm{sign}X = \sigma_{[a,1]}(\tilde{B}),$$
$$\tau_{a+0}(\tilde{X}) - \mathrm{sign}X = \sigma_{(a,1]}(\tilde{B}).$$

For every $a \in (-1,1)$, the following inequality is obtained from these identities in [10].

$$(3.1) \qquad |\sigma_{(a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \le |\mathrm{sign}X| + \hat{\beta}_2(X) \le 2\hat{\beta}_2(X)\,,$$

where $\kappa_1(\tilde{B})$ denote the Q-dimension of the kernel of the homomorphism $t-1 : H_1(\tilde{B};Q) \to H_1(\tilde{B};Q)$.

This inequality is used to prove the infiniteness of the second rational homologies $H_2(U;Q)$ of a universe and a punctured universe U in Section 4. As suggested in the introduction, we cannot detect the infiniteness of $\beta_2(U)$ only by this argument. We use a property of a homology 3-torus generalizing a property of the 3-torus T^3 introduced from now. For the 3-torus $T^3 = S^1 \times S^1 \times S^1$, we take 3 disjoint circles $C_i\,(i = 1, 2, 3)$ embedded in T^3 representing a Z-basis for $H_1(T^3; Z)$ such that C_1, C_2, C_3 are isotopic to $S^1\times 1\times 1$, $1\times S^1\times 1$, $1 \times 1 \times S^1$ in T^3, respectively. Let $N(C_i)$ be a tubular neighborhood of C_i in T^3 with a fixed meridian-longitude system for $i = 1, 2, 3$. A *homological 3-torus* is a closed 3-manifold $M = M(k_1, k_2, k_3) \in \mathbb{M}$ obtained from T^3 and any 3 knots k_1, k_2, k_3 in S^3 by replacing $N(C_1), N(C_2), N(C_3)$ with the compact knot exteriors $E(k_1), E(k_2), E(k_3)$ so that the meridian-longitude system of $\partial N(C_i)$ is identified with the longitude-meridian system of k_i in $E(k_i)$ for $i = 1, 2, 3$. The cup product $a\cup b\cup c \in H^3(M; Z)$ of a Z-basis a, b, c of $H^1(M; Z)$ representing the dual elements of the meridians of $k_i\,(i = 1, 2, 3)$ is a generator of $H^3(M; Z) \cong Z$, which is a property inherited from a well-known property of the 3-torus T^3. It is convenient to note that the cup product $a' \cup b' \cup c' \in H^3(M; Q)$ of any Q-base change a', b', c' of a, b, c in $H^1(M; Q)$ is a generator of $H^3(M; Q) \cong Q$ and hence the elements $a'\cup b', b'\cup c', c'\cup a' \in H^2(M; Q)$ form a Q-basis of $H^2(M; Q)$ orthogonally dual to the Q-basis c', a', b' of $H^1(M; Q)$, respectively [To see this, note that $u\cup v = -v\cup u$ and , in particular, $u\cup u = 0$ for all $u, v \in H^1(M; Q)$]. For an integer $m > 0$, let $\mathbb{T}_m$ be the collection consisting of the connected sum $M = \#_{i=1}^m M_i \in \mathbb{M}$ of m homological 3-tori $M_i = M(k_{i,1}, k_{i,2}, k_{i,3})\,(i = 1, 2, 3, \dots, m)$.

For an application of the signature invariants $\sigma_{[a,1]}(\tilde{B})$, let

$$B = M \times 1 \cup M \times (-1)$$

for a closed 3-manifold $M \in \mathbb{T}_m$, where $M \times 1$ is regarded as the copy of M, but $M \times (-1)$ is the copy of M with the opposite orientation of M. A homomorphism $\dot\gamma : H_1(B; Z) \to Z$ is *asymmetric* if there is no system of elements $x_1, x_2, \dots, x_n \in H_1(M; Z)\,(n = 3m)$ representing a Q-basis for $H_1(M; Q)$ such that $\dot\gamma(x_i) = \pm\alpha_*(x_i)$ for all i, where α denotes the standard orientation-reversing involution on B switching $M \times 1$ to $M \times (-1)$.

The following calculation is used in our argument.

Lemma 3.1. For positive integers d and m, let $(k_{i,1}, k_{i,2}, k_{i,3})\,(i = 1, 2, \dots, m)$ be a sequence of triplets of knots used for the closed 3-manifold $M \in \mathbb{T}_m$ such that

(1) the signature invariants $\sigma_{[a,1]}(k_{i,1}), \sigma_{[a,1]}(k_{i,2}), \sigma_{[a,1]}(k_{i,3})$ are critical for all $i\,(i = 1, 2, \dots, m)$, and

(2) $|\sigma_{[a,1]}(k_{1,1})| > 2d+4m$, and for all $i, i', j, j'\,(i, i' = 2, 3, \ldots, 3m; j, j' = 1, 2, 3)$,

$$|\sigma_{[a,1]}(k_{i,j})| > \sum_{(i,j)>(i',j')} |\sigma_{[a,1]}(k_{i',j'})| + 2d + 4m,$$

where $(i,j) > (i',j')$ denotes the dictionary order.

Then for any asymmetric homomorphism $\gamma : H_1(B; Z) \to Z$, there is a number $b \in (-1, 1)$ such that

$$\kappa_1(\tilde{B}) \leq 4m \quad \text{and} \quad |\sigma_{[b,1]}(\tilde{B})| > 2d + 4m.$$

Example 3.2 Let k be a trefoil knot. Then the connected sum $k_{1,1}$ of $d + 2m + 1$ copies of k has the critical signature invariant $\sigma_{[\frac{1}{2},1]}(k_{1,1})| = \pm(2d + 4m + 2)$. Further continuing connected sums of copies of k, we obtain a sequence of triplets of knots $(k_{i,1}, k_{i,2}, k_{i,3})\,(i = 1, 2, \ldots, m)$ used for the closed 3-manifold $M \in \mathbb{T}_m$ satisfying the assumptions (1) and (2) of Lemma 3.1 with $a = \frac{1}{2}$.

Proof of Lemma 3.1. Note that the infinite cyclic covering space $\tilde{M}$ of a homological 3-torus M associated with a non-trivial element $\dot{\gamma} \in H^1(M; Z)$ has $\kappa_1(\tilde{M}) = 2$ by a property inherited from the 3-torus. Hence we have $\kappa_1(\tilde{B}) \leq 4m$. For a positive integer n, let $\sigma^{(n)}_{[a,1]}(k)$ denote the signature invariant $\sigma_{[a,1]}(\tilde{M}^{(n)})$ for the infinite cyclic covering space $\tilde{M}^{(n)}$ of the homology handle $M = M(k)$ associated with n multiple element $n\dot{\gamma}_M$ of a generator $\dot{\gamma}_M \in H^1(M; Z)$. Since $\gamma \in H^1(B; Z)$ is a asymmetric homomorphism, $\sigma[b,1](\tilde{B})$ is the sum of the signature invariants

$$\sigma^{(n_{i,1})}_{[b,1]}(k_{i,1}) - \sigma^{(n'_{i,1})}_{[b,1]}(k_{i,1}),\ \sigma^{(n_{i,2})}_{[b,1]}(k_{i,2}) - \sigma^{(n'_{i,2})}_{[b,1]}(k_{i,2}),\ \sigma^{(n_{i,3})}_{[b,1]}(k_{i,3}) - \sigma^{(n'_{i,3})}_{[b,1]}(k_{i,3})$$

for positive integers $n_{i,1}, n'_{i,1}, n_{i,2}, n'_{i,2}, n_{i,3}, n'_{i,3}$ such that some of $n_{i,1} - n'_{i,1}, n_{i,2} - n'_{i,2}, n_{i,3} - n'_{i,3}$ are not zero.

Let $a = \cos(\theta_a)$ for $\theta_a \in (0, \pi)$. By [6, Lemma 1.3 (1)], we have

$$\sigma^{(n)}_{[\cos(\theta_a),1]}(k) = \sigma_{[\cos(n\theta_a),1]}(k).$$

Assume that the signature invariant $\sigma_{[a,1]}(k)$ is critical. Then for $a_n = \cos(\frac{\theta_a}{n})$, we have $\sigma^{(n)}_{[a_n,1]}(k) = \sigma_{[a,1]}(k)$ and for any positive integer $n' < n$,

$\sigma^{(n')}_{[a_n,1]}(k) = 0$. By these properties, we can find a number $b \in (-1,1)$ such that
$$\kappa_1(\tilde{B}) \leq 4m \quad \text{and} \quad |\sigma_{[b,1]}(\tilde{B})| > 2d + 4m.$$
$\square$

The following Q-dimensional estimate on a Q-subspace of the first cohomology $H^1(M;Q)$ of a closed 3-manifold $M \in \mathbb{T}_m$ is technically useful:

Lemma 3.4. Let Δ be a Q-subspace of $H^1(M;Q)$ of codimension $c(= 3m - \dim_Q \Delta)$, and $\Delta^{(2)}$ the Q-subspace of $H^2(M;Q)$ consisting of the cup product $u \cup v \in H^2(M;Q)$ for all $u, v \in \Delta$. Then $\dim_Q \Delta^{(2)} \geq 2m - c$.

We call the Q-space $\Delta^{(2)}$ the *cup product space* of the Q-space Δ.

Proof. Note that the cohomology ring $H^1(M;Q)$ splits into the cohomology rings $H^1(M_i;Q)$ of the homology torus connected summands $M_i\,(i = 1,2,\ldots,m)$. Let $\{a_i, b_i, c_i\}$ be a Q-basis for $H^1(M_i;Q)$, whose cup product $a_i \cup b_i \cup c_i$ is a Q-generator of $H^3(M_i;Q) \cong Q$. By a base change argument, if necessary, changing the roles of a, b, c and changing the index $i\,(i = 1,2,3,\ldots,m)$, the Q-subspace Δ of $H^1(M;Q)$ has a Q-basis
$$\bar{a}_i, \bar{b}_i, \bar{c}_i\,(0 \leq i \leq s),\ \bar{a}_j, \bar{b}_j\,(s < j \leq s'),\ a_k\,(s' < k \leq s''),$$
where
$$\begin{aligned}
\bar{a}_i &= a_i + \sum_{s<j\leq s'} x^a_j + \sum_{s'<k\leq s''} y^a_k,\\
\bar{b}_i &= b_i + \sum_{s<j\leq s'} x^b_j + \sum_{s'<k\leq s''} y^b_k,\\
\bar{c}_i &= c_i + \sum_{s<j\leq s'} x^c_j + \sum_{s'<k\leq s''} y^c_k,\\
\bar{a}_j &= a_j + \sum_{s'<k\leq s''} z^a_k,\\
\bar{b}_j &= b_j + \sum_{s<j\leq s'} z^b_k
\end{aligned}$$
for some elements $x^a_j, x^b_j, x^c_j \in H^1(M_j;Q)\,(s < j \leq s')$ and $y^a_k, y^b_k, y^c_k, z^a_k, z^b_k \in H^1(M_k;Q)\,(s' < k \leq s'')$. Then the identity $s + s' + s'' = 3m - c$ is obtained by

counting the Q-dimension of Δ. Since $s'' \leq m$, we have $s+s' = 3m-c-s'' \geq 2m-c$. The cup product space $\Delta^{(2)}$ has Q-linearly independent elements

$$\bar{a}_i \cup \bar{b}_i,\ \bar{b}_i \cup \bar{c}_i,\ \bar{c}_i \cup \bar{a}_i\,(0 \leq i \leq s),\ \bar{a}_j \cup \bar{b}_j\,(s < j \leq s').$$

Hence we have

$$\dim_Q \Delta^{(2)} \geq 3s + (s'-s) = 2s + s' \geq s + s' \geq 2m - c,$$

completing the proof. □

The following corollary is used to confirm the non-vanishing of the second rational homology of a bounded Samsara 4-manifold.

Corollary 3.5. For $m > 0$, assume that a closed 3-manifold $M \in \mathbb{T}_m$ is a boundary component of a (possibly non-compact) oriented 4-manifold X. Let d be the Q-dimension of the kernel of the natural homomorphism $i_* : H_1(M;Q) \to H_1(X;Q)$. Then we have $\beta_2(X) \geq \max\{2m-d, d\} \geq m$.

Proof. Let D be the kernel of the natural homomorphism $i_* : H_1(M;Q) \to H_1(X;Q)$, and $x_i\,(i = 1,2,3,\ldots,d)$ a Q-basis for D with $y_i\,(i = 1,2,3,\ldots,d)$ the Poincaré-dual elements in $H_2(M;Q)$ such that $\mathrm{Int}_M(x_i, y_j) = \delta_{i,j}$ for all i,j. Since $H_1(M;Q)$ is a direct summand of $H_1(\partial X;Q)$, regard D as a Q-subspace of $H_1(\partial X;Q)$. Then there are d homology classes $\bar{x}_i\,(i = 1,2,3,\ldots,d)$ in $H_2(X,\partial X;Q)$ sent to $x_i\,(i = 1,2,3,\ldots,d)$ under the boundary homomorphism $\partial_* : H_2(X,\partial X;Q) \to H_1(\partial X;Q)$, so that we have

$$\mathrm{Int}_X(\bar{x}_i, y_j) = \delta_{i,j}$$

for all i,j by regarding the elements $y_i\,(i = 1,2,3,\ldots,d)$ as elements of $H_2(X;Q)$. Thus, $\beta_2(X) \geq d$.

Next, since the quotient Q-space $H_1(M;Q)/D$ injects to $H_1(X;Q)$ by the induced map of i_*, there is an exact sequence

$$H^1(X;Q) \xrightarrow{i^*} H^1(M;Q) \longrightarrow D' \to 0,$$

where D' is the Q-dual space of D. This implies that the image $\mathrm{Im}(i^*)$ of the natural map $i^* : H^1(X;Q) \to H^1(M;Q)$ is a Q-subspace of codimension d. Since $i^*(u_X) \cup i^*(v_X) = i^*(u_X \cup v_X) \in H^2(X;Q)$ for $u_X, v_X \in H^1(X;Q)$, we see from Lemma 3.4 that $\beta_2(X) \geq \dim_Q \mathrm{Im}(i^*)^{(2)} \geq 2m - d$. Thus,

$$\beta_2(X) \geq \max\{d, 2m-d\} \geq m. \quad \square$$

4. Infiniteness of the second rational homology

In this section, we show the following theorem.

Theorem 4.1. Let X be a non-compact oriented 4-manifold with the second Betti number $\beta_2(X) < +\infty$. Then there is a punctured 3-manifold $M^0 \in \mathbb{M}^0$ which is not embeddable in X.

The following corollary is directly obtained from Theorem 4.1.

Corollary 4.2. For every 4D punctured universe or 4D universe U, we have $\beta_2(U) = +\infty$.

Proof of Theorem 4.1. Let $\beta_2(X) = d < +\infty$. We show that there is $M \in \mathbb{M}$ such that M^0 is not embeddable in X. Suppose M^0 is embedded in X for an $M \in \mathbb{M}$ with $\beta_1(M) = n$. The 2-sphere $K = \partial M^0$ is a null-homologous 2-knot in X. Let X_M be the 4-manifold obtained from X by replacing a tubular neighborhood $N(K) = S^2 \times D^2$ of K in X by the product $D^3 \times S^1$. Then we have

$$\beta_2(X_M) = \beta_2(X) = d$$

and the closed 3-manifold M is embedded in X_M by a type 1 embedding. We show that there is an $M \in \mathbb{T}_m$ non-embeddable in X_M by a type 1 embedding.

Let X' be the 4-manifold obtained from X_M by splitting along M, and $B = \partial X' = M \times 1 \cup M \times (-1)$.

For the homomorphisms $i'_*, i_* : H_2(M; Q) \to H_2(X'; Q)$ induced from the natural maps $i' : M \to M \times (-1) \to X'$, $i : M \to M \times 1 \to X'$, let

$$C = \mathrm{im} i'_* \cap \mathrm{im} i_* \subset H_2(X'; Q),\ C'_* = (i'_*)^{-1}(C),\ C_* = (i_*)^{-1}(C).$$

We show the following lemma:

Lemma 4.3. Every closed 3-manifold $M \in \mathbb{T}_m$ with $m > d$ satisfies one of the following (1)-(3).

(1) The homomorphism i'_* or i_* is not injective,

(2) The homomorphisms i'_* and i_* are injective and $C'_* = C_* = 0$ or $C'_* \neq C_*$.

(3) The homomorphisms i'_* and i_* are injective and $C'_* = C^* \neq 0$ which has no Q-basis $x_1, x_2, \ldots, x_s$ with $i'_*(x_i) = \pm i_*(x_i)$ for all i.

By assuming Lemma 4.3, the proof of Theorem 4.1 is completed as follows. If i'_* and i_* are injective and $C_* = C'_* = 0$, then the natural homomorphism $H_2(M;Q) \to H_2(X_M;Q)$ is injective. Since

$$\beta_1(M) = n = 3m > \beta_2(X_M) = \beta_2(X) = d,$$

we have a contradiction. Hence (2) implies $C'_* \neq C_*$. Then in either case of (1)-(3), there is an end-trivial homomorphism $\gamma : H_1(X';Z) \to Z$ such that the restriction $\dot{\gamma} : H_1(B;Z) \to Z$ of γ is asymmetric. To see this, we use an analogous argument of [10, Section 5]. The inclusion $k : B \to X'$ is called a *loose embedding* if the homomorphism $k_* : H_2(B;Z) \to H_2(X';Q)$ is not injective. In either case of (1)-(3) of Lemma 4.3, the inclusion k is a loose embedding and there is a closed oriented 2-manifold F in B, called a *null-surface*, such that F bounds a compact connected oriented 3-manifold V in X' and the Poincaré dual element $\dot{\gamma} \in H^1(B;Z)$ of the homology class $[F] \in H_2(B;Z)$ is asymmetric. Then the 3-manifold V defines an end-trivial homomorphism $\gamma : H_1(X';Z) \to Z$ by the intersection number $\mathrm{Int}_{X'}(x,[V]) \in Z$ for every $x \in H_1(X';Z)$. The element $\dot{\gamma} \in H^1(B;Z)$ is taken as the restriction of γ. Since

$$\hat{\beta}_2(X') \leq \hat{\beta}_2(X) \leq \beta_2(X) = d,$$

the inequality (3.1) of the signature theorem implies

$$|\sigma_{[a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \leq 2d$$

for all $a \in (-1,1)$, where recall that $\kappa_1(\tilde{B})$ is the Q-dimension of the kernel of the homomorphism $t-1 : H_1(\tilde{B};Q) \to H_1(\tilde{B};Q)$. By a choice of a closed 3-manifold $M \in \mathbb{T}_m$ in Lemma 3.1, there is a number $b \in (-1,1)$ such that $|\sigma_{[b,1]}(\tilde{B})| - \kappa_1(\tilde{B}) > 2d$, which is a contradiction. This completes the proof of Theorem 4.1 except for the proof of Lemma 4.3.

Proof of Lemma 4.3. Let $\tilde{X}$ be the infinite cyclic cover of X associated with the fundamental region $(X'; M \times (-1), M \times 1)$. Let $n = 3m$. Suppose that the following assertion is true:

(*) The homomorphisms i_* and i'_* are injective and $C'_* = C^* \neq 0$, which has a Q-basis $x_1, x_2, \ldots, x_s$ with $i'_*(x_i) = \pm i_*(x_i)$ for all i.

Then by the Mayer-Vietoris exact sequence, we have

$$H_2(\tilde{X};Q) \cong \Gamma^{d'} \oplus (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)},$$

for some non-negative integers d' and $c(\pm)$ such that

$$\dim_Q C = c(+) + c(-) \leq n,\ n - (c(+) + c(-)) \leq d',\ d' + c(-) \leq d,$$

so that $n - c(+) \leq d$. Let Y be a compact 4-manifold such that $M \subset Y \subset X$ and the Γ-torsion part $\mathrm{Tor}_\Gamma H_2(\tilde{Y};Q)$ of the homology Γ-module $H_2(\tilde{Y};Q)$ has

$$\mathrm{Tor}_\Gamma H_2(\tilde{Y};Q) = \mathrm{Tor}_\Gamma H_2(\tilde{X};Q) \cong (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)}.$$

By the duality in [3], we have

$$\mathrm{Tor}_\Gamma H_1(\tilde{Y}, \partial\tilde{Y};Q) \cong (\Gamma/(t+1))^{c(+)} \oplus (\Gamma/(t-1))^{c(-)}.$$

Let

$$H_*(\tilde{Y}, \partial\tilde{Y};Q) = \mathrm{Tor}_\Gamma H_*(\tilde{Y}, \partial\tilde{Y};Q) \oplus FH_*(\tilde{Y}, \partial\tilde{Y};Q)$$

be any splitting of a finitely generated Γ-module into the Γ-torsion part and Γ-free part, and

$$H^*(\tilde{Y}, \partial\tilde{Y};Q) = T^*(\tilde{Y}, \partial\tilde{Y};Q) \oplus F^*(\tilde{Y}, \partial\tilde{Y};Q)$$

the Q-dual splitting with

$$\begin{aligned} T^*(\tilde{Y}, \partial\tilde{Y};Q) &= \hom_Q(\mathrm{Tor}_\Gamma H_*(\tilde{Y}, \partial\tilde{Y};Q), Q), \\ F^*(\tilde{Y}, \partial\tilde{Y};Q) &= \hom_Q(FH_*(\tilde{Y}, \partial\tilde{Y};Q), Q). \end{aligned}$$

Let $T^1(\tilde{Y}, \partial\tilde{Y};Q)_{t+1}$ be the $(t+1)$-component of $T^1(\tilde{Y}, \partial\tilde{Y};Q)$. For the natural homomorphism

$$k^* : T^*(\tilde{Y}, \partial\tilde{Y};Q) \to H^*(M;Q),$$

consider the following commutative square on cup products:

$$\begin{array}{ccc} T^1(\tilde{Y}, \partial\tilde{Y};Q)_{t+1} \times T^1(\tilde{Y}, \partial\tilde{Y};Q)_{t+1} & \xrightarrow{\cup} & H^2(\tilde{Y}, \partial\tilde{Y};Q) \\ k^* \otimes k^* \downarrow & & k^* \downarrow \\ H^1(M;Q) \times H^1(M;Q) & \xrightarrow{\cup} & H^2(M;Q). \end{array}$$

Let Ω be the Q-subspace of $H^2(M;Q)$ generated by the elements

$$k^*(u \cup v) \in H^2(M;Q)$$

for all $u, v \in T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1}$.

Let

$$(k')^* : T^2(\tilde{Y}; Q) \to H^2(M;Q)$$

be the natural homomorphism for the Q-dual splitting

$$H^*(\tilde{Y}; Q) = T^*(\tilde{Y}; Q) \oplus F^*(\tilde{Y}; Q)$$

of any splitting $H_*(\tilde{Y}; Q) = \mathrm{Tor}_\Gamma H_*(\tilde{Y}; Q) \oplus FH_*(\tilde{Y}; Q)$.

By a transfer argument of [4, Lemma 1.4], the homomorphism

$$(k')^* : T^2(\tilde{Y}; Q) \to H^2(M;Q)$$

is injective. For the natural homomorphism $j^* : H^2(\tilde{Y}, \partial\tilde{Y}; Q) \to H^2(\tilde{Y}; Q)$, we have $(k')^* j^* = k^*$ and

$$tj^*(u \cup v) = j^*(tu \cup tu) = j^*(-u \cup -v) = j^*(u \cup v)$$

for all $u, v \in T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1}$. Hence

$$k^*(u \cup v) = (k')^* j^*(u \cup v) \in (k')^* T^2(\tilde{Y}; Q)_{t-1}$$

for all $u, v \in T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1}$, so that

$$\Omega \cap (k')^* T^2(\tilde{Y}; Q)_{t+1} = 0.$$

Hence the quotient map

$$\Omega \to H^2(M;Q)/(k')^* T^2(\tilde{Y}; Q)_{t+1}$$

is injective. Since $T^2(\tilde{Y}; Q)_{t+1} \cong (\Gamma/(t+1))^{c(+)}$, we have

$$\dim_Q \Omega \le \dim_Q H^2(M;Q)/(k')^* T^2(\tilde{Y}; Q)_{t+1} = 3m - c(+) \le d.$$

On the other hand, by a transfer argument of [4, Lemma 1.4], the homomorphism

$$k^* : T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1} \to H^1(M;Q)$$

is injective. Since $\dim_Q T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1} = c(+)$, the image

$$\Delta = k^* T^1(\tilde{Y}, \partial\tilde{Y}; Q)_{t+1}$$

of the homomorphism k^* is a Q-subspace of $H^1(M;Q)$ of codimension $d' = 3m - c(+) \leq d$. Since the cup product space $\Delta^{(2)}$ of Δ is equal to Ω, we have

$$\dim_Q \Omega \geq 2m - d' \geq 2m - d.$$

Hence $2m - d \leq d$, that is $m \leq d$. This contradicts the inequality $m > d$. Thus, the assertion (*) is false. This shows Lemma 4.3. □

This completes the proof of Theorem 4.1. □

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