On alternation numbers of links

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Abstract

We construct infinitely many hyperbolic links with x-distance far from the set of (possibly, splittable) alternating links in the concordance class of every link. A sensitive result is given for the concordance class of every (possibly, split) alternating link. Our proof uses an estimate of the τ-distance by an Alexander invariant and the topological imitation theory, both established earlier by the author.

Key words: Alternating link, x-distance, Alternation number, Concordance, Alternating Laurent polynomial, Semi-classical Alexander polynomial, Link
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1 Alternation number and τ-alternation number

In this paper, links are always oriented links in the oriented 3-sphere S^3 and a knot is regarded as a link of one component. An *alternating link* is a link with an alternating diagram, a link diagram such that an over-crossing and an under-crossing appear alternatively along every knot diagram component. Let $\mathbb{A}$ be the set of (possibly, splittable) alternating links. After the solution of the Tait flype conjecture on alternating links by W. W. Menasco and M. B. Thistlethwaite in [16], it became an important question to ask how a non-alternating link is "close to" or "far from" the set $\mathbb{A}$ under a suitable metric. The x-*distance* (or *Gordian distance*) $d^{\mathrm{x}}(L, L')$ between links L and L' with the same number of components is the minimal number of cross-changes transforming a diagram of L into a diagram of L'. A *zero-linking twist* of an oriented link L is an operation on links to obtain a link L' from L by a twist along a trivial knot O such that $L \cap O = \emptyset$ and the linking number $\mathrm{Link}(L, O) = 0$,

[1] Dedicating this paper to Professor Takao Matumoto on his sixtieth birthday.
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and the τ-*distance* $d^\tau(L, L')$ between links L and L' with the same number of components is the minimal number of zero-linking twists transforming L into L' (cf. [12]). For links L, L' with different numbers of components, we define $d^\tau(L, L') = d(L, L') = +\infty$. Since the crossing change is a zero-linking twist and every link diagram is transformed into a diagram of a trivial link by crossing changes, we have

$$0 \leqq d^\tau(L, L') \leqq d^{\mathrm{x}}(L, L') < +\infty$$

for all links L, L' with the same number of components. The x-distance and the τ-distance are metric functions on the set of links with r components for every $r \geqq 1$. An estimate of $d^\tau(L, L')$ was done in [12] in terms of local link signatures and a localized version of Nakanishi index (cf. [14]). For our use, the latter invariant will be useful and explained in Section 2. Here is an example showing that the τ-distance is distinct from the x-distance in general.

Example 1.1. For the 2-bridge knot 7_4 and the trivial knot O, we have $d^\tau(7_4, O) = 1 < 2 = d^{\mathrm{x}}(7_4, O)$. It is direct to see that $d^\tau(7_4, O) = 1$ and $d^{\mathrm{x}}(7_4, O) \leqq 2$. By a result of T. Kanenobu and H. Murakami [7], we have $d^{\mathrm{x}}(7_4, O) > 1$ and hence $d^{\mathrm{x}}(7_4, O) = 2$. For a link example, let L be a link obtained from the $(2, 2n)$-torus link for $n > 1$ by reversing the orientation of one component, and O^2 the trivial link of two components. Then we have $d^\tau(L, O^2) = 1 < n = d^{\mathrm{x}}(L, O^2)$.

We put the following definition:

Definition 1.2. The *alternation number* $\mathrm{alt}(L)$ and the τ-*alternation number* $\mathrm{alt}^\tau(L)$ of a link L are the numbers $d^{\mathrm{x}}(L, \mathbb{A})$ and $d^\tau(L, \mathbb{A})$, respectively.

It is direct to see that

$$0 \leqq \mathrm{alt}^\tau(L) \leqq \mathrm{alt}(L) < +\infty$$

for every link L. Two links $L_i\,(i = 0, 1)$ are *concordant*(= *link-cobordant*) if there is a smooth embedding $c : L \times [0, 1] \to S^3 \times [0, 1]$ for a closed oriented 1-manifold L such that $c(L \times i) = L_i$ $(i = 0, 1)$ (cf. [14]). The concordance relation is an equivalence relation on the set of links. Let $[L]$ be the *concordance class* of a link L, namely the set of links concordant to L. The *concordance-alternation number* $\mathrm{alt}[L]$ and the *concordance-τ-alternation number* $\mathrm{alt}^\tau[L]$ of a link L are defined by:

$$\mathrm{alt}[L] = \min_{L' \in [L]} \mathrm{alt}(L') \quad \text{and} \quad \mathrm{alt}^\tau[L] = \min_{L' \in [L]} \mathrm{alt}^\tau(L'),$$

which are uniquely determined by the concordance class $[L]$. In this paper, *we construct infinitely many hyperbolic links* L^* *in the concordance class of every link* L *such that* $\mathrm{alt}(L^*)$ *is equal to any previously given integer* $n \geqq \mathrm{alt}[L]$.

In the case of the concordance class of every (possibly, splittable) alternating link L, we can let this link L^* have the property that $\mathrm{alt}(L^*) = \mathrm{alt}^\tau(L^*) = d^\tau(L, L^*) = d^\times(L, L^*)$. These results will be given in Section 3.

2 Algebraic alternation number and algebraic τ-alternation number

Let $\Lambda = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ be the Laurent polynomial ring. A non-zero Laurent polynomial $f(t)$ is *alternating* if the coefficients of the Laurent polynomial $f(-t)$ in t are nonzero integers with the same sign. Let $\tilde{E}(L)$ be the infinite cyclic covering over the compact exterior $E(L)$ of an oriented link L in S^3 which is induced from the epimorphism $\gamma_L : \pi_1(E(L)) \to \mathbf{Z}$ sending each oriented meridian of L to $1 \in \mathbf{Z}$. Then $H_1(\tilde{E}(L))$ is naturally regarded as a finitely generated Λ-module, called the Λ-*module* of L and denoted by $H(L)$. The *torsion-Alexander polynomial* $A^T(L;t)$ of a link L is the zeroth characteristic polynomial of the Λ-torsion part $TH(L)$ of the Λ-module $H(L)$ of L (see [13]). To state it more explicitly, let M be a Λ-presentation matrix of size (q,p) for $TH(L)$ with $p \geqq q$. That is, let M be a matrix given by $\phi(x_1, x_2, \ldots, x_p) = (y_1, y_2, \ldots, y_q)M$ for a Λ-exact sequence

$$\Lambda^p \xrightarrow{\phi} \Lambda^q \to TH(L) \to 0 \quad (p \geqq q)$$

with respect to Λ-bases $x_i (i = 1, 2, \ldots, p)$ and $y_j (j = 1, 2, \ldots, q)$ of Λ^p and Λ^q, respectively. Then $A^T(L;t)$ is defined to be a generator of the smallest principal ideal containing the ideal generated by q-minors of M. In other words, for the finite maximal Λ-module $D(L)$ of $H(L)$, the Λ-torsion module $TH(L)/D(L)$ has a non-degenerate square Λ-presentation matrix, whose determinant is not 0 and equals to $A^T(L;t)$ up to units of Λ (see [10]). The torsion-Alexander polynomial $A^T(L;t)$ is always a non-zero Laurent polynomial and an invariant of L up to units of Λ. By definition, the classical Alexander polynomial $A(L;t)$ of L is $A^T(L;t)$ or 0 according to whether $TH(L) = H(L)$ or $TH(L) \neq H(L)$. The torsion-Alexander polynomial $A^T(L;t)$ is *semi-classical* if every knot component K of L belongs to a sublink L_K of L such that $A(L_K;t)$ is a factor of $A^T(L;t)$. Here are some examples on semi-classical torsion-Alexander polynomials.

Example 2.1.
(1) If $TH(L) = H(L)$, then $A^T(L;t) = A(L;t)$ and it is semi-classical. In particular, if L is a knot, then it is semi-classical.
(2) If L has only trivial components, then $A^T(L;t)$ is semi-classical.
(3) If L is a connected sum or a split sum of two links $L_i (i = 1, 2)$ with $A^T(L_i;t)$ $(i = 1, 2)$ semi-classical, then $A^T(L;t)$ is semi-classical.
(4) Let K be a knot such that $A(K;t)$ is a non-unit of Λ, and L a 2-component

parallel link of K with mutually opposite orientations and with the linking number 0. Then L bounds an annulus as a Seifert surface whose associated Seifert matrix is the zero matrix (0). Hence we have $H(L) \cong \Lambda$ and $A^T(L;t)$ is a unit of Λ which cannot have $A(K;t)$ as a factor. Thus, $A^T(L;t)$ is not semi-classical.

Let $\mathbb{A}_a$ be the set of links L such that $A^T(L;t)$ is alternating and semi-classical. The following lemma is essentially a well-known result by R. H. Crowell [3] and K. Murasugi [17]:

Lemma 2.2. $\mathbb{A} \subset \mathbb{A}_a$.

Proof. Every alternating link L is a split union of non-split alternating links $L_i (i = 1, 2, \ldots, s)$. We see from [3] or [17] that $A(L_i;t)$ is an alternating Laurent polynomial for all i. Since we have $TH(L_i) = H(L_i)$ and

$$H(L) \cong \Lambda^{s-1} \oplus H(L_1) \oplus H(L_2) \oplus \ldots \oplus H(L_s),$$

we have that $A^T(L;t) = A(L_1;t)A(L_2;t)\ldots A(L_s;t)$, implying that $A^T(L;t)$ is alternating and semi-classical. □

The number $s-1$ in the proof of Lemma 2.2 is called the *splitting number* of the alternating link L. We note that the alternating torsion-Alexander polynomial of a link is not always realizable by an alternating link. It is an unsolved open problem to characterize the Alexander polynomials of alternating links. A related conjecture is the *trapezoidal conjecture* proposed by R. H. Fox [4], asking that the Alexander polynomial $A(L;t) = \sum_{i=k}^{k+m} a_i t^i$ of every non-split alternating link L has the following properties: Namely,

(1) $|a_k| \leqq |a_{k+1}| \leqq |a_{k+2}| \leqq \cdots \leqq |a_{k+[m/2]}|$ and
(2) if $|a_{k+i}| = |a_{k+i+1}|$ for some i, then we have $|a_{k+i}| = |a_{k+j}|$ for every $j = i+1, i+2, \ldots, [m/2]$.

This conjecture is known to be true for many classes of links including the class of 2-bridge links, and more recently proved for all alternating knots of genus up to 2 by P. Ozsváth and Z. Szabó [18] and I. Jong [5]. On the other hand, K. Murasugi [17] and I. Jong [5] observed that there are alternating Laurent polynomials of degree 4 which are realized by knots and satisfy the trapezoidal conjecture, but are not realizable by any alternating knots. For example, the Alexander polynomial $A(9_{44};t) = 1 - 4t + 7t^2 - 4t^3 + t^4$ is such an example, in addition satisfying the Ozsváth-Szabó condition in [18]. In the following proposition, we investigate which knot in the knot table with up to 10 crossings belongs to $\mathbb{A}$ or $\mathbb{A}_a$:

Proposition 2.3.
(1) If the crossing number $\mathrm{cr}(K) \leqq 7$, then $K \in \mathbb{A}$.

(2) Among the knots K with $\mathrm{cr}(K) = 8$, $K \in \mathbb{A}_a$ if and only if $K \neq 8_{19}$, and $K \in \mathbb{A}$ if and only if $K \neq 8_{19}, 8_{20}, 8_{21}$.
(3) Among the knots K with $\mathrm{cr}(K) = 9$, we have always $K \in \mathbb{A}_a$, and $K \in \mathbb{A}$ if and only if $K \neq 9_i\,(42 \leqq i \leqq 49)$.
(4) Among the knots K with $\mathrm{cr}(K) = 10$, $K \in \mathbb{A}_a$ if and only if $K \neq 10_{124}$, 10_{128}, 10_{139}, 10_{145}, 10_{152}, 10_{154}, 10_{161}, 10_{162}, and $K \in \mathbb{A}$ if and only if $K \neq 10_i$ $(124 \leqq i \leqq 166)$.

Proof. We see (1)-(4) from the data on Alexander polynomials and diagrams of the knot table except how to determine a knot which is in $\mathbb{A}_a$ but not in $\mathbb{A}$. For example, we can see it by using the data on Kauffman polynomials $F_K(a, x)$ in [14] [2], because for an alternating knot K the span of $F_K(a, x)$ on a is known to be equal to the crossing number $\mathrm{cr}(K)$ by Y. Yokota [21]. □

The *algebraic alternation number* $\mathrm{alt}_a(L)$ (or the *algebraic τ-alternation number* $\mathrm{alt}^\tau_a(L)$, respectively) of a link L is defined to be $d^\times(L, \mathbb{A}_a)$ (or $d^\tau(L, \mathbb{A}_a)$, respectively). The *algebraic concordance-alternation number* $\mathrm{alt}_a[L]$ (or the *algebraic concordance-τ-alternation number* $\mathrm{alt}^\tau_a[L]$, respectively) of a link L is the minimal number of $\mathrm{alt}_a(L')$ (or $\mathrm{alt}^\tau_a(L')$, respectively) for all links $L' \in [L]$, which is uniquely determined by the concordance class $[L]$. A *multiplicative subset* of Λ is a subset $S \subset \Lambda\backslash\{0\}$ such that

(1) the units $\pm t^i (i \in \mathbf{Z})$ of Λ are in S,
(2) the product $f(t)g(t)$ of any elements $f(t)$ and $g(t)$ of S is in S, and
(3) every prime factor of any element $g(t) \in S$ is in S.

For the quotient field $Q(\Lambda)$ of Λ and a multiplicative subset S of Λ, let Λ_S be the subring $\{f(t)/g(t) \in Q(\Lambda) | f(t) \in \Lambda,\ g(t) \in S\}$ of $Q(\Lambda)$. For a Λ-module H, let H_S be the Λ_S-module $H \otimes_\Lambda \Lambda_S$. For a finitely generated Λ-module H and a multiplicative subset S of Λ, let $e_S(H)$ be the least number of Λ_S-generators of the Λ_S-module H_S. (We take $e_S(H) = 0$ when $H = 0$). Let $e_S(L) = e_S(H(L))$ for the Λ-module $H(L)$ of L, which is equal to the Nakanishi index of L if S is the set of units $\pm t^i (i \in \mathbf{Z})$ of Λ (cf. [14]). Our basic tool is the following *estimation lemma*, which is proved in [12]:

Lemma 2.4. For arbitrary two links $L_i (i = 0, 1)$ with the same number of components and every multiplicative subset S of Λ, we have

$$+\infty > d^\times(L_0, L_1) \geqq d^\tau(L_0, L_1) \geqq |e_S(L_0) - e_S(L_1)|.$$

The following lemma is direct from definitions:

[2] The table of $F_K(a, x)$ made there has an ambiguity on the multiples of a although the span of $F_K(a, x)$ on a is uniquely determined since it was computed without counting the writhe of a knot diagram.

Lemma 2.5. We have the following inequalities:

$$\mathrm{alt}_a^\tau(L) \leqq \mathrm{alt}_a(L) \leqq \mathrm{alt}(L), \quad \mathrm{alt}_a^\tau(L) \leqq \mathrm{alt}^\tau(L) \leqq \mathrm{alt}(L),$$

where everything is taken 0 if L is an alternating link.

The alternation number, τ-alternation number, algebraic alternation number and algebraic τ-alternation number for the non-alternating knots with up to 10 crossings are calculated as follows:

Example 2.6. For every non-alternating knot K with $\mathrm{cr}(K) \leqq 10$, we have $\mathrm{alt}(K) = 1$ by checking the list of non-alternating knots in Proposition 2.3, so that we have $\mathrm{alt}^\tau(K) = \mathrm{alt}(K) = 1$ and $\mathrm{alt}_a(K) = \mathrm{alt}_a^\tau(K) \leqq 1$ for all non-alternating knots with $\mathrm{cr}(K) \leqq 10$. More explicitly, checking the Alexander polynomials, we have $\mathrm{alt}^\tau(K) = \mathrm{alt}(K) = 1$ if and only if $K = 8_i\,(i = 19, 20, 21)$, $9_i\,(42 \leqq i \leqq 49)$ or $10_i\,(124 \leqq i \leqq 166)$, and $\mathrm{alt}_a^\tau(K) = \mathrm{alt}_a(K) = 1$ if and only if $K = 8_{19}$ or $10_{124}, 10_{128}, 10_{139}, 10_{145}, 10_{152}, 10_{154}, 10_{161}, 10_{162}$, among the non-alternating knots K with $\mathrm{cr}(K) \leqq 10$.

3 Main theorems and the proofs

As a result for an alternating link, we obtain the following theorem:

Theorem 3.1. For every $n \geqq 1$, every (possibly, split) alternating link $L_\alpha \in \mathbb{A}$ is concordant to infinitely many hyperbolic links L_n such that

$$\mathrm{alt}(L_n) = \mathrm{alt}^\tau(L_n) = \mathrm{alt}_a(L_n) = \mathrm{alt}_a^\tau(L_n) = d^{\mathrm{x}}(L_n, L_\alpha) = d^\tau(L_n, L_\alpha) = n.$$

Proof. We use a slice knot K such that $d^{\mathrm{x}}(K, O) = 1$ for a trivial knot O and $A(K;t)$ has a negative root. For example, the knot K in Fig. 1 is a ribbon knot such that $d^{\mathrm{x}}(K, O) = 1$ is confirmed by the crossing change at the point p indicated in Fig.1 and the Alexander polynomial $A(K;t) = 3 - (t^2 + t^{-2})$ has the negative roots $t = (-\sqrt{5} \pm 1)/2$. Let K^n be the n-fold connected sum $K\#K\#\ldots\#K$ of K. Since K^n is a ribbon knot, we see that a connected sum $L_\alpha\#K^n$ is concordant to L_α. Using that $d^{\mathrm{x}}(K^n, O) \leqq n$,

Fig. 1.

we have $d^{\mathrm{x}}(L_\alpha \# K^n, L_\alpha) \leqq n$. Our desired links L^n are constructed from the link $L_\alpha \# K^n$ by the "topological imitation"technique in [11]. In fact, we can construct from $L_\alpha \# K^n$ infinitely many hyperbpolic links L^n with an AID (=almost identical) imitation $q : (S^3, L^n) \to (S^3, L_\alpha \# K^n)$ such that the link $L_\alpha \# K^{n-1}$ is obtained from L^n by a crossing change. Then the link L^n is concordant to $L_\alpha \# K^n$ by a property of an imitation and hence to L_α, and we have $d^{\mathrm{x}}(L^n, L_\alpha) \leqq n$. We show that $\mathrm{alt}^\tau_a(L^n) \geqq n$. Then the proof will be completed by Lemma 2.5. Let $m = \mathrm{alt}^\tau_a(L^n) = d^\tau(L^n, L')$ for a link $L' \in \mathbb{A}_a$. Let $\bar{K}^n$ be the component of $L_\alpha \# K^n$ containing K^n as a connected summand. Since $A^T(L'; t)$ is semi-classical, we have a sublink $C' \subset L'$ such that $A(C'; t)$ is a factor of $A^T(L'; t)$ and C' changes into a sublink $C^n \subset L^n$ containing the component $q^{-1}(\bar{K}^n)$ by m times of zero-linking twists. Using that the link C^n is an (AID) imitation of the link $q(C^n)$ by the imitation map defined by q, we see from a property of an imitation that $H(C^n) = H(q(C^n))$. Since $q(C^n)$ contains K^n as a connected summand, the Λ-module $H(C^n)$ has $H(K^n) = H(K)^n$, a direct sum of n copies of $H(K)$, as a direct summand. Let S be the subset of Λ consisting of a Laurent polynomial $f(t)$ which has no negative root. We see that S is a multiplicative subset of Λ. It is important to note that every alternating Laurent polynomial $f(t) = \sum_{i=k}^{k+m} a_i t^i \in \Lambda$ is in S. In fact, if r is a negative number, then we see from the definition of an alternating Laurent polynomial that the signs of $a_i r^i$ for all i are the same, so that $f(r) \neq 0$. Since $H(K^n) = H(K)^n$ and $A(K; t) \notin S$, we see that $e_S(K^n) \geqq n$ (see [10,12] for a calculation). Thus, we have $e_S(C^n) \geqq n$. On the other hand, since $A^T(L; t)$ is an alternating Laurent polynomial and $A(C'; t)$ is a factor of $A^T(L; t)$, we see that $A(C'; t) \in S$. Then we show that $e(C')_S = 0$. To see this, we note that there is a Λ-exact sequence $\Lambda^h \to \Lambda^h \to H(C') \to 0$ for a positive integer h(cf. [10]), which induces a Λ_S-exact sequence $\Lambda^h_S \to \Lambda^h_S \to H(C')_S \to 0$. By the definition of the Alexander polynomial, $A(C'; t)$ is equal to the determinant of a matrix representing the Λ-homomorphism $\Lambda^h \to \Lambda^h$ up to units of Λ, which is a unit in Λ_S. Hence we have $H(C')_S = 0$ and $e_S(C') = 0$. By the estimation lemma, we have

$$m \geqq d^\tau(C^n, C') \geqq |e_S(C^n) - e_S(C')| = e_S(C^n) \geqq n,$$

implying that $m = \mathrm{alt}^\tau_a(L^n) \geqq n$. This completes the proof of Theorem 3.1. □

Here is a reason why we exclude the case $n = 0$ in Theorem 3.1.

Remark 3.2. The Λ-rank $\mathrm{rank}_\Lambda H(L)$ is a concordance invariant of a link L (cf. [8,9,13]), so that the splitting numbers of concordant alternating links are equal. Hence any split alternating link is not concordant to any non-split (and hence hyperbolic) alternating link, although it is always concordant to a non-split (more strongly hyperbolic) link. Thus, we cannot take $n = 0$ in Theorem 3.1.

As a result for a general link, we obtain the following theorem:

Theorem 3.3. For every link L and every integer $n \geqq \mathrm{alt}[L]$ except that $n = \mathrm{alt}[L] = 0$ and $[L]$ is represented by a split link, we have infinitely many hyperbolic links L^n such that L^n is concordant to L and $\mathrm{alt}(L^n) = n$.

Proof. If $\mathrm{alt}[L] = 0$, then the concordance class $[L]$ contains an alternating link, and the desired result follows from Theorem 3.1 for $n > 0$. Let $d = \mathrm{alt}[L] > 0$. First, we show that there are infinitely many hyperbolic links L^d with $[L^d] = [L]$ and $\mathrm{alt}(L^d) = d$. For a link L' with $\mathrm{alt}(L') = d$ representing $[L]$, choose a crossing point of a diagram of L' to obtain a link L'_1 with $\mathrm{alt}(L'_1) = d - 1$ by the crossing change. By the "topological imitation"technique in [11] applied to this crossing point, we have infinitely many hyperbolic links L^d with an AID imitation $q : (S^3, L^d) \to (S^3, L')$ such that L'_1 is obtained from L^d by a crossing change. Then we have $\mathrm{alt}(L^d) \leqq \mathrm{alt}(L'_1) + 1 = d$. Since L^d is concordant to L' by the definition of an imitation, we have $\mathrm{alt}(L^d) \geqq d$ and hence $\mathrm{alt}(L^d) = d$. Let $n > d$. Let K be the knot used in the proof of Theorem 3.1, and K^h the h-fold connected sum $K\#K\#\dots\#K$ of K. For every positive integer h, we have a hyperbolic link L^{d+h} with an AID imitation $q : (S^3, L^{d+h}) \to (S^3, L^d\#K^h)$ which is a composite of AID imitations $q^i : (S^3, L^{d+i}) \to (S^3, L^{d+i-1}\#K)$ $(i = 1, 2, 3, \dots, h)$, constructed by using [11], such that L^{d+i-1} is obtained from L^{d+i} by a crossing change. Then we have

$$\mathrm{alt}(L^{d+h}) \leqq \mathrm{alt}(L^{d+h-1}\#K) \leqq \mathrm{alt}(L^{d+h-1}) + 1.$$

By properties of an AID imitation, we note that there are infinitely many families of mutually distinct hyperbolic links L^{d+i} $(i = 0, 1, 2, \dots, h)$ for every h. For every such family, we show that $\lim_{h\to+\infty} \mathrm{alt}(L^{d+h}) = +\infty$. To see this, let L_α be an alternating link such that $\mathrm{alt}(L^{d+h}) = d^{\times}(L^{d+h}, L_\alpha)$. By the estimation lemma, we have $\mathrm{alt}(L^{d+h}) \geqq |e_S(L^{d+h}) - e_S(L_\alpha)|$, where S is the multiplicative set used in the proof of Theorem 3.1. Since $H(L^{d+h}) \cong H(L^d\#K^h)$ which has $H(K^h) = H(K)^h$ as a direct summand, we have $e_S(L^{d+h}) = e_S(L^d\#K^h) \geqq e_S(K^h) \geqq h$. Since $e_S(L_\alpha)$ is equal to the splitting number of L_α which is smaller than the component number of L_α and hence of L, we see that $\lim_{h\to+\infty} \mathrm{alt}(L^{d+h}) = +\infty$ and there is a positive integer j so that $\mathrm{alt}(L^{d+j}) > n$. To complete the proof, it suffices to show that there is an integer h with $0 < h < j$ such that $\mathrm{alt}(L^{d+h}) = n$. To see this, suppose such an h does not exist. Since $\mathrm{alt}(L^d) = d < n$, we can take the maximal integer h such that $0 \leqq h < j$ and $\mathrm{alt}(L^{d+h}) \leqq n$. Using $\mathrm{alt}(L^{d+h}) < n$ and $\mathrm{alt}(L^{d+j}) > n$, we have $\mathrm{alt}(L^{d+h+1}) \leqq \mathrm{alt}(L^{d+h}) + 1 \leqq n$ and $0 < h + 1 < j$, which contradicts the maximalness of h. □

Here is a note on the exceptional case of Theorem 3.3.

Remark 3.4. Although the same assertion of Theorem 3.3 using alt^τ, alt_a or alt_a^τ instead of alt also holds by a similar argument, we need a remark on the exceptional case that $\mathrm{alt}[L] = 0$ and $[L]$ is represented by a split link. By Remark 3.2, any alternating link representing $[L]$ is split and thus we must take

$n > 0$ for Theorem 3.3 and its alt^τ version. Let L be a split alternating link. By applying the "topological imitation" technique in [11] to a trivial crossing change of L into L, we obtain a hyperbolic link L^* given by an AID imitation $q : (S^3, L^*) \to (S^3, L)$ such that L is obtained from L^* by a crossing change. By properties of an AID imitation, we have $A^T(L^*; t) = A^T(L; t)$ which is a semi-classical alternating Laurent polynomial, so that $\mathrm{alt}_a(L^*) = 0$. Thus, in the statement of the alt_a or alt_a^τ version of Theorem 3.3, we can take $n = 0$. We note that, since $\mathrm{alt}(L^*) \leqq 1$, $[L^*] = [L]$ and L^* is non-split, we have $\mathrm{alt}(L^*) = \mathrm{alt}^\tau(L^*) = 1$.

From our viewpoint it is an important problem to calculate the value $\mathrm{alt}[L]$ of a link L. For this purpose, it would be an interesting question to ask *which link L has $\mathrm{alt}(L) = \mathrm{alt}[L]$*. For example, *does any torus link $T(p, q)$ have $\mathrm{alt}(T(p, q)) = \mathrm{alt}[T(p, q)]$* ? The following remark concerns a recent development of this question.

Remark 3.5. Let $\sigma(K)$ be the (-1)-multiple of the signature of a knot K so that the positive trefoil knot takes $+2$, and $s(K)$ the Rasmussen invariant or the twice of the Ozsváth-Szabó invariant which is an additive concordance knot invariant (cf. [19,20]). By using C. Livingston's observation in [15], T. Abe observed in [1] the inequality $\mathrm{alt}(K) \geqq |\sigma(K) - s(K)|/2$, by which T. Abe proved that every torus knot $K = T(p, q)$ $(p > 2, q > 3)$ except $(p, q) = (3, 4), (3, 5)$ has $\mathrm{alt}[K] > 1$, meaning that the almost alternating torus knots are just $T(3, 4)$ and $T(3, 5)$, confirming a conjecture by C. Adams in [2]. For further calculations on the alternation numbers of torus knots, see T. Kanenobu [6]. Since $|\sigma(K)-s(K)|$ is a concordance invariant, Abe's inequality actually implies the inequality

$$\mathrm{alt}[K] \geqq |\sigma(K) - s(K)|/2,$$

which is useful to know the value $\mathrm{alt}[K]$, although Abe's inequality does not detect the assertions of Theorems 3.1 and 3.3 because of its concordance invariance. For example, let T^m be the m-fold connected sum of $T(4, 5)$. Then we have $\mathrm{alt}(T^m) = \mathrm{alt}[T^m] = 2m$ by Kanenobu's calculation. Thus, if a knot K is concordant to T^m, then we have $\mathrm{alt}(K) \geqq 2m$, and conversely for every integer $n \geqq 2m$, there is a hyperbolic knot K which is concordant to T^m such that $\mathrm{alt}(K) = n$ by Theorem 3.3.

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