On 4-dimensional universe for every 3-dimensional manifold *

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Abstract

A boundary-less connected oriented 4-manifold is called a universe for every 3-manifold if every closed connected oriented 3-manifold is embedded in it, and a punctured universe if every punctured 3-manifold is embedded in it, which is known to be an open 4-manifold. We introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe and investigate some relationships among them. After introducing some topological invariants for every (possibly non-compact) oriented 4-manifold which we call the topological indexes, we show infinity and independence on some topological indexes of every universe.

Keywords: Universe, Punctured universe, Topological index, 3-manifold, Punctured 3-manifold, Type 1 embedding, Type 2 embedding, Signature theorem

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1. Introduction

Throughout this paper, by a closed 3-manifold we mean a closed connected oriented 3-manifold and by a punctured 3-manifold a punctured manifold of a closed connected oriented 3-manifold. Then we know that for every compact oriented 4-manifold, there is a closed 3-manifold whose punctured 3-manifold is not embeddable in it (see [4]) and hence any oriented 4-manifold with every punctured 3-manifold embedded must be non-compact. This motivates us to put the following definition:

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1 The non-orientable version is also known in [11], but we do not discuss it here. Also, by an embedding we will mean a smooth or piecewise-linear embedding.
Definition. A universe is an open connected oriented 4-manifold $U$ with every closed 3-manifold $M$ embedded. A punctured universe is an open connected oriented 4-manifold $U$ with every punctured 3-manifold $M^0$ embedded.

Then we ask a question: *What topological shapes a universe and a punctured universe have?*

In this question, we introduce the following topological indexes $^\hat{d}(Y)(d = 1, 2)$, $\delta(Y)$, $\delta_i(Y) (i = 0, 1, 2)$, $\rho(Y)$, $\rho_i(Y) (i = 0, 1, 2)$ of every (possibly, non-compact) oriented 4-manifold $Y$, which are obtained from homological arguments and are topological invariants of $Y$ with values taken in $\{0, 1, 2, \ldots, +\infty\}$. We apply these invariants to a punctured universe, a universe and their refined universes, namely types 1, 2 and full universes to obtain our main result (Theorem 3.3) which is stated as follows:

For a punctured universe $U$, we show that one of the topological indexes $^\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ is $+\infty$. Further, in every case, there is a punctured spin universe $U$ with the other topological indexes taken 0.

For a type 1 universe $U$, we show that one of the topological indexes $^\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ is $+\infty$. We have always $^\hat{\beta}_1(U) \geq 1$, but in the case of $\rho_1(U) = +\infty$, we can add the condition that $^\hat{\beta}_1(U) = +\infty$. Further, in every case, there is a type 1 spin universe $U$ with the other topological indexes on $^\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ taken 0.

For a type 2 universe $U$, we show that one of the topological indexes $^\hat{\beta}_2(U)$, $\delta_2(U)$ is $+\infty$. Further, in every case, there is a type 2 spin universe $U$ with the other topological index taken 0.

For a universe $U$, we show that one of the topological indexes $^\hat{\beta}_2(U)$, $\delta(U)$, $\rho(U)$ is $+\infty$. In the case of $\rho(U) = +\infty$, we can add the condition that $^\hat{\beta}_1(U) = +\infty$. Further, in every case, there is a spin universe $U$ with the other topological indexes on $^\hat{\beta}_2(U)$, $\delta(U)$ and $\rho(U)$ taken 0.

For a full universe $U$, we show that one of the topological indexes $^\hat{\beta}_2(U)$, $\delta(U)$ is $+\infty$. We have always $^\hat{\beta}_1(U) \geq 1$. Further, in every case, there is a full spin universe $U$ with the other topological index on $^\hat{\beta}_2(U)$ and $\delta(U)$ taken 0.

In Section 2, we introduce types 1, 2 and full universes as refined notions of a universe and a punctured universe. We explain some relationships among them in Theorem 1.1. In Section 3, the topological indexes of every oriented 4-manifold are defined and our main result (Theorem 3.3) is stated. The existence part of universes in our main result (Theorem 3.3) is shown in this section with some examples. In
Section 4, we establish a non-compact 4-manifold version of the signature theorem for an infinite cyclic covering of a compact oriented manifold given in [3], which is needed to prove the infinity of some topological indexes stated in Theorem 3.3. In Section 5, we introduce a notion of a loose embedding needed as a tool connecting an embedding argument with an argument of an infinite cyclic covering. In Section 6, we complete the proof of Theorem 3.3.

Figure 1: Universes of types 1 and 2

2. Types 1, 2 and full universes as refined notions of a universe and a punctured universe

Let $\mathcal{M}$ be the set of closed 3-manifolds $M$, and $\mathcal{M}^0$ the set of punctured 3-manifolds $M^0$. It is useful to denote the members of $\mathcal{M}$ and $\mathcal{M}^0$ by $M_i$ $(i = 1, 2, 3, \ldots)$ and $M^0_i$ $(i = 1, 2, 3, \ldots)$, respectively. For a connected open oriented 4-manifold $U$, we note that there are two types of embeddings $k : M \to U$. An embedding $k : M \to U$ is of type 1 if $U \setminus k(M)$ is connected, and of type 2 if $U \setminus k(M)$ is disconnected (see Fig. 1). If there is a type 1 embedding $k : M \to U$, then there is an element $x \in H_1(U; Z)$ with the intersection number $\text{Int}_U(x, k(M)) = +1$, so that the intersection form $\text{Int}_U : H_1(U; Z) \times H_3(U; Z) \to Z$ induces an epimorphism

$$I_d : H_d(U; Z) \to Z$$

for $d = 1, 3$ such that the composite $I_3k_* : H_3(M; Z) \to H_3(U; Z) \to Z$ is an isomorphism and the composite $I_1k_* : H_1(M; Z) \to H_1(U; Z) \to Z$ is the 0-map. This suggests that we must consider more refined universes as follows:

**Definition.** The universe $U$ is a type 1 universe if every closed 3-manifold is type 1 embeddable in $U$, a type 2 universe if every $M$ is type 2 embedded in $U$, and a full universe if $U$ is a type 1 universe and a type 2 universe.
Our central problem is to characterize the topological shapes of punctured, types 1, 2 and full universes. We note that a full universe is obtained from a type 2 universe by taking a connected sum with $S^1 \times S^3$ (see Fig. 2). We first establish the following theorem.

**Theorem 2.1.** The following assertions hold.

1. Type 1 universe $\to$ Full universe $\to$ Universe $\to$ Punctured universe $\to$ Type 2 universe

2. Type 1 universe $\not\to$ Full universe.

3. Type 2 universe $\not\to$ Full universe.

4. Universe $\not\to$ Type 1 universe.

5. Universe $\not\to$ Type 2 universe.

6. Punctured universe $\not\to$ Universe.

**Proof.** (1) is obvious by definition. To see (3) and (4), we note that the stable 4-space $SR^4 = R^4 \#_{i=1}^{+\infty} S^2 \times S^2$ considered in [5] is a type 2 spin universe because every closed 3-manifold $M$ bounds a simply connected spin 4-manifold whose double is the connected sum of some copies of $S^2 \times S^2$. Since $H_1(SR^4; Z) = 0$, we see that...
any closed 3-manifold cannot be type 1 embedded in \( SR^4 \), showing (3) and (4). To see (2) and (5), we consider a type 1 spin universe

\[
U_{SP} = R^4 \#_{i=1}^{+\infty} M_i \times S^1
\]

which we call the \( S^1 \)-product universe. We use a notion of a linking form, namely a non-singular symmetric bilinear form \( \ell : G \times G \to Q/Z \) on a finite abelian group \( G \). The linking form \( \ell \) is split if \( \ell \) is hyperbolic, i.e., \( G \) is a direct sum \( H' \oplus H'' \) with \( \ell(H',H') = \ell(H'',H'') = 0 \) or \( \ell \) is the orthogonal sum of a linking form \( \ell_H : H \times H \to Q/Z \) and its inverse \(-\ell_H : H \times H \to Q/Z\). Then we have the following lemma:

**Lemma 2.2.** If a closed 3-manifold \( M \) with \( H_1(M; Z) \) a finite abelian group is type 2 embeddable in the product universe \( U_{SP} \), then the linking form

\[
\ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z
\]

is split.

Before proving Lemma 2.2, the proof of Theorem 2.1 will be completed by using Lemma 2.2. In fact, the lens space \( L(p,q) \) with \( p \neq 0, \pm 1 \) is not type 2 embeddable in \( U_{SP} \) by Lemma 2.2, showing (2) and (5). To see (6), for \( I = [0,1] \) we consider a punctured spin universe

\[
U_{IP} = R^4 \#_{i=1}^{+\infty} \text{int}(M_i^0 \times I),
\]

which we call the \( I \)-product punctured universe. Suppose that there is an embedding \( k : M \to U_{IP} \) for a closed 3-manifold \( M \in M \). We note that every element of \( H_1(U_{IP}; Z) \) is represented by the sum of 1-cycles in \( \text{int}(M_i^0 \times I) \) for a finite number of \( i \) which can be moved to be disjoint from \( k(M) \). This means that the intersection number \( \text{Int}(M, H_1(U_{IP}; Z)) = 0 \), showing that the embedding \( k \) is not of type 1 and hence \( k \) must be of type 2. Regarding \( I \subset S^1 \), we can consider \( U_{IP} \subset U_{SP} \). Then the composite embedding \( M \xrightarrow{k} U_{IP} \subset U_{SP} \) is still of type 2, because the boundary \( \partial(M_i^0 \times I) \) is connected. Thus, if \( H_1(M; Z) \) is a finite abelian group, then the linking form \( \ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z \) splits by Lemma 2.2. Thus, the lens space \( L(p,q) \) with \( p \neq 0, \pm 1 \) is not embeddable in \( U_{IP} \), implying that \( U_{IP} \) is not any universe, showing (6). This completes the proof of Theorem 2.1 except the proof of Lemma 2.2.

The proof of Lemma 2.2 is given as follows:

**Proof of Lemma 2.2.** By an \( S^1 \)-semi-product 4-manifold, we mean a 4-manifold which is the connected sum of \( S^1 \)-products \( M_i \times S^1 \) \( (i = 1, 2, \ldots, m) \) for some \( m \). Assume that \( M \) is type 2 embedded in \( U_{SP} \). Then \( M \) is type 2 embedded in an \( S^1 \)-semi-product 4-manifold. We show the following assertion:
(2.2.1) If $H_1(M; Z)$ is a finite abelian group, then $M$ is type 2 embedded in an $S^1$-semi-product 4-manifold $X$ consisting of the connected summands $M_i \times S^1$ ($i = 1, 2, \ldots, m$) such that there is a point $p_i \in S^1$ with $M_i \times p_i \cap M = \emptyset$ for every $i$.

Proof of (2.2.1). We see that $M$ is embedded in an $S^1$-semi-product 4-manifold $X_U = \#^m_{j=1} M_j \times S^1$. The $n$-fold cyclic covering $M_1 \times S^1 \to M_1 \times S^1$ associated with the $n$-fold cyclic covering $S^1 \to S^1$ induces an $n$-fold cyclic covering $X^{(1)} \to X_U$ such that $X^{(1)}$ is an $S^1$-semi-product 4-manifold consisting of the connected summand $M_1$ and the trivial lifts of the other connected summands $M_i \times S^1$ ($i = 2, \ldots, m$). Since $H_1(M; Z)$ is finite, the manifold $M$ is trivially lifted to $X^{(1)}$. We note that $M$ is type 2 embedded in $X_U$ if and only if $\text{Int}_{X_U} (M, H_1(X_U; Z)) = 0$. Since $H_1(X^{(1)}; Z)$ is generated by loops which are lifts of loops in $X_U$, we see that any trivial lift $M'$ of $M$ to $X^{(1)}$ has $\text{Int}_{X^{(1)}} (M', H_1(X^{(1)}; Z)) = 0$ and hence is type 2 embedded. Taking $n$ sufficiently large, we have $M_i \times p_1 \cap M = \emptyset$ for a point $p_1 \in S^1$. Applying the same arguments inductively to $M_i \times S^1$ ($i = 2, 3, \ldots, s$), we obtain the conclusion of (2.2.1).

By (2.2.1), for $I = [0,1]$ we may consider that $M$ is type 2 embedded in the connected sum

$$Y = M_1 \times I \# M_2 \times I \# \ldots \# M_m \times I,$$

so that $M$ splits $Y$ into two compact 4-manifolds $A$ and $B$ whose boundaries $\partial A$ and $\partial B$ have the form

$$\partial A = M \cup \partial_A Y, \quad \partial B = (-M) \cup \partial_B Y,$$

where

$$\partial_A Y = M_1 \times \partial I \cup M_2 \times \partial I \cup \cdots \cup M_s \times \partial I,$$

$$\partial_B Y = M_{s+1} \times \partial I \cup M_{s+2} \times \partial I \cup \cdots \cup M_m \times \partial I.$$

We show the following assertion:

(2.2.2) The following natural sequence

$$0 \to \text{tor} H_2 (A, M \cup \partial_A Y; Z) \xrightarrow{\partial_A} \text{tor} H_1 (M \cup \partial_A Y; Z) \xrightarrow{\partial_B} \text{tor} H_1 (A; Z) \to 0$$

on the homology torsion parts is a split exact sequence.

By (2.2.2), the linking form

$$\ell^+: \text{tor} H_1 (M \cup \partial_A Y; Z) \times \text{tor} H_1 (M \cup \partial_A Y; Z) \to \mathbb{Q}/\mathbb{Z}$$
is split. This is because $\text{im}\partial_s$ is a direct summand of $\text{tor}H_1(M \cup \partial_A Y; Z)$ and $(\text{im}\partial_s)^\perp = \text{im}\partial_s$ with respect to $\ell^+$. Since the linking form
\[
\ell_0 : \text{tor}H_1(\partial_A Y; Z) \times \text{tor}H_1(\partial_A Y; Z) \to Q/Z
\]
is split and the linking form $\ell^+$ is an orthogonal sum of the linking forms $\ell$ and $\ell_0$, we see from [9] that the linking form $\ell : H_1(M; Z) \times H_1(M; Z) \to Q/Z$ is split. This completes the proof of Lemma 2.2 except the proof of (2.2.2).

**Proof of (2.2.2).** Let
\[
\partial_A^0 Y = M_1 \times 0 \cup M_2 \times 0 \cup \cdots \cup M_s \times 0, \\
\partial_B^0 Y = M_{s+1} \times 0 \cup M_{s+2} \times 0 \cup \cdots \cup M_m \times 0.
\]
Further, let $\partial^0 Y = \partial_A^0 Y \cup \partial_B^0 Y$. Since $H_2(M; Z) = H_2(Y, \partial^0 Y; Z) = 0$, the Mayer-Vietoris exact sequence
\[
H_2(M; Z) \to H_2(A, \partial_A^0 Y; Z) \oplus H_2(B, \partial_B^0 Y; Z) \to H_2(Y, \partial^0 Y; Z)
\]
implies that
\[
H_2(A, \partial_A^0 Y; Z) = H_2(B, \partial_B^0 Y; Z) = 0.
\]
Since $H_1(M \cup \partial_A^0 Y, \partial_A^0 Y; Z) = H_1(M; Z)$ is finite and $H_2(A, \partial_A^0 Y; Z) = 0$, we see from the exact sequence
\[
H_2(A, \partial_A^0 Y; Z) \to H_2(A, M \cup \partial_A^0 Y; Z) \to H_1(M \cup \partial_A^0 Y, \partial_A^0 Y; Z)
\]
that $H_2(A, M \cup \partial_A^0 Y; Z)$ is finite. Because $j_*$ passes through the finite abelian group $H_2(A, M \cup \partial_A^0 Y; Z)$, we see that the image of the homomorphism $j_* : H_2(A; Z) \to H_2(A, M \cup \partial_A Y; Z)$ is finite. Thus, the semi-exact sequence
\[
\text{tor}H_2(A, M \cup \partial_A Y; Z) \xrightarrow{\partial} \text{tor}H_1(M \cup \partial_A Y; Z) \xrightarrow{i_*} \text{tor}H_1(A; Z)
\]
is exact. We construct a monomorphism
\[
\pi : \text{tor}H_1(A; Z) \to \text{tor}H_1(M \cup \partial_A Y; Z)
\]
with the identity
\[
i_* \pi = 1 : \text{tor}H_1(A; Z) \xrightarrow{\pi} \text{tor}H_1(M \cup \partial_A Y; Z) \xrightarrow{i_*} \text{tor}H_1(A; Z).
\]
Then we see that the sequence ($\#$) is a split exact sequence, because $i_*$ is onto and $\partial_*$ is injective, for $\partial_* : \text{tor}H_2(A, M \cup \partial_A Y; Z) \to \text{tor}H_1(M \cup \partial_A Y; Z)$ is Poincaré dual to the epimorphism $i_* : \text{tor}H_1(M \cup \partial_A Y; Z) \to \text{tor}H_1(A; Z)$. 

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To construct a monomorphism $\pi$, we note that the Mayer-Vietoris exact sequence

$$0 = H_2(Y, \partial^0 Y; Z) \to H_1(M; Z) \to H_1(A, \partial^0_A Y; Z) \oplus H_1(B, \partial^0_B Y; Z) \to H_1(Y, \partial^0 Y; Z) = Z^{m-1}$$

induces a natural isomorphism

$$j_*^A + j_*^B : H_1(M; Z) \cong \text{tor} H_1(A, \partial^0_A Y; Z) \oplus \text{tor} H_1(B, \partial^0_B Y; Z).$$

Then we can construct a monomorphism

$$\pi^A : \text{tor} H_1(A, \partial^0_A Y; Z) \to H_1(M; Z)$$

so that

$$j_*^A \pi^A = 1 : \text{tor} H_1(A, \partial^0_A Y; Z) \xrightarrow{\pi^A} H_1(M; Z) \xrightarrow{j_*^A} \text{tor} H_1(A, \partial^0_A Y; Z).$$

Since $j_*^A$ passes through the natural homomorphisms $j_*^M : H_1(M; Z) \to \text{tor} H_1(A; Z)$ and $j_*^0 : \text{tor} H_1(A; Z) \to \text{tor} H_1(A, \partial^0_Y; Z)$, we have

$$j_*^0(j_*^M \pi^A) = j_*^A \pi^A = 1.$$ 

Using that $H_2(A, \partial^0_A Y; Z) = 0$ and $j_*^0$ is onto, we see that the sequence

$$0 \to \text{tor} H_1(\partial^0_Y; Z) \xrightarrow{j_*^0} \text{tor} H_1(A; Z) \xrightarrow{j_*^0} \text{tor} H_1(A, \partial^0_Y; Z) \to 0$$

obtained from the homology sequence of the pair $(A, \partial^0_Y)$ is a split exact sequence, by which we can define a homomorphism

$$\pi^0 : \text{tor} H_1(A; Z) \to \text{tor} H_1(\partial^0_Y; Z)$$

with the identity $i_*^0 \pi^0 = 1 - j_*^M \pi^A j_*^0$. We define the homomorphism

$$\pi' = \pi^A j_*^0 + \pi^0 : \text{tor} H_1(A; Z) \to H_1(M; Z) \oplus \text{tor} H_1(\partial^0_Y; Z).$$

This homomorphism

$$j_*^M + i_*^0 : H_1(M; Z) \oplus \text{tor} H_1(\partial^0_Y; Z) \to \text{tor} H_1(A; Z)$$

has the identity

$$(j_*^M + i_*^0)\pi' = 1 : \text{tor} H_1(A; Z) \to \text{tor} H_1(A; Z).$$

In fact, we have

$$(j_*^M + i_*^0)\pi'(x) = j_*^M \pi^A j_*^0(x) + i_*^0 \pi^0(x) = j_*^M \pi^A j_*^0(x) + x - j_*^M \pi^A j_*^0(x) = x$$
Thus, we have a desired monomorphism of the homology tor $H_1(M; Z)$ and $H_1(A; Z)$ to the natural homomorphism $\partial^0_\Lambda$ and $\partial^0_\Lambda$ to the direct summand $\partial^0_\Lambda$ of $\partial_\Lambda$ with $i_*\pi = 1$ where the value of $\pi$ on the direct summand $\partial_\Lambda$ is taken $0$. Thus, we have a desired homomorphism $\pi$, showing (2.2.2).

This completes the proof of Lemma 2.2.

3. Homology of a universe and a punctured universe

Let $Y$ be an orientable possibly non-compact 4-manifold. For the intersection form

$$\text{Int} : H_d(Y; Z) \times H_{4-d}(Y; Z) \to Z,$$

we define the $d$th null homology of $Y$ to be the subgroup

$$O_d(Y; Z) = \{ x \in H_d(Y; Z) | \text{Int}(x, H_{4-d}(Y; Z)) = 0 \}$$

of the $d$th homology group $H_d(Y; Z)$ and the $d$th non-degenerate homology of $Y$ to be the quotient group

$$\hat{H}_d(Y; Z) = H_d(Y; Z)/O_d(Y; Z).$$

We have the following lemma:

**Lemma 3.1.** $\hat{H}_d(Y; Z)$ is a free abelian group.

**Proof.** We first note that the induced intersection form

$$\text{Int} : \hat{H}_d(Y; Z) \times \hat{H}_{4-d}(Y; Z) \to Z$$

is non-degenerate and $\hat{H}_*(Y; Z)$ is a torsion-free abelian group. Thus, if $Y$ is compact, then $\hat{H}_*(Y; Z)$ is a free abelian group. Assume that $Y$ is non-compact. Let

$$Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \cdots$$

be an ascending sequence of compact 4-submanifolds $Y_n$ of $Y$ with $\bigcup_{n=1}^{\infty} Y_n = Y$. We find elements $x_i^Q \in H_d(Y_i; Q)$ ($i = 1, 2, \ldots, m_1$) representing a $Q$-basis for $\hat{H}_d(Y_1; Z) \otimes Q$ and elements $y_i^Q \in H_{4-d}(Y_1; Q)$ representing a $Q$-basis for $\hat{H}_{4-d}(Y_n; Z) \otimes Q$ ($i = 1, 2, \ldots, m_1$) with the $Q$-intersection numbers $\text{Int}_Q(x_i^Q, y_j^Q) = \delta_{ij}$ for all $i, j$. Then the elements $x_i^Q$ ($i = 1, 2, \ldots, m_1$) and $y_i^Q$ ($i = 1, 2, \ldots, m_1$) are regarded as linearly independent elements of $H_d(Y_2; Q)$ and $H_{4-d}(Y_2; Q)$, respectively. Taking the orthogonal complements of the $Q$-subspaces generated by these elements with
For an ascending sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \ldots$ of compact 4-submanifolds $Y_n$ for a non-compact oriented 4-manifold $Y$ with $\cup_{n=1}^{+\infty} Y_n = Y$, we have
\[ \hat{\beta}_d(Y_n) \leq \hat{\beta}_d(Y_{n+1}) (n = 1, 2, 3, \ldots) \quad \text{and} \quad \lim_{n \to +\infty} \hat{\beta}_d(Y_n) = \hat{\beta}_d(Y). \]

For an abelian group $G$, let $G^{(2)} = \{x \in G \mid 2x = 0\}$, which is a direct sum of some copies of $Z_2$. For $M^0 \in M^0$, let $\delta(M^0 \subset Y)$ be the minimal Z-rank of the image $\text{im}[k^0_* : H_2(M^0; Z) \to H_2(Y; Z)]$, and $\rho(M^0 \subset Y)$ the minimal $Z_2$-rank of $\text{im}[k^0_* : H_2(M^0; Z) \to H_2(Y; Z)]^{(2)}$, for all embeddings $k^0 : M^0 \to Y$. By taking the value $0$ for the non-embeddable case, we define the following topological invariants of $Y$:
\[
\delta_0(Y) = \sup \{ \delta(M^0 \subset Y) \mid M^0 \in M^0 \}, \\
\rho_0(Y) = \sup \{ \rho(M^0 \subset Y) \mid M^0 \in M^0 \}.
\]
Let $\delta(M \subset Y)$ be the minimal Z-rank of the image $\text{im}[k_* : H_2(M; Z) \to H_2(Y; Z)]$, and $\rho(M \subset Y)$ the minimal $Z_2$-rank of $\text{im}[k_* : H_2(M; Z) \to H_2(Y; Z)]^{(2)}$, for all embeddings $k : M \to Y$. By taking the value $0$ for the non-embeddable case, we define the following invariants of $Y$:
\[
\delta(Y) = \sup \{ \delta(M \subset Y) \mid M \in M \}, \\
\rho(Y) = \sup \{ \rho(M \subset Y) \mid M \in M \}.
\]
Restricting all embeddings $k : M \to Y$ to all embeddings $k : M \to Y$ of type $i$ for $i = 1, 2$, we obtain the topological indexes $\delta_i(Y)$ and $\rho_i(Y)$ ($i = 1, 2$) of $Y$ in place of $\delta(Y)$ and $\rho(Y)$. 

The Z-rank $\hat{\beta}_d(Y)$ of $\hat{H}_d(Y; \mathbb{Z})$ is our first topological index of $Y$. The proof of Lemma 3.1 also implies the following corollary.

**Corollary 3.2.** For an ascending sequence $Y_1 \subset Y_2 \subset \cdots \subset Y_n \subset \ldots$ of compact 4-manifolds $Y_n$ for a non-compact oriented 4-manifold $Y$ with $\cup_{n=1}^{+\infty} Y_n = Y$, we have
\[ \hat{\beta}_d(Y_n) \leq \hat{\beta}_d(Y_{n+1}) (n = 1, 2, 3, \ldots) \quad \text{and} \quad \lim_{n \to +\infty} \hat{\beta}_d(Y_n) = \hat{\beta}_d(Y). \]
Our main result concerns a behavior on the topological indexes for a punctured universe and a universe, and their refined universes, namely types 1, 2 and full universes, which are stated as follows:

**Theorem 3.3.**

(1) If \( U \) is a punctured universe, then one of the following cases (1.1)-(1.3) holds.

(1.1) \( \tilde{\beta}_2(U) = +\infty \).

(1.2) \( \delta_0(U) = +\infty \).

(1.3) \( \rho_0(U) = +\infty \).

Further, in every case, there is a punctured spin universe \( U \) with the other topological indexes taken 0.

(2) If \( U \) is a type 1 universe, then one of the following cases (3.1)-(3.3) holds.

(3.1) \( \tilde{\beta}_2(U) = +\infty \) and \( \tilde{\beta}_1(U) \geq 1 \).

(3.2) \( \delta_1(U) = +\infty \) and \( \tilde{\beta}_1(U) \geq 1 \).

(3.3) \( \rho_1(U) = +\infty \) and \( \tilde{\beta}_1(U) = +\infty \).

Further, in every case, there is a type 1 spin universe \( U \) with the other topological indexes on \( \tilde{\beta}_2(U) \), \( \delta_1(U) \) and \( \rho_1(U) \) taken 0.

(3) If \( U \) is a type 2 universe, then one of the following cases (3.1) and (3.2) holds.

(3.1) \( \tilde{\beta}_2(U) = +\infty \).

(3.2) \( \delta_2(U) = +\infty \).

Further, in every case, there is a type 2 spin universe \( U \) with the other topological index taken 0.

(4) If \( U \) is a universe, then one of the following cases (4.1)-(4.3) holds.

(4.1) \( \tilde{\beta}_2(U) = +\infty \).

(4.2) \( \delta(U) = +\infty \).

(4.3) \( \rho(U) = +\infty \) and \( \tilde{\beta}_1(U) = +\infty \).

Further, in every case, there is a spin universe \( U \) with the other topological indexes on \( \tilde{\beta}_2(U) \), \( \delta(U) \) and \( \rho(U) \) taken 0.

(5) If \( U \) is a full universe, then one of the following cases (5.1) and (5.2) holds.

(5.1) \( \tilde{\beta}_2(U) = +\infty \) and \( \tilde{\beta}_1(U) \geq 1 \).

(5.2) \( \delta(U) = +\infty \) and \( \tilde{\beta}_1(U) \geq 1 \).

Further, in every case, there is a full spin universe \( U \) with the other topological index on \( \tilde{\beta}_2(U) \) and \( \delta(U) \) taken 0.
In the following Examples 3.4-3.6, we give some examples on a punctured universe, a universe, and their refined universes, namely types 1, 2 and full universes, which are sufficient to see the existence assertions on (1)-(5) of Theorem 3.3.

Example 3.4. The stable 4-space $SR^4 = R^4\oplus \bigoplus_{i=1}^{\infty} S^2 \times S^2_i$ has the following property:

(3.4.1) For every $M \in M$, there is a type 2 embedding $k : M \to SR^4$ inducing the trivial homomorphism $k_* : H_2(M; Z) \to H_2(SR^4; Z)$.

Thus, $U = SR^4$ is a punctured and type 2 spin universe with $\hat{\beta}_2(U) = +\infty$, $\hat{\beta}_1(U) = 0$, $\delta_0(U) = \delta_2(U) = 0$ and $\rho_0(U) = \rho_2(U) = 0$. Further, $U_S = S^1 \times S^3#SR^4$ is a punctured, type 1, type 2, full spin universe with

$\hat{\beta}_2(U_S) = +\infty$, $\hat{\beta}_1(U_S) = 1$,
$\delta_0(U_S) = \delta_1(U_S) = \delta_2(U_S) = \delta(U_S) = 0$,
$\rho_0(U_S) = \rho_1(U_S) = \rho_2(U_S) = \rho(U_S) = 0$.

Proof of (3.4.1). Let $W$ be a simply connected spin 4-manifold with $\partial W = M$ whose double $DW$ is homeomorphic to the connected sum $X$ of some copies of $S^2 \times S^2$. Since the natural homomorphism $i_* : H_2(M; Z) \to H_2(W; Z)$ is injective, we can represent a basis of the image of $i_*$ by mutually disjoint 2-spheres $S_i (i = 1, 2, \ldots, m)$ in $W$ which we can find in the factors $S^2 \times p$'s of the connected summands $S^2 \times S^2$'s of the $S^2 \times S^2$-decomposition of $DW$, if necessary, by taking connected sums with some copies of $S^2 \times S^2$. By the surgeries of $DW$ on $S_i (i = 1, 2, \ldots, m)$, we obtain the connected sum $X'$ of some copies of $S^2 \times S^2$ such that the inclusion $M \to X'$ induces the zero map $H_2(M; Z) \to H_2(X'; Z)$. Since the stable 4-space $SR^4$ is constructed from a punctured manifold of $X'$, we have (3.4.1). □

Example 3.5. For $I = [0, 1]$, let $W_i$ be a spin 4-manifold obtained from $M_i \times I$ by attaching 2-handles on $M_i \times 1$ along a basis for $H_1(M_i \times 1; Z)$ (torsions) to obtain that $H_1(W_i; Z)$ is a torsion abelian group. Then the natural homomorphism $H_2(M_i \times I; Z) \to H_2(W_i; Z)$ is an isomorphism, so that $H_2(W_i; Z)$ is a free abelian group. Let $W_i (i = 1, 2, \ldots)$ be the 4-manifolds corresponding to the 3-manifolds $M_i \in M (i = 1, 2, \ldots)$. We construct the open 4-manifolds

$U_T = R^4\oplus \bigoplus_{i=1}^{\infty} \text{int} W_i$ and $U_{ST} = S^1 \times S^3#U_T$.

The open 4-manifold $U_T$ is a punctured and type 2 spin universe with

$\hat{\beta}_2(U_T) = \hat{\beta}_1(U_T) = 0$,
$\delta_0(U_T) = \delta_2(U_T) = +\infty$,
$\rho_0(U_T) = \rho_2(U_T) = 0$.

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The open 4-manifold $U_{ST}$ is a punctured, type 1, type 2 and full spin universe with
\[
\begin{align*}
\hat{\beta}_2(U_{ST}) = 0, & \quad \hat{\beta}_1(U_{ST}) = 1, \\
\delta_0(U_{ST}) = \delta_1(U_{ST}) = \delta_2(U_{ST}) = \delta(U_{ST}) = +\infty, & \quad \\
\rho_0(U_{ST}) = \rho_1(U_{ST}) = \rho_2(U_{ST}) = \rho(U_{ST}) = 0.
\end{align*}
\]

**Example 3.6.** Let $Z/2 = Z[1/2]$ be a subring of $Q$. The 4-dimensional solid torus with three meridian disks is a spin 4-manifold $D(T^3)$ with boundary the 3-dimensional torus $T^3$ which is obtained from the 4-disk $D^4$ by attaching the three 0-framed 2-handles along the Borromean rings $L_B$ (see [8, 10]). For $s \geq 2$, let $D(sT^3)$ be the disk sum of $s$ copies of $D(T^3)$. Then the boundary $\partial D(sT^3)$ is the connected sum $#sT^3$ of $s$ copies of $T^3$. For $s = 0$, we understand $D(sT^3) = S^4$ and $#sT^3 = \emptyset$. Let
\[
\Sigma = S^1 \times S^3 # D(sT^3) \quad \text{and} \quad \tilde{\Sigma} = S^4 # D(sT^3) = D(sT^3).
\]

A Samsara 4-manifold on $M \in \mathbb{M}$ is a compact oriented spin 4-manifold $\Sigma$ with $\partial \Sigma = #sT^3$ and with $Z/2$-homology of $\Sigma$ for some $s \geq 0$ such that there is a type 1 embedding $k : M \to \Sigma$ inducing the trivial homomorphism
\[
k_* = 0 : H_2(M; Z/2) \to H_2(\Sigma; Z/2).
\]

We also call $\Sigma$ the standard Samsara 4-manifold on $S^3$. In [8], we showed that there is a Samsara 4-manifold $\Sigma_i$ on every $M_i \in \mathbb{M}$ ($i = 1, 2, 3, \ldots$). Let $R_i^4$ be the upper-half 4-space with boundary the 3-space $R_i^3$. Let
\[
\Sigma R_i^4 = R_i^4 \#^\infty \Sigma_i
\]
be the 4-manifold obtained from $R_i^4$ by making the connected sums with the closed $\Sigma_i$'s and the disk sums with the bounded $\Sigma_i$'s. We call the open 4-manifold $U_{SM} = \text{int}(\Sigma R_i^4)$ a Samsara universe, which is a punctured and type 1 spin universe with
\[
\begin{align*}
\hat{\beta}_2(U_{SM}) = 0, & \quad \hat{\beta}_1(U_{SM}) = +\infty, \\
\delta_0(U_{SM}) = \delta_1(U_{SM}) = 0, & \quad \\
\rho_0(U_{SM}) = \rho_1(U_{SM}) = +\infty.
\end{align*}
\]

Let $\Sigma R^4$ be the 4-manifold obtained from $R^4$ by making the connected sums with countably many copies of $S^1 \times S^3$ and the disk sums with countably many copies of $D(T^3)$, and
\[
\Sigma R^4 = \text{int}(\Sigma R_i^4).
\]

Every Samsara universe $U_{SM}$ has the same $Z/2$-homology as $\Sigma R^4$. By Theorem 3.3, we can see that any Samsara universe $U_{SM}$ is not any type 2 universe. A reduced Samsara 4-manifold on $M^0 \in \mathbb{M}^0$ is a compact oriented spin 4-manifold $\tilde{\Sigma}$ with $\partial \tilde{\Sigma} = #sT^3$.
and with $Z_2$-homology of $\Sigma$ for some $s \geq 0$ such that there is a punctured embedding $k^0 : M^0 \to \Sigma$ inducing the trivial homomorphism $k^0_* = 0 : H_2(M^0; Z_2) \to H_2(\Sigma; Z_2)$.

We also call $\Sigma$ the \textit{standard reduced Samsara 4-manifold} on the punctured 3-sphere $(S^3)^0$. We obtain a reduced Samsara 4-manifold $\Sigma_i$ on $M_i^0$ from a Samsara 4-manifold $\Sigma_i$ on $M_i$ by a surgery of $\Sigma_i$ killing a generator of $\hat{H}_1(\Sigma_i; Z) = Z$, and conversely we obtain a Samsara 4-manifold $\Sigma_i$ on $M_i$ from a reduced Samsara 4-manifold $\Sigma_i$ on $M_i^0$ by the surgery of $\Sigma_i$ along the 2-knot $S^2_i = \partial M^0_i$ (see [8]).

Let $\hat{\Sigma} R^4_+ = R^4_+ \bigoplus_{i=1}^\infty \hat{\Sigma}_i$ be the 4-manifold obtained from $R^4_+$ by making the connected sums with the closed $\Sigma_i$’s and the disk sums with the bounded $\Sigma_i$’s. We call the open 4-manifold $U_{RS} = \text{int}(\hat{\Sigma} R^4_+)$ a \textit{reduced Samsara universe}, which is a punctured spin universe with the following topological indexes

\[ \hat{\beta}_2(U_{RS}) = \hat{\beta}_1(U_{RS}) = 0, \]
\[ \delta_0(U_{RS}) = 0, \]
\[ \rho_0(U_{RS}) = +\infty. \]

Let $\hat{\Sigma} R^4_+$ be the 4-manifold obtained from $R^4_+$ by making the disk sums with countably many copies of $D(T^3)$, and $\hat{\Sigma} R^4 = \text{int}(\hat{\Sigma} R^4_+)$. Every reduced Samsara universe $U_{RS}$ has the same $Z_2$-homology as $\hat{\Sigma} R^4$. By [8, (3.1.4.1)], we can show that if a closed 3-manifold $M$ with $H_1(M; Z)$ a finite abelian group is embedded in $U_{RS}$, then the linking form $\ell_p : H_1(M; Z)_p \times H_1(M; Z)_p \to Q/Z$ restricted to the $p$-primary component $H_1(M; Z)_p$ of $H_1(M; Z)$ for every odd prime $p$ is hyperbolic. Thus, $U_{RS}$ is not any universe. Further, from [8, 3.1(4)], we can see that $\Sigma R^4$ and $\hat{\Sigma} R^4$ are not any punctured universe.

4. A non-compact version of the signature theorem for an infinite cyclic covering

We need a non-compact 4-manifold version of the signature theorem in [3] to prove Theorem 3.3 which is explained in this section.

Let $Y$ be a non-compact oriented 4-manifold with boundary a closed 3-manifold $B$. Assume that $\hat{\beta}_2(Y) < +\infty$. We say that a homomorphism $\gamma : H_1(Y; Z) \to Z$
is end-trivial if there is a compact submanifold $Y'$ of $Y$ such that the restriction $\gamma|_{\text{cl}(Y\setminus Y')} : H_1(Y\setminus Y'; Z) \to Z$ is the zero map. For any end-trivial homomorphism $\gamma : H_1(Y; Z) \to Z$, we take the infinite cyclic covering $(\tilde{Y}, \tilde{B})$ of $(Y, B)$ associated with $\gamma$. Then $H_2(\tilde{Y}; Q)$ is a (possibly, infinitely generated) $\Gamma$-module for the principal ideal domain $\Gamma = Q[t, t^{-1}]$ of Laurent polynomials with rational coefficients. Consider the $\Gamma$-intersection form

$$\text{Int}_\Gamma : H_2(\tilde{Y}; Q) \times H_2(\tilde{Y}; Q) \to \Gamma$$

defined by $\text{Int}_\Gamma(x, y) = \sum_{m=-\infty}^{+\infty} \text{Int}(x, t^{-m}y)t^m$ for $x, y \in H_2(\tilde{Y}; Q)$. Then we have the identities:

$$\text{Int}_\Gamma(\overline{f(t)}x, y) = \text{Int}_\Gamma(x, f(t)y) = f(t)\text{Int}_\Gamma(x, y), \quad \text{Int}_\Gamma(y, x) = \overline{\text{Int}_\Gamma(x, y)},$$

where $\overline{\cdot}$ denotes the involution of $\Gamma$ sending $t$ to $t^{-1}$. Let

$$O_2(\tilde{Y}; Q)_\Gamma = \{x \in H_2(\tilde{Y}; Q) | \text{Int}_\Gamma(x, H_2(\tilde{Y}; Q)) = 0\}$$

and

$$\tilde{H}_2(\tilde{Y}; Q)_\Gamma = H_2(\tilde{Y}; Q)/O_2(\tilde{Y}; Q)_\Gamma,$$

which is a torsion-free $\Gamma$-module. We show the following lemma:

**Lemma 4.1.** If $\hat{\beta}_2(Y) < +\infty$, then $\tilde{H}_2(\tilde{Y}; Q)_\Gamma$ is a free $\Gamma$-module of finite rank.

**Proof.** We split $Y$ by a compact 4-submanifold $Y' \supset B$ of $Y$ and $Y'' = \text{cl}(Y\setminus Y')$ such that $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ and $Y''$ is trivially lifted to $\tilde{Y}$. Then we note that $\tilde{H}_2(\tilde{Y''}; Q)_\Gamma = 0$. For $B_0 = Y' \cap Y''$, since $H_2(\tilde{Y'}; Q)$ and $H_1(B_0; Q)$ are finitely generated $\Gamma$-modules, the Mayer-Vietoris sequence

$$H_2(\tilde{Y'}; Q) \oplus H_2(\tilde{Y''}; Q) \to H_2(\tilde{Y}; Q) \to H_1(B_0; Q)$$

shows that $\tilde{H}_2(\tilde{Y}; Q)_\Gamma$ is a finitely generated, torsion-free $\Gamma$-module, so that it is a free $\Gamma$-module of finite rank. \hfill $\square$

Let $A(t)$ be a $\Gamma$-Hermitian matrix representing the $\Gamma$-intersection form $\text{Int}_\Gamma$ on $\tilde{H}_2(\tilde{Y}; Q)_\Gamma$. For $x \in (-1, 1)$ let $\omega_x = x + \sqrt{1-x^2}i$, which is a complex number of norm one. For $a \in (-1, 1)$ we define the signature invariant of $\tilde{Y}$ by

$$\tau_{a\geq 0}(\tilde{Y}) = \lim_{x \to a \geq 0} \text{sign}A(\omega_x).$$

The signature invariants $\sigma_a(\tilde{B})$ ($a \in [-1, 1]$) of $\tilde{B}$ are also defined in [1, 2] by the quadratic form

$$b : \text{tor}_\Gamma H_1(\tilde{B}; Q) \times \text{tor}_\Gamma H_1(\tilde{B}; Q) \to Q$$

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on the $\Gamma$-torsion part $\text{tor}_\Gamma H_1(\hat{B}; Q)$ of $H_1(\hat{B}; Q)$. For $a \in [-1, 1]$, let
\[
\sigma_{[a, 1]}(\hat{B}) = \sum_{a \leq i \leq 1} \sigma_i(\hat{B}),
\]
\[
\sigma(a, i](\hat{B}) = \sum_{a < i \leq 1} \sigma_i(\hat{B}).
\]

We show the following theorem which is a non-compact version of the signature theorem given in [3].

**Theorem 4.2 (A non-compact version of the signature theorem).**

\[
\tau_{a-0}(\hat{Y}) - \text{sign} Y = \sigma_{[a, 1]}(\hat{B}),
\]
\[
\tau_{a+0}(\hat{Y}) - \text{sign} Y = \sigma_{[a, 1]}(\hat{B}).
\]

**Proof of Theorem 4.2.** As it is discussed in Lemma 4.1, we split $Y$ by a compact 4-submanifold $Y' \supset B$ and $Y'' = \text{cl}(Y \setminus Y')$ such that $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ (see Corollary 3.2) and $Y''$ is trivially lifted to $\hat{Y}$. We use a variant argument of the proof of the Novikov addition theorem for infinite cyclic coverings (see [3]). We consider the homology over the quotient field $Q(\Gamma)$ of $\Gamma$. For $B_0 = Y' \cap Y''$, let $K_1(\hat{B}_0; Q(\Gamma))$ be the kernel of the natural homomorphism
\[
H_1(\hat{B}_0; Q(\Gamma)) \to H_1(\hat{Y'}; Q(\Gamma)) \oplus H_1(\hat{Y''}; Q(\Gamma))
\]
in the Mayer-Vietoris sequence of $(Y; Y', Y''; B_0)$. Let $x_i (i = 1, 2, \ldots, m)$ be a $Q(\Gamma)$-basis of the $Q(\Gamma)$-vector space $K_1(\hat{B}_0; Q(\Gamma))$. This basis is extended to a $Q(\Gamma)$-basis $x_i (i = 1, 2, \ldots, m, m + 1, \ldots, n)$ for $H_1(\hat{B}_0; Q(\Gamma))$. A $Q(\Gamma)$-basis $y_i (i = 1, 2, \ldots, n)$ for $H_2(\hat{B}_0; Q(\Gamma))$ is taken so that the $Q(\Gamma)$-intersection number $\text{Int}_{Q(\Gamma)}(x_i, y_j) = \delta_{ij}$ in $\hat{B}_0$ (see [1]). Let $z_i (i = 1, 2, \ldots, m)$ be “suspension elements” of $x_i (i = 1, 2, \ldots, m)$ in $H_2(\hat{Y}; Q(\Gamma))$ (which are constructed from the $Q(\Gamma)$-basis $x_i (i = 1, 2, \ldots, m)$ of $K_1(\hat{B}_0; Q(\Gamma))$ by using 2-chains in $Y'$ and $Y''$ whose boundary cycles representing $x_i$). We regard $y_i (i = 1, 2, \ldots, m)$ as elements of $H_2(\hat{Y}; Q(\Gamma))$ under the natural homomorphism $H_2(\hat{B}_0; Q(\Gamma)) \to H_2(\hat{Y}; Q(\Gamma))$. Then we have $\text{Int}_{Q(\Gamma)}(z_i, y_j) = \delta_{ij}$ and $\text{Int}_{Q(\Gamma)}(y_i, y_j) = 0$ in $\hat{Y}$. Let $y_{i'}^0 (i' = 1, 2, \ldots, n')$ be elements of $H_2(\hat{Y'}; Q(\Gamma))$ such that $y_i (i = 1, 2, \ldots, m)$ and $y_{i'}^0 (i' = 1, 2, \ldots, n')$ form a $Q(\Gamma)$-basis for $H_2(\hat{Y'}; Q(\Gamma))$ and $y_{i'}^0 (i' = 1, 2, \ldots, n')$ are orthogonal to the elements $y_i, z_i (i = 1, 2, \ldots, m)$ with respect to the $Q(\Gamma)$-intersection form $\text{Int}_{Q(\Gamma)}$ in $\hat{Y}$. Similarly, let $y_{i''}^0 (i'' = 1, 2, \ldots, n'')$ be elements of $H_2(\hat{Y''}; Q(\Gamma))$ such that $y_i (i = 1, 2, \ldots, m)$ and $y_{i''}^0 (i'' = 1, 2, \ldots, n'')$ form a $Q(\Gamma)$-basis for $H_2(\hat{Y''}; Q(\Gamma))$ and $y_{i''}^0 (i'' = 1, 2, \ldots, n'')$ are orthogonal to the elements $y_i, z_i (i = 1, 2, \ldots, m)$ with respect to the $Q(\Gamma)$-intersection form $\text{Int}_{Q(\Gamma)}$ in $\hat{Y}$. Since $H_2(\hat{Y''}; Q(\Gamma)) = 0$, we see that
\[
\tau_{a\pm 0}(\hat{Y}) = \tau_{a\pm 0}(\hat{Y'}).
By the version with $t = 1$ of this argument, we also have $\text{sign}Y = \text{sign}Y'$. Thus, by the compact version of the signature theorem in [3], we have
\[
\tau_{a-0}(\tilde{Y}) - \text{sign}Y = \tau_{a-0}(\tilde{Y}') - \text{sign}Y' = \sigma_{[a,1]}(\tilde{B} \cup \tilde{B}_0) = \sigma_{[a,1]}(\tilde{B}),
\]
\[
\tau_{a+0}(\tilde{Y}) - \text{sign}Y = \tau_{a-0}(\tilde{Y}') - \text{sign}Y' = \sigma_{[a,1]}(\tilde{B} \cup \tilde{B}_0) = \sigma_{[a,1]}(\tilde{B}),
\]
because $\sigma_x(\tilde{B}_0) = 0$ for all $x \in [-1,1]$. 

Let $\kappa_1(\tilde{B})$ denote the $Q$-dimension of the kernel of the homomorphism $t - 1 : H_1(\tilde{B}; Q) \to H_1(\tilde{B}; Q)$. Then we have the following corollary:

**Corollary 4.3.** For every $a \in (-1,1)$,
\[
|\sigma_{[a,1]}(\tilde{B})| - \kappa_1(\tilde{B}) \leq |\text{sign}Y| + \hat{\beta}_2(Y) \leq 2\hat{\beta}_2(Y).
\]

**Proof.** In the proof of Theorem 4.2, we have
\[
\sigma_{[a,1]}(\tilde{B}) + \text{sign}Y = \tau_{a+0}(\tilde{Y}) = \tau_{a+0}(\tilde{Y}').
\]
On the other hand, in [4, Theorem 1.6], it is shown that
\[
|\tau_{a+0}(\tilde{Y}'))| - \kappa_1(\tilde{Y}') \leq \hat{\beta}_2(Y').
\]
Since $\hat{\beta}_2(Y') = \hat{\beta}_2(Y)$ and $\partial \tilde{Y}' = \tilde{B} \cup \tilde{B}_0$ with $\sigma_{[a,1]}(\tilde{B}_0) = \kappa_1(\tilde{B}_0) = 0$, we have the desired inequalities. 

5. **Loose embedding**

Let $M'$ be a compact connected oriented 3-manifold $M'$, and $U$ a possibly non-compact connected oriented 4-manifold. We say that an embedding $k' : M' \to U$ is *loose* if the kernel $K(M') = \ker(k'_* : H_2(M'; Z) \to H_2(U; Q)) \neq 0$. It is known that if the boundary $\partial M'$ of $M'$ is $\emptyset$ or connected, then every indivisible $x \in K(M')$ is represented by a closed connected oriented surface $F$ in $M'$ which we call a *null-surface* of the loose embedding $k'$ (see [6]). Then we have $sk'_*[F] = 0$ in $H_2(U; Z)$ for a positive integer $s$, which is assumed to be taken to be the smallest positive integer.

We consider a loose embedding $k^0 : M^0 \to U$ for $M^0 \in M^0$ which is regarded as the inclusion map $k^0 : M^0 \subset U$, and $F$ as a null-surface of $k^0$. We use the following lemma:

**Lemma 5.1.** Assume that for a tubular neighborhood $N_F$ of $F$ in $U$, there is a compact connected oriented 3-manifold $V$ in $\text{cl}(U \setminus N_F)$ such that $[\partial V] = s[F]$ in $H_2(N_F; Z)$. 

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Proof. Choose a compact connected 4-submanifold $U'$ of $U$ with $N_F \subset M^0 \times [-1,1] \subset U' \subset U$ and $M^0 \times 0 = M^0$ such that $K(M^0) = \ker(k^0_p : H_2(M^0;Z) \to H_2(U';Q))$ and $sk^0_p[F] = 0$ in $H_2(U';Z)$. Let $E' = cl(U' \setminus N_F)$. Then there is an indivisible element $z \in H_3(E', \partial N_F; Z) = H_3(U', N_F; Z)$ with $\partial_s z = s[F] \in H_2(N_F; Z)$ under $\partial_s : H_3(U', N F; Z) \to H_2(N_F; Z)$. Since $H_3(E', \partial N_F; Z) = H^1(E', \partial U'; Z)$, we have a compact oriented 3-manifold $V''$ in $E' \subset cl(U \setminus N_F)$ such that $z = [V''] \in H_3(E', \partial N_F; Z)$, and $\partial V'' = s''F'' \subset \partial N_F$ for a closed connected surface $F''$ and a factor $s'' > 0$ of $s$ such that $[\partial V''] = s''[F''] = s[F] \in H_2(N_F; Z)$. Replace $V''$ by a connected non-closed component $V$ of $V''$. Then we still have $[\partial V] = s''[F''] = s[F] \in H_2(N_F; Z)$.

Let $E_M = cl(U \setminus M^0 \times [-1,1]) \subset E = cl(U \setminus N_F)$. For a null-surface $F$ of a loose embedding $k^0 : M^0 \subset U$, we define a homomorphism

$$\gamma : H_1(E_M;Z) \xrightarrow{i_*} H_1(E;Z) \xrightarrow{\text{Int}} Z,$$

where $i_*$ is a natural homomorphism and $\text{Int}_V$ is defined by the identity $\text{Int}_V(x) = \text{Int}(x,V)$ for $x \in H_1(E;Z)$. We have the following lemma:

Lemma 5.2. $i_*$ and $\text{Int}_V$ are onto, so that $\gamma$ is onto.

We call $\gamma$ a null-epimorphism (associated with an null-surface $F$) of a loose embedding $k^0$.

Proof. Since $M^0 \setminus F$ is connected, every simple loop $l$ in $U \setminus F$ meeting $M^0$ transversely is deformed in $U \setminus F$ into a simple loop $l'$ in $U \setminus M^0$. Hence, $i_*$ is onto. Then we have the $Q$-linking number $\text{Link}_Q(F,m) = +1$ for a meridian $m$ of $F$ in $\partial N_F$ and hence we see that $m$ meets $V$ with the intersection number $s$ in $E$. Since $V$ is connected, $m$ is used to construct a simple loop $m'$ in $E$ meeting $V$ transversely at just one point. Hence, $\text{Int}_V$ is onto.

We also need the following lemma:

Lemma 5.3. Every null-epimorphism $\gamma : H_1(E_M;Z) \to Z$ of a loose embedding $k^0 : M^0 \to U$ is end-trivial.

Proof. The infinite cyclic covering $\tilde{E}$ induced from the epimorphism $\text{Int}_V$ is constructed from the infinite copies of $cl(E \setminus V \times [0,1])$ by attaching them along the infinite copies of a bi-collar $V \times [0,1]$ of $V$ in $E$. Thus, the restriction of $\text{Int}_V$ to the non-compact part $cl(E \setminus V \times [0,1])$ is the 0-map. Since the infinite cyclic covering $\tilde{E}_M \to E_M$ induced from $\gamma$ is a restriction of the infinite cyclic covering $\tilde{E} \to E$, we see that $\gamma$ is end-trivial.
Let $\alpha$ be the reflection on the double $DM^0(= \partial E_M)$ of $M^0$ exchanging the two copies of $M^0$ orientation-reversely. A meridian $m$ of $F$ in $M^0 \times [-1, 1]$ is deformed in $M^0 \times [-1, 1]$ into a loop $m'$ in $DM^0 = \partial E_M$ with $\alpha(m') = -m'$. Since $\text{Int}_V([m]) = s$, the following lemma is directly obtained:

**Lemma 5.4.** We have $\tilde{\gamma}(x_F) = s$ and $\alpha_*(x_F) = -x_F$ for the element $x_F = [m'] \in H_1(\partial E_M; Z)$ and the restriction $\tilde{\gamma} : H_1(DM^0; Z) \to Z$ of $\gamma$.

**Corollary 5.5.** If $s$ is odd, then the $Z_2$-reduction $\tilde{\gamma}_2 : H_1(DM^0; Z) \to Z_2$ of $\tilde{\gamma}$ is not $\alpha$-invariant.

A homomorphism $\tilde{\gamma} : H_1(DM^0; Z) \to Z$ satisfying the conclusion of Corollary 5.5 is called a $Z_2$-asymmetric homomorphism in [4, 8].

**Proof.** We can write $x_F$ as $x' - \alpha_*(x')$ for the element $x' \in H_1(DM^0; Z)$ represented by a loop in $M^0$. Then

$$\tilde{\gamma}(x_F) = \tilde{\gamma}(x') - \tilde{\gamma} \alpha_*(x') = s \equiv 1 \pmod{2},$$

which shows that $\gamma_2$ is not $\alpha$-invariant. \qed

6. Completion of the proof of Theorem 3.3

Throughout this section, we make the proof of the remaining part of Theorem 3.3.

**Completion of the proof of (1).** For any positive integers $n$, $c$, we take $n$ knots $K_i (1 \leq i \leq n)$ whose signatures $\sigma(K_i) (1 \leq i \leq n)$ have the condition that

$$|\sigma(K_i)| > 2c \quad \text{and} \quad |\sigma(K_i)| > \sum_{j=1}^{i-1} |\sigma(K_j)| + 2c \quad (i = 2, 3, \ldots, n).$$

Let $M_i = \chi(K_i, 0)$ and $M = M_1 \# M_2 \# \ldots \# M_n$. We call $M$ a $c$-efficient 3-manifold of rank $n$. The following calculation is made in [4, Lemma 1.3]:

**(6.1.1)** Every $c$-efficient 3-manifold $M$ of any rank $n$ has

$$|\sigma_{(-1,1)}(\overline{DM^0})| > 2c$$

for every $Z_2$-asymmetric homomorphism $\hat{\gamma} : H_1(DM^0; Z) \to Z$.

Suppose that a punctured universe $U$ has

$$\hat{\beta}_2(U) = c < +\infty, \quad \delta_0(U) = b < +\infty, \quad \rho_0(U) = b' < +\infty.$$
Let $M$ be a $c$-efficient 3-manifold of any rank $n > b + b'$. Suppose that $M^0$ is embedded in $U$. For the inclusion $k^0 : M^0 \subset U$, the kernel
\[ K(M^0) = \ker[k^0 : H_2(M^0; \mathbb{Z}) \to H_2(U; \mathbb{Q})] \]
is a free abelian group of rank $d > b'$. Then there is a basis $x_i$ $(i = 1, 2, \ldots, n)$ of $H_2(M^0; \mathbb{Z})$ such that $x_i$ $(i = 1, 2, \ldots, d)$ is a basis of $K(M^0)$. Since $\rho_0(U) = b' < d$, we can find an indivisible element $x$ in the basis $x_i$ $(i = 1, 2, \ldots, d)$ such that the multiplied element $rx$ for an odd integer $r$ is represented by the boundary cycle of a 3-chain in $U$. Taking a closed connected oriented surface $F$ in $M^0$ representing $x$, we have a null-epimorphism $\gamma : H_1(E_M; \mathbb{Z}) \to \mathbb{Z}$ (associated with an null-surface $F$) of the loose embedding $k^0$ whose restriction $\hat{\gamma} : H_1(DM^0; \mathbb{Z}) \to \mathbb{Z}$ is a $\mathbb{Z}_2$-asymmetric homomorphism. Then we obtain from (6.1.1) a contradiction that 
\[ 2c < |\sigma_{(-1,1)}(\widetilde{DM^0})| \leq 2c \]
because $\hat{\beta}_2(E_M) \leq \hat{\beta}_2(U) = c$ and $\kappa_1(DM^0) = 0$. Thus, at least one of $\hat{\beta}_2(U)$, $\delta_0(U)$, $\rho_0(U)$ must be $+\infty$. $\square$

**Completion of the proof of (2).** Let $U$ be a type 1 universe. We always have $\hat{\beta}_1(U) \geq 1$. Since $U$ is also a punctured universe, at least one of $\hat{\beta}_2(U)$, $\delta_1(U)$, $\rho_1(U)$ must be $+\infty$ by (1). Suppose that a type 1 universe $U$ has 
\[ b = \hat{\beta}_2(U) < +\infty, \ c = \delta_1(U) < +\infty, \ s = \hat{\beta}_1(U) < +\infty. \]

Then we show that there is a 3-manifold $M$ which is not type 1 embeddable in $U$. Let $\hat{H}_1(U; \mathbb{Z}) = Z^s$. Let $U_u$ $(u = 1, 2, \ldots, 2^s - 1)$ be the connected double coverings of $U$ induced from the epimorphisms $Z^s \to Z_2$. Let $M_u$ be the subset of $M$ consisting of $M$ such that a type 1 embedding $k : M \to U$ is trivially lifted to $k_u : M \to U_u$. Since every type 1 embedding $M \to U$ lifts to $U_u$ trivially for some $u$, we see that 
\[ \bigcup_{u=1}^{2^s-1} M_u = M. \]

Let $U'$ be a compact 4-submanifold of $U$ such that $U'' = \text{cl}(U \setminus U')$ is trivially lifted to $U_u$ for all $u$. Let $U'_u$ and $U''_u$ be the lifts of $U'$ and $U''$ to $U_u$. Let 
\[ b' = \max\{\beta_2(U'_u)| u = 1, 2, \ldots, 2^{s-1}\}. \]

(6.2.1) $\text{rank}(\text{im}(k_u),_s) \leq b + b'$ for any $u$.

**Proof of (6.2.1).** Let $K(M) = \ker(k_u : H_2(M; \mathbb{Z}) \to H_2(U; \mathbb{Q}))$. Let $F_j$ $(j = 1, 2, \ldots, m)$ be a system of closed connected surfaces representing a basis for $K(M)$. 20
Let $V_j$ be a compact oriented 3-manifold in $U$ such that $\partial V_j = r_j k(F_j)$ for a positive integer $r_j$. Let $V''_j = V_j \cap U''$ be a compact orientable 3-manifold. Then $\partial V''_j = \hat{V}'_j \cup r_j F''_j$ where $F''_j = k(F_j) \cap U''$ and $V'_j = V_j \cap \partial U'$. Since $V''_j$ is trivially lifted to $U_u$, we see that the 2-cycle $k_u(F_j)$ is $Q$-homologous to the rational 2-cycle

$$\frac{1}{r_j} [r_j k_u(\text{cl}(F_j \backslash F''_j)) + k_u(\hat{V}'_j)]$$

in $U'_u$ for all $j$. This means that $(k_u)_*(K(M))$ is in the image of the natural homomorphism $H_2(U'_u; Q) \to H_2(U_u; Q)$. Hence we have $\text{rank}( (k_u)_*(K(M)) ) \leq b'$. Since $\text{rank}(\text{im}(k_u)) \leq b$, we have $\text{rank}(\text{im}(k_u)_*) \leq b + b'$.

For any positive integers $n$, $c$, we take $n$ knots $K_i$ $(1 \leq i \leq n)$ whose local signatures $\sigma_{\{a,1\}}(K_i)$ $(1 \leq i \leq n)$ have the condition that there are numbers $a_i \in (-1, 1)$ $(i = 1, 2, \ldots, n)$ such that

$$|\sigma_{\{a,1\}}(K_1)| > 2c, \quad |\sigma_{\{a,1\}}(K_i)| > \sum_{j=1}^{i-1} |\sigma_{\{a,1\}}(K_j)| + 2c \quad (i = 2, 3, \ldots, n)$$

for every $a \in (-1, 1)$ (see [7]). Let $M_i = \chi(K_i, 0)$ be the 0-surgery manifold along $K_i$, and $M = M_1 \# M_2 \# \cdots \# M_n$. We call $M$ a strongly $c$-efficient 3-manifold of rank $n$. For this 3-manifold $M$, we say that a homomorphism $\hat{\gamma} : H_1(DM^0; Z) \to Z$ is symmetric if $\hat{\gamma}|_{\alpha(M_i)} = \pm \hat{\gamma}|_{\bar{M_i}}$ for all $i$, where $\alpha$ is the reflection on the double $DM^0$. Otherwise, $\hat{\gamma}$ is said to be an asymmetric homomorphism. The following calculation is also seen from [4, Lemma 1.3]:

(6.2.2) For every strongly $c$-efficient 3-manifold $M$ of any rank $n$ and every asymmetric homomorphism $\hat{\gamma} : H_1(DM^0; Z) \to Z$, we have a number $a \in (-1, 1)$ such that

$$|\sigma_{\{a,1\}}(\text{DM}^0)| > 2c.$$

For example, if $M$ is constructed from the knots $K_i$ $(i = 1, 2, \ldots, n)$ with $K_i$ the $ic^+_j$-fold connected sum of the trefoil knot for any fixed integer $c^+_j > c$, then $M$ is a strongly $c$-efficient 3-manifold of rank $n$. We show that every strongly $c$-efficient 3-manifold $M$ of rank $> b + b'$ is not type 1 embedded in $U$. Suppose that $M$ is type 1 embedded in $U$ and lifts trivially in $U_u$. Let $U(M)$ and $U_u(M) = U(M) \cup tU(M)$ be the 4-manifolds respectively obtained from $U$ and $U_u$ by splitting along $M$, where $t$ denotes the double covering involution. Let $\partial U(M) = M_0 \cup -M_1$ and $\partial U_u(M) = M_0 \cup -M_2$, where $M_0, M_1, M_2$ are the copies of $M$. Since the natural homomorphism $H_2(M; Z) \to H_2(U; Q)$ is not injective, there is a non-zero element $[C] \in H_2(M; Z)$ such that $C = \partial C$ for a 3-chain $D$ in $U_u$ and $C = \partial D_*$ for a 3-chain $D_*$ in $U$ which is

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the image of $D$ under the covering projection $U_u \to U$. The 3-chains $D$ and $D_\ast$ define 3-chains $D', D''$ and $D'''$ in $U(M)$ such that

$$\partial D' = C''_1 - (C_0 + C'_0),$$
$$\partial D'' = C'_1 - C''_0,$$
$$\partial D''' = (C'_1 + C''_1) - (C_0 + C'_0 + C''_0)$$

for some 2-cycles $C_u, C'_u, C''_u$ in $M_u$ ($u = 0, 1$) (see Fig. 3). Since $\hat{\beta}_2(U(M)) \leq c$, the non-zero end-trivial homomorphism $\hat{\gamma} : H_1(DM^0; Z) \to Z$ defined by any 3-chain in $U(M)$ must be symmetric by Corollary 4.3 and (6.2.2) because every strongly $c$-efficient 3-manifold $M$ has $\kappa_1(DM^0) = 0$. Let

$$[C] = \sum_{i=1}^m a_i x_i, \quad [C'] = \sum_{i=1}^m a'_i x_i, \quad [C''] = \sum_{i=1}^m a''_i x_i$$

in $H_1(M; Z)$ with $x_i$ a generator of $H_1(M_i; Z) \cong Z$. By the symmetry conditions on $D', D''$ and $D'''$, we have the following relations:

$$a''_i = \varepsilon_i (a_i + a'_i), \quad a'_i = \varepsilon'_i a''_i, \quad a'_i + a''_i = \varepsilon''_i (a_i + a'_i + a''_i),$$

where $\varepsilon_i, \varepsilon'_i, \varepsilon''_i = \pm 1$ for all $i$. Then we have

$$(1 + \varepsilon'_i)a''_i = \varepsilon''_i (\varepsilon_i + 1)a''_i.$$

If $\varepsilon_i \varepsilon'_i = -1$, then we have $a''_i = a'_i = a_i = 0$ for all $i$. If $\varepsilon_i \varepsilon'_i = 1$, then we have $a_i = 0$ for all $i$. Hence we have $[C] = 0$, contradicting that $[C] \neq 0$. Hence $M$ is not type 1 embeddable in $U$.

Figure 3: A situation of 3-chains
Completion of the proof of (3). Let $U$ be a type 2 universe. Suppose that

$$\beta_2(U) = c < +\infty, \quad \delta_2(U) = b < +\infty.$$  

Let $M \in \mathbb{M}$ be a $c$-efficient 3-manifold of any rank $n > b$. Let $k : M \subset U$ be a type 2 embedding which is a loose embedding. Let $U'$ and $U''$ be the 4-manifolds obtained from $U$ by splitting along $M$. For $U'$ or $U''$, say $U'$, we have a null-surface $F$ in $M$ and a positive (not necessarily odd) integer $r$ such that the natural homomorphism $H_2(M; \mathbb{Z}) \to H_2(U'; \mathbb{Z})$ sends $r[F]$ to 0. Taking the minimal positive integer $r$, we have a compact connected oriented 3-manifold $V$ in $U'$ with $\partial V = rF$. This 3-manifold $V$ defines an end-trivial epimorphism $\varphi : H_1(U'; \mathbb{Z}) \to \mathbb{Z}$ whose restriction $\varphi : H_1(M; \mathbb{Z}) \to \mathbb{Z}$ is equal to $r\hat{\gamma}_F$ for the epimorphism $\hat{\gamma}_F : H_1(M; \mathbb{Z}) \to \mathbb{Z}$ defined by $F$. Let $\bar{M}$ and $\bar{M}_F$ denote the infinite cyclic coverings of $M$ induced from $\hat{\gamma}$ and $\hat{\gamma}_F$, respectively. Let $(1 \leq i_1 < i_2 < \cdots < i_s(\leq n))$ be the enumeration of $i$ such that the $\mathbb{Z}_2$-reduction of $\hat{\gamma}_F$ restricted to the connected summand $M_i$ of $M$ is non-trivial. By a calculation made in [4, Lemma 1.3], we have

$$\sigma_{(-1,1)}(\bar{M}_F) = \sum_{j=1}^s \sigma(K_{i_j}),$$

so that $|\sigma_{(-1,1)}(\bar{M}_F)| > 2c$. By [4, Lemma 1.3], we also have

$$\sigma_{(-1,1)}(\bar{M}_F) = \sigma_{(a,1)}(\bar{M})$$

for some $a \in (-1, 1)$. Then, since $\hat{\beta}_2(U') \leq \hat{\beta}_2(U) = c$ and $\kappa_1(\bar{M}) = 0$, we obtain from Corollary 4.3 a contradiction that

$$2c < |\sigma_{(a,1)}(\bar{M})| \leq 2c.$$  

Hence $\hat{\beta}_2(U)$ or $\delta_2(U)$ must be $+\infty$.  

Completion of the proof of (4). Let $U$ be a universe. Assume that

$$\hat{\beta}_2(U) = c < +\infty \quad \text{and} \quad \delta(U) = b < +\infty.$$  

By the proof of (3), for every infinite family of strongly $c$-efficient 3-manifolds of infinitely many ranks $n$ any member must be type 1 embeddable to $U$. By the proof of (2), we have $\rho(U) = +\infty$ and $\hat{\beta}_1(U) = +\infty$.  

Completion of the proof of (5). Since a full universe $U$ is a type 1 and type 2 universe, the desired result follows from (2) and (3).  

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References


