1. What Is Knot Theory? Why Is It In Mathematics?

In this chapter, we briefly explain some elementary foundations of knot theory. In 1.1, we explain about knots, links and spatial graphs together with several scientific examples. In 1.2, we discuss diagrams of knots, links and spatial graphs and equivalences on knots, links and spatial graphs. Basic problems on knot theory are also explained there. In 1.3, a brief history on knot theory is stated. In 1.4, we explain how the first non-trivial knot is confirmed. In 1.5, the linking number useful to confirm a non-trivial link and the linking degree which is the absolute value of the linking number are explained. In particular, we show that the linking degree is defined directly from an unoriented link. In 1.6, some concluding remarks on this chapter are given. In 1.7, some books on knot theory are listed as general references.

1.1 Knots, links, and spatial graphs

A *knot* is a tangled string in Euclidean 3-space R^3 which is usually considered as a closed tangled string in R^3, and a *link* is the union of some mutually disjoint knots (see Figure 1). The AYATORI game (= Cat's cradle play) let us know that a given knot can be deformed into various forms and we feel that it is a difficult problem to judge whether any given two knots are actually the same knot or not.

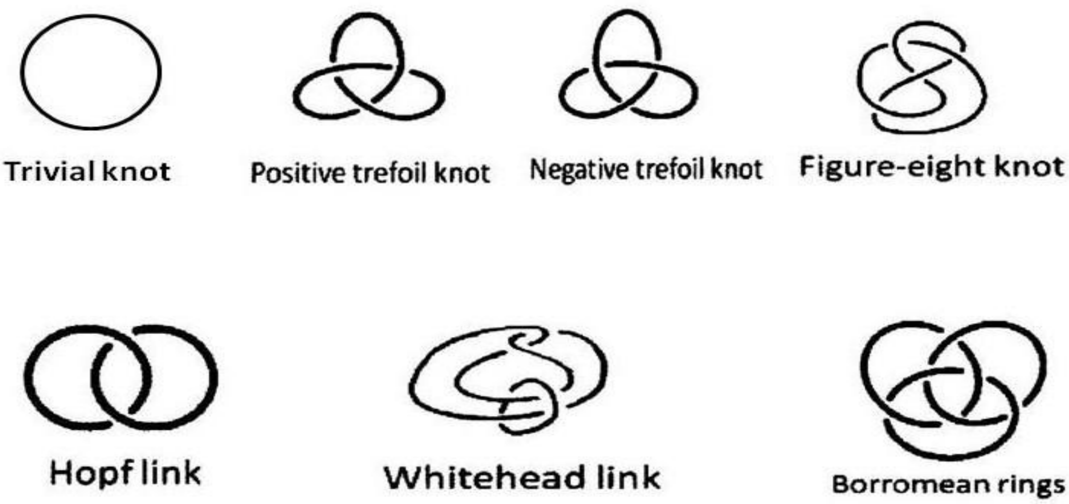

Figure 1: Knots and links

A *spatial graph* is the union Γ of some strings with endpoints in R^3 which are mutually disjoint except the endpoints. A *vertex* of a spatial graph Γ is a point in Γ gathering more than 3 strings and an *edge* of Γ is a connected component of the strings obtained from Γ by cutting along all the vertices of Γ. For example, Kinoshita's θ-curve in Figure 2 is a spatial graph with two vertices and three edges.

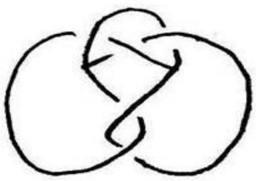

Figure 2: Kinoshita's θ-curve

To find out a knot or a link phenomenon in natural science, at first a setting of an object which one can consider as a string becomes important. Here, we show some examples about such objects.

Example 1: It is possible to think a chain (see Figure 3) as one string if roughly seeing, but also as a link which are twined round one after another like a string if a little more minutely seeing.

Figure 3: A chain

Example 2： A *3-braid* is a knitting of three strings (see, for example, Figure 4). Since this knitting pattern is known to be used in a pottery of the JOMON period (an ancient time) in Japan, we see that the people of the JOMON period might understand the fact that the 3-braid is a technology of making a long, strong string like a rope from some short strings like straws. Then, by joining a and a´, and b and b´ in Figure 4, and then by deforming it (without changing near the ends), we can make AWABI MUSUBI (= the abalone knot) of "MIZUHIKI" in Figure 5 used for the custom of the present in Japan from ancient times. Also, although it is a little more difficult, we can also make the same knot by joining b and b´, and c and c´ in Figure 4, and then by deforming it (without changing near the ends). In this way, the knot is also an interested study object for cultural anthropology.

Figure 4 : A 3-braid

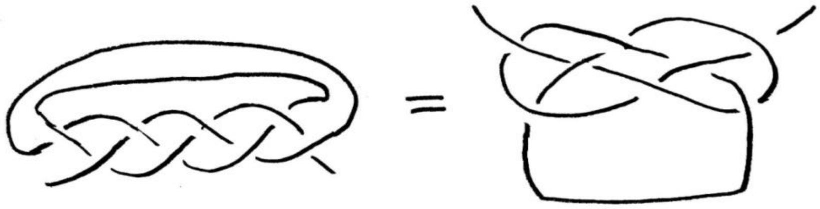

Figure 5 : AWABI MUSUBI(=abalone knot) of "MIZUHIKI"

Example 3 : Assume that there are n particles moving without colliding in the plane with a time parameter. Then the track of the motion forms an *n-braid* in the three-dimensional space which is the product space of the plane and the time axis. In the the case of 3 particles, we have a 3-braid like the example in Figure 4. Every knot and link can cause from an n-braid by taking the closure.

Example 4 : When we consider DNA as one string long rope, there is one which becomes a closed curve called a *DNA knot* (see Figure 6).

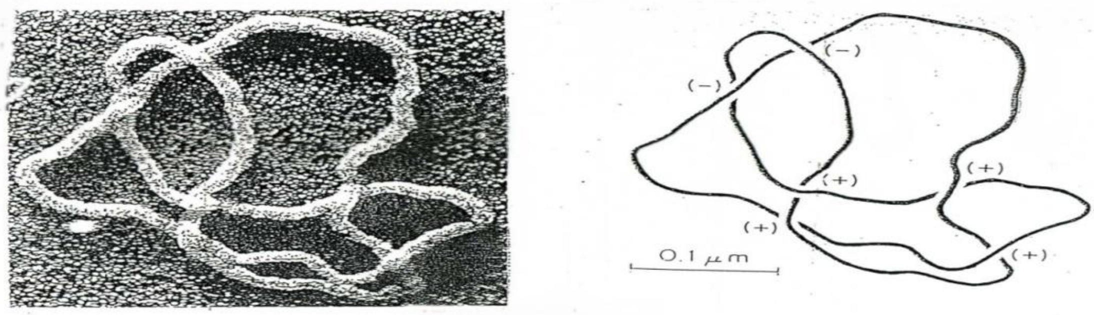

Figure 6 : A DNA knot (Acknowledgement :Professor N. R. Cozzarelli)

Example 5 : A *molecular graph* on a molecule in chemistry is a spatial graph whose vertices correspond to the atoms in the molecule and whose edges express the combination data between the atoms by bonds. Topology of molecular graphs has begun to attract attention in researches of the synthetic-chemistry.

Example 6 : A protein molecule is considered to be a string consisting of amino acid bases. Some protein molecules such as a prion protein of the Creutzfeldt-Jacob disease and an amyloid β protein of the Alzheimer's disease appear to be more or less related to knot theory.

Example 7 : A large scale structure on the cosmos is recently known to the astrophysicists.

Example 8: The seismometer is a machine which draws the track of an earthquake-motion of an observation point as a spatial curve, called an *earthquake curve* with the time parameter. The analysis of this earthquake curve can be considered as knot theory in the wide sense.

1.2 Diagrams and equivalence on knots, links, and spatial graphs

Figure 7 : A crossing

A knot is in the three-dimensional space and we think that it is made of a very thin string. We present it by a plane curve with only double crossings as they are shown in Figure 7, which we call a *knot diagram* or simply a *diagram*. For a link, it is similarly presented and called a *link diagram* or simply a *diagram*. By knots and links, we mean their diagrams unless making confusion. A spatial graph is also presented to the plane with only double points on the edges which we call a *spatial graph diagram* (see Figure 3). For two knots, we say that they are the *same knot* or *equivalent knots* if we can deform them into the same shape in the manner of AYATORI game (= Cat's cradle play), i.e., by a finite number of Reidemeister moves I-III in Figure 8. For two links, we say similarly that they are the *same link* or *equivalent links* if we can deform them into the same shape in the manner of AYATORI game, namely by a finite number of Reidemeister moves I-III. For two spatial graphs, we say that they are the *same graph* or *equivalent graphs* if we can deform them into the same shape in the manner of AYATORI game or in other word by a finite number of Reidemeister moves I-V in Figure 8. A knot is called a *trivial knot* if it is equivalent to a circle in the plane like a rubber band. Also, a link is called a *trivial link* if it is equivalent to an union of separated trivial knots.

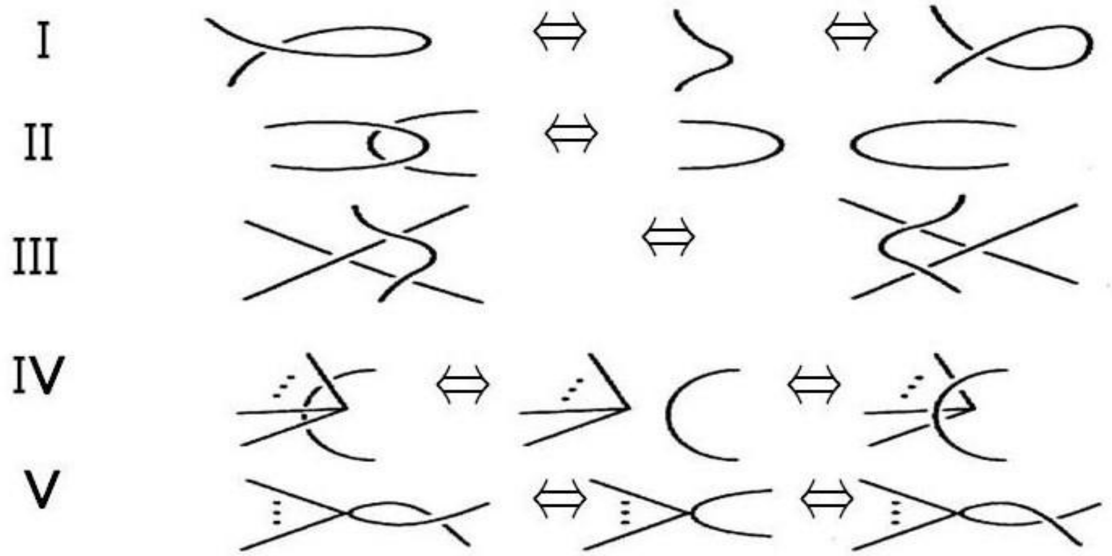

Figure 8: Reidemeister moves

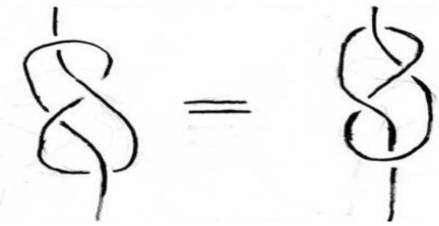

Figure 9： The same arc knot

In daily life, we normally think of an arc knot in Figure 9. At that point, we consider that the end points extend virtually long and endless. If we can deform them into the same shape in the manner of AYATORI game without moving the parts that are extended, then they are considered to be the same knot. A link with two end points can be considered similarly to the case of an arc knot (See Figure 10). If a link has more than two end points, then such a link no longer has any meaning unless the data extending the end points are definitely given.

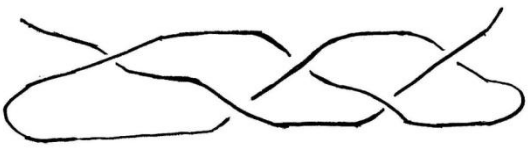

Figure 10：A link with two end points

The main purpose of knot theory to solve the following two problems (which are related to each other):

Equivalence Problem : Given two knots (or links, spatial graphs), determine whether or not they are equivalent.

Classification Problem : Enumerate all the knots (or links, spatial graphs) up to equivalences.

To solve these problems, it is very important to develop *topological invariants*, (namely, quantities which are invariant under the Reidemeister moves) of knots, links and spatial graphs.

1.3 A brief history of knot theory

Knot theory is now believed that a scientific study to be associated with the atomic theory of vortex atoms in ether around the end of the nineteenth century. However, it is can be traced back to a note by J. B. Listing, a disciple of Gauss in 1849. In the note, it has been that the mirror image of the figure-eight knot is the same knot (Figure 11). Up until the 1930s, important researches were made by K. Reidemeister and H. Seifert in Germany and J. W. Alexander in U.S.A., etc. From 1940 until the 1970's, one may say that basic mathematical theory on knot theory was established with R. H. Fox in U.S.A. as a center. In Japan, from around 1960, H. Terasaka, T. Homma, S. Kinoshita (later, moving to U.S.A.) and K. Murasugi (later, moving to Canada), F. Hosokawa, etc. have begun to make contributions to knot theory. From around 1980, knot theory came to attention not only in almost all areas of mathematics, but also in the fields of science that will be cutting-edge researches, such as gene synthesis, quantum statistical mechanics, soft matter physics, biochemistry, polymer network, applied chemistry···. The international conference "Knot theory and related topics" received the world's first was held at Osaka as a satellite conference of ICM Kyoto in 1990, from whose proceedings "Knots 90" (A. Kawauchi, e.d., Walter de Gruyter, 1992) one may feel a fever of an expansion of knot theory.

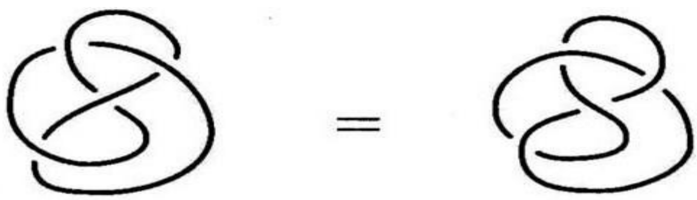

Figure 11 : Equivalence of the figure-eight knot and the mirror image

1.4 The first non-trivial knots

Of knots and related concerns that are normally used casually in everyday life, let me say here the simplest mathematical proof that there is a non-trivial knot. (This proof uses an argument of the 3-colorability of a knot which is well-known to the experts of knot theory as, algebraically, the representation theory of the knot group to the symmetric group of degree three or as, topologically, the theory of three-fold branched coverings of a knot.) Given a knot diagram, we color all the edges connecting the crossings by using three colors (e.g., red, blue and yellow) by imposing in the vicinity of every crossing the condition that we color the upper arc by the same color and color the lower two edges by the same color or different colors so that one color or three colors are used in the vicinity of every crossing. By this method, we can always color every knot diagram by one color. For some knot diagrams, we can also color them by 3 colors (see Figure 12). Such a knot is called a *3-colorable knot*.

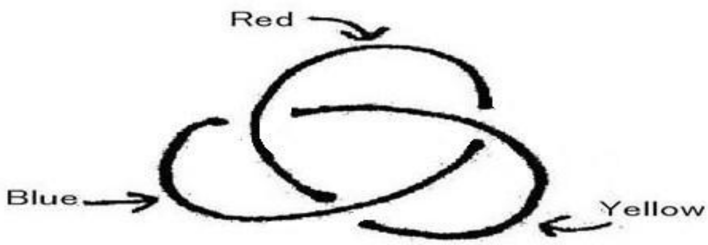

Figure 12：The trefoil knot is 3-colorable

Theorem： Every 3-colorable knot is a non-trivial knot.

The reason why this is true is because we can easily check that any knot diagram D' transformed from a 3-colorable knot diagram D by Reidemeister moves I, II, III (see Figure 8) is also 3-colorable. We recommend to confirm this fact for various knots.

1.5 Understanding the linking number

The linking number of a link of two oriented knot components is the most fundamental topological invariant in knot theory. However, because it takes a value in the integers, this invariant cannot de defined without the notion of a negative integer. In this section, we first define the linking number. Next, we introduce the absolute value of the linking number, called the linking degree and computable directly from a link diagram without using the notion of a negative integer.

Figure 13： A link diagram

1.5.1. Linking number

We shall explain how to define the linking number of an oriented link. We consider a link L of two oriented knot components as in Figure 13. Every crossing between the distinct knot components coincides with one of the four crossings shown in Figure 14, where the signs of two crossings in the left hand side are defined to be +1 and the signs of two crossings in the right hand side are defined to be -1. Let m be the sum of signs of all the crossings between the distinct knot components. We see that this integer m is always even and does not change under Reidemeister moves I, II, III (see Figure 8), namely m is a topological invariant. Then we define the *linking number* v of the link L to be the integer m/2. When we reverse the orientation of one component of L, the sign of every crossing between the distinct components is changed so that the linking number v of L is changed into –v. For example, the link diagram in Figure 13 has the two (+1)-crossings and the four (-1)-crossings and the linking number v is given by

$$v = (2 - 4) / 2 = -1.$$

Figure 14： The sign of a crossing

1.5.2. Linking degree

We consider a diagram of an unoriented link L with knot components J and K as in Figure 15. To distinguish between J and K, we denote K by a bold line. We attach an orientation to one of the components J and K, say J as shown in Figure 16. Every crossing between the oriented component J and the unoriented component K coincides with one of the two crossings in Figure 17. We construct a *meridian loop*, an oriented loop surrounding K one time in every crossing between J and K as in Figure 18. For example, we obtain the left-sided diagram in Figure 19 from Figure 16, from which we obtain the right-sided diagram in Figure 19 by sliding these meridian loops along K.

Figure 15： An unoriented diagram

Figure 16： A link diagram with the component J oriented

Figure 17： The situations of a crossing

We denote by (P,Q) a pair of the subsets of loops with the same orientations in the set of meridian loops around K obtained from the oriented knot J. Let p and q be the numbers of the elements in P and Q, respectively, where we take $p \leqq q$. Then let $n=q-p$, and the *linking degree* d of a link L consisting of the unoriented knot components J and K is defined by

$$d = n/2 = (q-p)/2.$$

For example, the linking degree d of the link L in Figure 15 is computed from Figure 19 to be

$$d = (4-2)/2 = 1.$$

If we take the reversed orientation on J, then the orientations of all the meridian loops are reversed and the difference n between the numbers of the meridian loops with the same orientations is unchanged, so that the linking degree d is independent of a choice of any orientation on J.

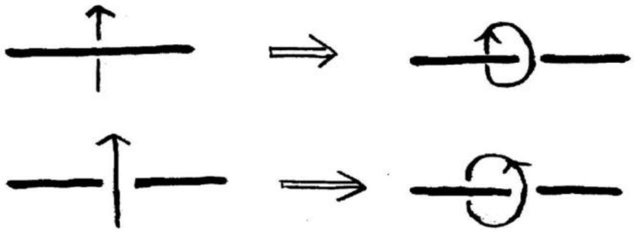

Figure 18: Constructing a meridian loop of K

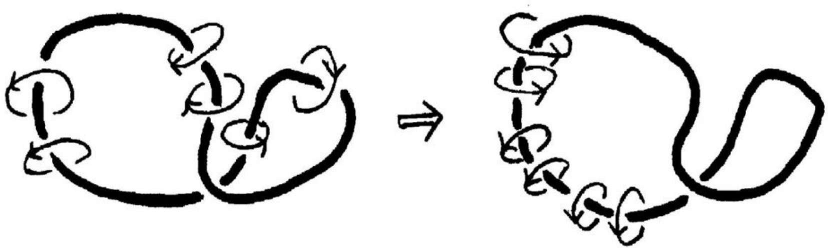

Figure 19

We show that the same number n is obtained as the difference of the numbers of the meridian loops with the same orientations even if we consider the meridian loops around J obtained from the component K oriented in any direction (instead of J). In fact, if we consider J and K as oriented knots, then we see that the orientation of the meridian loop is locally determined as it is shown in Figure 20, so that a pair of the numbers of the meridian loops around J with the same orientations obtained from the oriented knot K coincides with the pair (up to ordering) of the numbers of the meridian loops around K with the same orientations obtained from the oriented knot J. This implies that the linking degree d of a link L is a non-negative rational number with denominator 2 which is independent of choices of the components and the orientations of J, K.

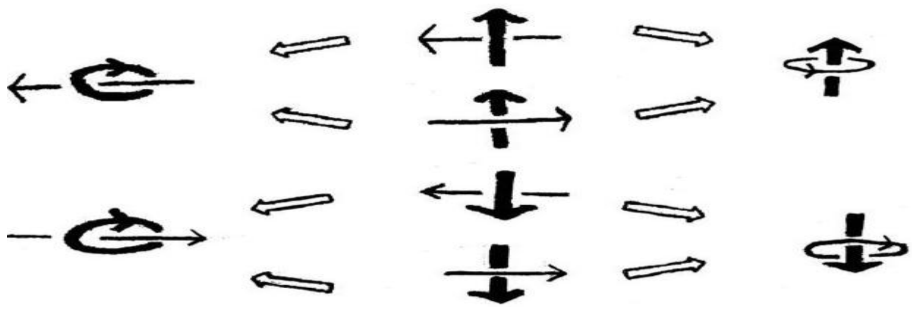

Figure 20: The orientation of a meridian loop is locally determined

We show the following (1) and (2).

(1) The linking degree d *is unchanged under Reidemeister moves I，II，III and hence it is a topological invariant.*

(2) The linking degree d *takes a value in the natural numbers or zero. When we orient a component J, the linking degree* d *is computed to be the difference of the numbers of the meridian loops with the same orientations around the other component K on the crossings of J which are upper than K, or the difference of the numbers of the meridian loops around K with the same orientations on the crossings of J which are lower than K.*

First, we show (1). The proof is made by considering a pair of the numbers of the meridian loops with the same orientations around K obtained from an oriented J. The numbers n and d are unchanged under Reidemeister moves except the moves relating to both J and K. In particular, they are unchanged by the Reidemeister move I. For the Reidemeister move II relating both J and K, we can see from Figure 21 that the numbers n, d are unchanged under the Reidemeister move II.

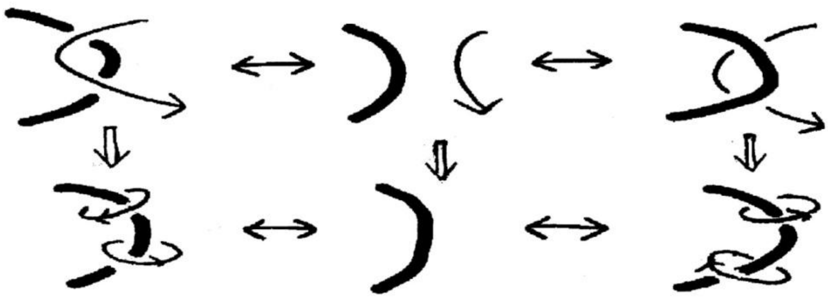

Figure 21: Reidemeister move II relating to J and K

For the Reidemeister move III relating both J and K, it is sufficient to consider the cases in the left-hand side of Figure 22, if necessary, by changing the roles of J and K. By examining the pictures in the right-hand side of Figure 22, we see also that the numbers n, d are unchanged under the Reidemeister move III, showing (1).

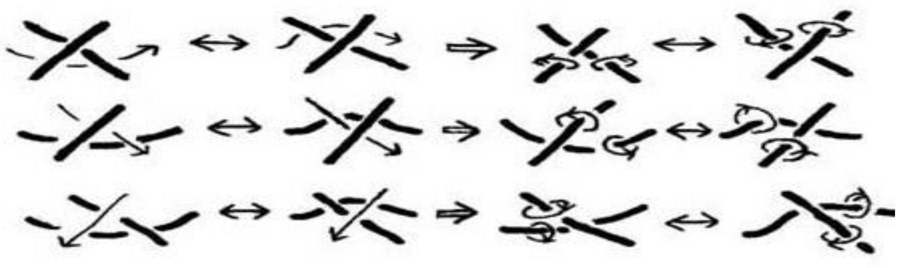

Figure 22: Reidemeister move III relating to J and K

Next, we show (2). We can see from Figure 20 and the definitions of the linking number and the linking degree that the linking degree d is the absolute value of the linking number v. Therefore, if we assume that the linking number v takes a value in the integers, we can see that the linking degree d takes a value in the natural numbers or zero. Here, we give a direct proof about it. Let (P,Q) be a pair of the subsets of loops with the same orientations in the set of meridian loops around K obtained from an oriented knot J. Let E and F be the subsets of P obtained from the crossings of J upper and lower than K, respectively. Let e and f be the numbers of the elements of E and F, respectively. Also, let G and H be the subsets of Q obtained from the crossings of J upper and lower than K, respectively, and g and f the numbers of the elements of G and H, respectively. Then for the numbers p, q of the elements of P and Q, where we assume $p \leqq q$, we have

$$p=e+f, \qquad q=g+h.$$

We change all the lower crossings of J into the upper crossings of J, so that all the crossings of J are upper than K by a crossing change operation in Figure 23. By this change, the numbers $p=e+f$ and $q=g+h$ of the meridian loops with the same orientations are changed into e+h and g+f, respectively (cf. Figure 18). Since now the knot diagram of J is upper than the diagram of K, we can move the knot diagrams of J and K by Reidemeister moves II, III so that they do not meet. By the topological invariance of the linking degree, we have

$$(e+h) - (g+f) = 0, \text{ namely } g-e = h-f.$$

Hence, we have

$$d = n/2 = (q-p)/2 = g-e = h-f$$

and (2) is proved.

Figure 23: A crossing-change operation

1.5.3. Computing examples of the linking degree using the meridian loops on the upper crossings

Here are some examples of computations on the linking degree.

Example 0. We introduce an orientation on J as in Figure 16 to compute the linking degree of a link in Figure 15. Then the meridian loops on the upper crossings belonging to J are the loops in Figure 24. Thus, we have d = 2-1 = 1.

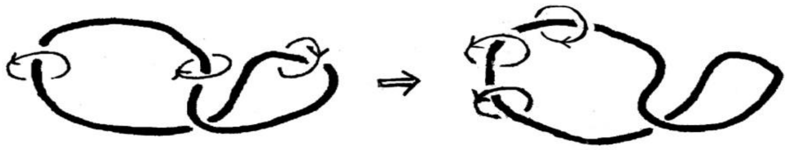

Figure 24: The meridian loops on the upper crossings of the diagram of Figure 16

Example 1 . The linking degree d of the Hopf link in Figure 25 is computed to be d = 1.

Figure 25: A computing process for the Hopf link

Example 2. The linking degree d of the Whitehead link in Figure 26 is computed to be d = 1-1 = 0.

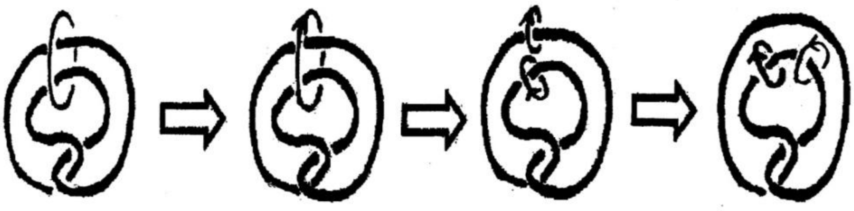

Figure 26: A computing process for the Whitehead link

Example 3. The linking degree d of a 2-braid link in Figure 27 is computed to be d = 2.

Figure 27: A computing process for a 2-braid link

Example 4. The linking degree d of the parallel link of a trefoil knot in Figure 28 is computed to be d = 3.

Figure 28: A computing process for the parallel link of a trefoil knot

Remark. The linking degree d of a twisted parallel link of a trefoil knot in Figure 29 is computed to be d = 3-3 = 0.

Figure 29: A twisted parallel link of a trefoil knot

Example 5. The linking degree d of a parallel link of the figure-eight knot in Figure 30 is computed to be d = 2-2 = 0.

Figure 30: A computing process for a parallel link of the figure-eight knot

1.6 Conclusion

It is an ultimate purpose of knot theory to clarify a topological difference of knot phenomena in mathematics and in science. In this study, a building power and a computational ability in mathematics are needed in addition to the intuition power having to do with a figure. We can watch a knot with eyes and our ability of space perception will be grown up by playing with it. Knot theory is a subject suitable for understanding nature deeply and desirable for learning in an early age.

1.7 References

There are many books on knot theory some of which are listed here. Among them, the first appearing book is “Knotentheorie” written in German in 1933 by K. Reidemeister.

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