PRESENTATION OF IMMERSED SURFACE-LINKS BY MARKED GRAPH DIAGRAMS

SEIICHI KAMADA, AKIO KAWAUCHI, JIEON KIM, AND SANG YOUL LEE

Abstract. It is well known that surface-links in 4-space can be presented by diagrams on the plane of 4-valent spatial graphs with makers on the vertices, called marked graph diagrams. In this paper we extend the method of presenting surface-links by marked graph diagrams to presenting immersed surface-links. We also give some moves on marked graph diagrams that preserve the ambient isotopy classes of their presenting immersed surface-links.

1. Introduction

A surface-link, or an *embedded* surface-link, is a closed surface embedded in Euclidean 4-space $\mathbb{R}^4$. An *immersed surface-link* is a closed surface immersed in $\mathbb{R}^4$ such that the multiple points are transverse double points. It is well known that surface-links can be presented by diagrams on the plane of 4-valent spatial graphs with makers on the vertices, called marked graph diagrams (cf. [1, 3, 6, 7, 8, 9, 10]).

In this paper we extend the method of presenting surface-links by marked graph diagrams to presenting immersed surface-links. We also give some moves on marked graph diagrams that preserve the ambient isotopy classes of their presenting immersed surface-links, which are extension of moves given by Yoshikawa [10] for presentation of embedded surface-links.

2. Marked graph diagrams of immersed surface-links

In this section, we introduce a marked graph presentation of immersed surface-links. First, we recall quickly the notion of marked graph diagrams and links with bands from [9, 10].

Let A be the square $\{(x, y) | -1 \leq x, y \leq 1\}$, X be the diagonals in A presented by $x^2 = y^2$, and M_h (or M_v) be a thick interval in A given by $\{(x, y) | -1/2 \leq x \leq 1/2, -\delta \leq y \leq \delta\}$ (or $\{(x, y) | -1/2 \leq y \leq 1/2, -\delta \leq x \leq \delta\}$), where δ is a small positive number.

A *marked graph* (in $\mathbb{R}^3$) is a spatial graph G in $\mathbb{R}^3$ which satisfies the following:

(1) G is a finite regular graph with 4-valent vertices.
(2) Each vertex v is rigid; that is, there is a neighborhood $N(v)$ of v which is identified with thickened A such that v corresponds to the origin and the edges restricted to $N(v)$ correspond to X.

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(3) Each v has a *marker*, which is a thick interval in $N(v)$ which corresponds to M_h or M_v under the identification in (2).

An *orientation* of a marked graph G is a choice of an orientation for each edge of G such that around every vertex v, two edges incident to v in a diagonal position are oriented toward v and the other two incident edges are oriented outward. For example, see Fig. 1.

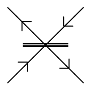

FIGURE 1. An orientation around a marked vertex

Not every marked graph admits an orientation. A marked graph is called *orientable* (or *non-orientable*) if it admits (or does not admit) an orientation. A marked graph depicted in Fig. 2 is non-orientable. An *oriented marked graph* is a marked graph equipped with an orientation. Two (oriented) marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with respect to markers as subsets of $\mathbb{R}^3$ (and the orientations).

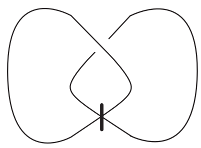

FIGURE 2. A non-orientable marked graph

A *banded link* $\mathcal{BL}$ (or a *link with bands*) is a pair $(L, \mathcal{B})$ of a link L in $\mathbb{R}^3$ and a set of mutually disjoint bands in $\mathbb{R}^3$ attached to L. It is called *oriented* if L is oriented and all bands are oriented coherently with respect to the orientation of L. In this case, the link obtained from L by surgery along the bands inherits an orientation, see Fig. 3. Two (oriented) banded links are *equivalent* if there is an ambient isotopy of $\mathbb{R}^3$ carrying the (oriented) link and (oriented) bands of one to those of the other.

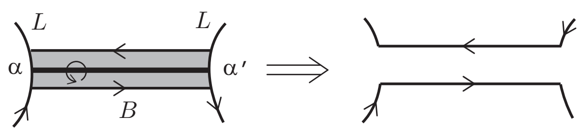

FIGURE 3. Surgery

For a marked graph G, we obtain a banded link $(L, \mathcal{B})$ by replacing a neighborhood of each 4-valent vertex with a band such that the core of the band corresponds to the marker as in Fig. 4 (b). The banded link is called the *banded link associated with* G and is denoted by $\mathcal{BL}(G)$. Conversely, a marked graph G is recovered from a banded link $\mathcal{BL}$ by shortening and replacing each band to a 4-valent vertex as in Fig. 4 (a). If G is oriented, then $\mathcal{BL}(G)$ is oriented, and vice versa.

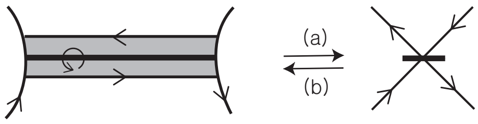

FIGURE 4. A band and a marked vertex

For a banded link $\mathcal{BL} = (L, \mathcal{B})$, the *lower resolution* $L_-(\mathcal{BL})$ is L and the *upper resolution* $L_+(\mathcal{BL})$ is the surgery result. For a marked graph G, the lower resolution $L_-(G)$ and the upper resolution $L_+(G)$ are defined to be those of the banded link $\mathcal{BL}(G) = (L, \mathcal{B})$ associated with G.

We present a marked graph by a diagram on the plane, which we call a *marked graph diagram*, in a usual way in knot theory.

Let D be a marked graph diagram. We denote by $\mathcal{BL}(D)$, $L_-(D)$, and $L_+(D)$ the banded link, the lower resolution and the upper resolution of the marked graph presented by D. See Fig. 5.

A link is called *H-trivial* if it is a split union of trivial knots and Hopf links [4]. A trivial link is regarded as an H-trivial link without Hopf links.

Definition 2.1. A marked graph diagram D (or a marked graph G) is *H-admissible* if both resolutions $L_-(D)$ and $L_+(D)$ (or $L_-(G)$ and $L_+(G)$) are H-trivial links.

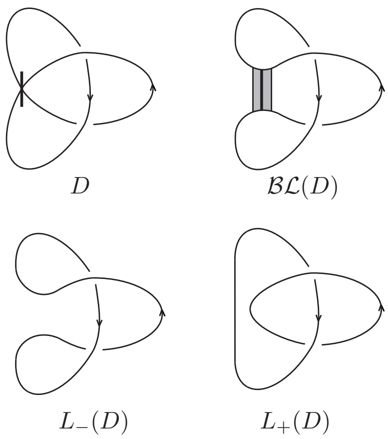

FIGURE 5. An H-admissible marked graph diagram

A marked graph diagram D (or a marked graph G) is called *admissible* if both resolutions $L_-(D)$ and $L_+(D)$ (or $L_-(G)$ and $L_+(G)$) are trivial links. By definition, an admissible marked graph (diagram) is H-admissible.

Now we discuss a marked graph presentation of an immersed surface-link.

For a subset $A \subset \mathbb{R}^3$ and an interval $I \subset \mathbb{R}$, let

$$AI = \{(x,t) \in \mathbb{R}^4 | x \in A, t \in I\}.$$

Let D be an H-admissible marked graph diagram, and $\mathcal{BL}(D) = (L, \mathcal{B})$ the banded link associated with D. Consider a surface $\mathcal{S}^1_{-1}$ in $\mathbb{R}^3[-1,1]$ satisfying

$$\mathcal{S}^1_{-1} \cap \mathbb{R}^3[t] = \begin{cases} L_+(D)[t] & \text{for } 0 < t \leq 1, \\ (L_-(D) \cup |\mathcal{B}|)[t] & \text{for } t = 0, \\ L_-(D)[t] & \text{for } -1 \leq t < 0, \end{cases}$$

where $|\mathcal{B}|$ denotes the union of the bands belonging to $\mathcal{B}$.

When D is oriented, we assume that the surface S^1_{-1} is oriented so that the orientation of $L_+(D)[1]$ as the boundary of S^1_{-1} coincides with the orientation of $L_+(D)$ induced from D.

Let L be an H-trivial link with trivial knot components O_i $(i = 1, \ldots, m)$ and Hopf link components H_j $(j = 1, \ldots, n)$, where $m \geq 0$ and $n \geq 0$. For an interval $[a, b]$, let $L_\wedge[a, b]$ be the union of disks Δ_i $(i = 1, \ldots, m)$ and n pairs of disks C_j $(j = 1, \ldots, n)$ in $\mathbb{R}^3[a, b]$ such that (1) $\partial\Delta_i = O_i[a]$ and $\partial C_j = H_j[a]$, (2) Δ_i has a unique maximal point, (3) each disk of C_j has a unique maximal point, and (4) the two disks of C_j intersect in a point transversely. We call Δ_i $(i = 1, \ldots, m)$ a *cone system* with base O_i $(i = 1, \ldots, m)$ and C_j $(j = 1, \ldots, n)$ a *cone system* with base H_j $(j = 1, \ldots, n)$.

We often assume an additional condition: (5) for each cone C_j over H_j, the intersection point of the two disks of C_j in condition (4) is the unique maximal point of each of the disks in condition (3). Similarly, for an H-trivial link L' with trivial knot components O'_i $(i = 1, \ldots, m')$ and Hopf link components H'_j $(j = 1, \ldots, n')$, where $m' \geq 0$ and $n' \geq 0$, let $L'_\vee[a, b]$ be the disjoint union of a cone system Δ'_i in $\mathbb{R}^3[a, b]$ with base O'_i in $\mathbb{R}^3[a]$ $(i = 1, \ldots, m')$ and a cone system C'_j in $\mathbb{R}^3[a, b]$ with base H'_j in $\mathbb{R}^3[a]$ $(i = 1, \ldots, n')$, where each disk in the cone system has a unique minimal point.

Let D be an H-admissible marked graph diagram. Consider the union

$$\mathcal{S}(D) = L'_\vee[-2, -1] \cup \mathcal{S}^1_{-1} \cup L_\wedge[1, 2],$$

which is an immersed surface-link in $\mathbb{R}^4$, where L and L' be upper and lower resolutions of D, respectively.

By an argument in [4, 5] it is seen that the ambient isotopy class of the immersed surface-link $\mathcal{S}(D)$ is uniquely determined from D. We call the immersed surface-link $\mathcal{S}(D)$ the *immersed surface-link constructed from D*.

Theorem 2.2. Let $\mathcal{L}$ be an immersed surface-link. There is an H-admissible marked graph diagram D such that $\mathcal{L}$ is ambient isotopic to $\mathcal{S}(D)$.

In the situation of this theorem, we say that $\mathcal{L}$ is *presented* by D.

Proof. The following argument is based on an argument in [5] where embedded and oriented surface-links are discussed (cf. [3]). Let $\mathcal{L}$ be an immersed surface-link. Let $d_1, \ldots, d_n$ be the double points of $\mathcal{L}$, and let $N(d_1), \ldots, N(d_n)$ be regular neighborhoods of them. Moving $\mathcal{L}$ by an ambient isotopy, we may assume the following conditions:

(1) All critical points of $\mathcal{L}$, except the double points, with respect to the projection $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are elementary critical points, that are maximal points, saddle points and minimal points.
(2) $\mathcal{L}$ is in $\mathbb{R}^3(-2, 2)$.
(3) All double points are in $\mathbb{R}^3[1]$.

(4) For each i ($i = 1, \dots, n$), $N(d_i) = N^3(d_i)[1-\epsilon, 1+\epsilon]$ for a 3-disk $N^3(d_i)$, and $N(d_i) \cap \mathcal{L}$ is the cone of a Hopf link $H_i \subset (\mathrm{int}N^3(d_i))[1-\epsilon]$ with the cone point $d_i \in \mathbb{R}^3[1]$. Here ϵ is a sufficiently small positive number. The 3-disks $N^3(d_1), \dots, N^3(d_n)$ are mutually disjoint.

Move double points into $\mathbb{R}^3[3]$ such that the condition (4) is preserved although the 3-disk $N^3(d_i)$ may change and the time level of d_i changes from 1 to 3, i.e.,

(1) All critical points of $\mathcal{L}$, except the double points, with respect to the projection $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are elementary critical points, that are maximal points, saddle points and minimal points.
(2) $\mathcal{L}$ is in $\mathbb{R}^3(-2, 4)$.
(3) All double points are in $\mathbb{R}^3[3]$. All maximal, saddle and minimal points are in $\mathbb{R}^3(-2, 2)$
(4) For each i ($i = 1, \dots, n$), $N(d_i) = N^3(d_i)[3-\epsilon, 3+\epsilon]$ for a 3-disk $N^3(d_i)$, and $N(d_i) \cap \mathcal{L}$ is the cone of a Hopf link $H_i \subset (\mathrm{int}N^3(d_i))[3-\epsilon]$ with the cone point $d_i \in \mathbb{R}^3[3]$. The 3-disks $N^3(d_1), \dots, N^3(d_n)$ are mutually disjoint.

Let $p_1, \dots, p_m$ be the maximal points of $\mathcal{L}$, $q_1, \dots, q_{m'}$ be the minimal points of $\mathcal{L}$, and let $N(p_1), \dots, N(p_m), N(q_1), \dots, N(q_{m'})$ be regular neighborhoods of them. Moving $\mathcal{L}$ by an ambient isotopy, we may assume the following conditions:

(1) All critical points of $\mathcal{L}$, except the double points, with respect to the projection $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ are elementary critical points, that are maximal points, saddle points and minimal points.
(2) $\mathcal{L}$ is in $\mathbb{R}^3(-4, 4)$.
(3) All double points and all maximal points are in $\mathbb{R}^3[3]$. All minimal points are in $\mathbb{R}^3[-3]$. All saddle points are in $\mathbb{R}^3(-2, 2)$.
(4) For each i ($i = 1, \dots, n$), $N(d_i) = N^3(d_i)[3-\epsilon, 3+\epsilon]$ for a 3-disk $N^3(d_i)$, and $N(d_i) \cap \mathcal{L}$ is the cone of a Hopf link $H_i \subset (\mathrm{int}N^3(d_i))[3-\epsilon]$ with the cone point d_i. The 3-disks $N^3(d_1), \dots, N^3(d_n)$ are mutually disjoint.
(5) For each i ($i = 1, \dots, m$), $N(p_i) = N^3(p_i)[3-\epsilon, 3+\epsilon]$ for a 3-disk $N^3(p_i)$, and $N(p_i) \cap \mathcal{L}$ is the cone of a trivial knot $O_i \subset (\mathrm{int}N^3(p_i))[3-\epsilon]$ with the cone point p_i. The 3-disks $N^3(p_1), \dots, N^3(p_m)$ are mutually disjoint, and also disjoint from $N^3(d_1), \dots, N^3(d_n)$.
(6) For each i ($i = 1, \dots, m'$), $N(q_i) = N^3(q_i)[-3-\epsilon, -3+\epsilon]$ for a 3-disk $N^3(q_i)$, and $N(q_i) \cap \mathcal{L}$ is the cone of a trivial knot $O'_i \subset (\mathrm{int}N^3(q_i))[-3+\epsilon]$ with the cone point q_i. The 3-disks $N^3(q_1), \dots, N^3(q_{m'})$ are mutually disjoint.

Finally, applying the argument in [5], we can move all saddle points into the same hyperplane $\mathbb{R}^3[0]$. Then we see the result. □

Remark 2.3. Theorem 1.4 of [4] states that any immersed and oriented surface-link is ambient isotopic to an immersed surface-link satisfying a certain condition. Applying the argument in [5], we can obtain an immersed surface-link required in Theorem 2.2.

3. Moves on marked graph diagrams

We discuss moves on marked graph diagrams which preserve the ambient isotopy classes of the immersed surface-links presented by the diagrams.

The moves depicted in Figs. 6 and 7 were introduced by Yoshikawa [10] as moves on marked graph diagrams which do not change the ambient isotopy classes of their

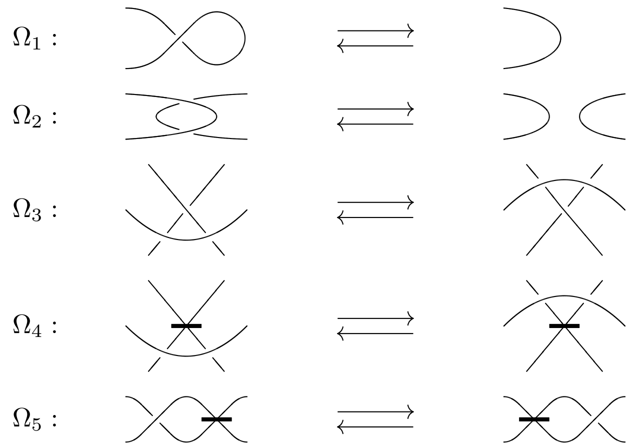

FIGURE 6. Moves of Type I

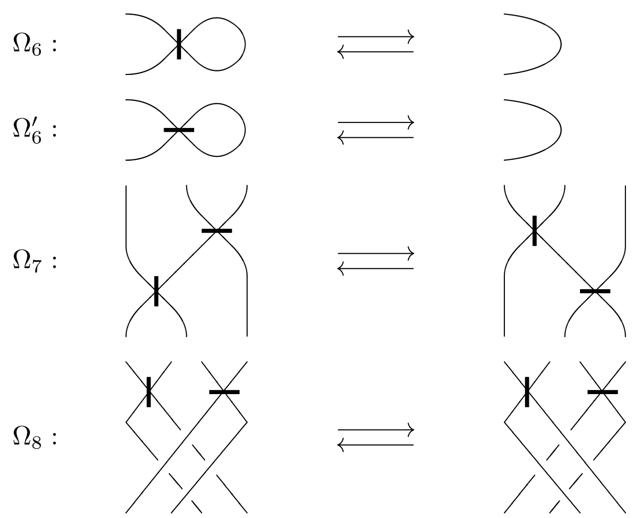

FIGURE 7. Moves of Type II

presenting surface-links. The moves and their mirror images are called *Yoshikawa moves*. Furthermore, we call the moves in Fig. 6 (Fig. 7) and their mirror images *moves of type I* (*moves of type II*). Moves of type I do not change the ambient isotopy classes of marked graphs in $\mathbb{R}^3$, and moves of type II do. Note that Yoshikawa moves preserve H-admissibility and admissibility.

It is known that two admissible marked graph diagrams present ambient isotopic surface-links if and only if they are related by Yoshikawa moves (cf. [7, 9, 10]).

Let D be a link diagram of an H-trivial link L. A crossing point p of D is an *unlinking crossing point* if it is a crossing between two components of the same Hopf link of L and if the crossing change at p makes the Hopf link into a trivial link.

Definition 3.1. Let D be an H-admissible marked graph diagram and let D_- and D_+ be the diagrams of the lower resolution $L_-(D)$ and the upper resolution $L_+(D)$, respectively. A crossing point p of D is a *lower singular point* (or an *upper singular point*, resp.) if p is an unlinking crossing point of D_- (or D_+, resp.).

We introduce new moves for H-admissible marked graph diagrams. They are the moves Ω_9, Ω'_9 and Ω_{10} in Fig. 8 and their mirror images, which we call *moves of type III*. Here we assume that the moves of type III are defined only if two diagrams appearing before and after the move are H-admissible. For example, for the move Ω_9 (or Ω'_9, resp.) in Fig. 8, we require that the component l in the resolution $L_+(D)$ (or $L_-(D)$, resp.) is trivial and that p is an upper (or lower, resp.) singular point.

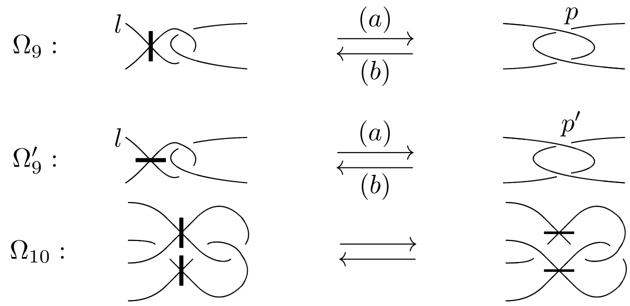

FIGURE 8. Moves of Type III: Ω_9, Ω'_9 and Ω_{10}

Definition 3.2. The *generalized Yoshikawa moves* are Yoshikawa moves (moves of type I and II) and moves of type III introduced above. Two marked graph diagrams are *stably equivalent* if they are related by a finite sequence of generalized Yoshikawa moves.

Theorem 3.3. Let $\mathcal{L}$ and $\mathcal{L}'$ be immersed surface-links presented by marked graph diagrams D and D', respectively. If D and D' are stably equivalent, then $\mathcal{L}$ and $\mathcal{L}'$ are ambient isotopic.

Proof. It suffices to show that $\mathcal{L}$ and $\mathcal{L}'$ are ambient isotopic when D' is obtained from D by a move of Ω_9, Ω'_9 or Ω_{10}. The moves Ω_9 and Ω'_9 correspond to a creation or removal of a saddle point, and the move Ω_{10} corresponds to a change the level of double point singularity. See Fig. 9, which shows partial pictures of broken surface diagrams in 3-space in the sense of [2]. (In [2], embedded surfaces are discussed. However, broken surface diagrams are considered for immersed surface-links and it is true that if two broken surface diagrams are ambient isotopic in 3-space then the immersed surface-links are ambient isotopic in 4-space.) Since the moves Ω_9, Ω'_9 and Ω_{10} do not change the ambient isotopy classes of broken surface diagrams in 3-space, we see that $\mathcal{L}$ and $\mathcal{L}'$ are ambient isotopic. □

Let Ω^*_9 and Ω'^*_9 be the moves depicted in Fig. 10 or their mirror images. They are equivalent to Ω_9 and Ω'_9 modulo Yoshikawa moves (of type I) as shown in Fig. 11.

We conclude the paper by proposing a question.

Question 3.4. Suppose that D and D' are marked graph diagrams presenting ambient isotopic immersed surface-links. Is D stably equivalent to D'?

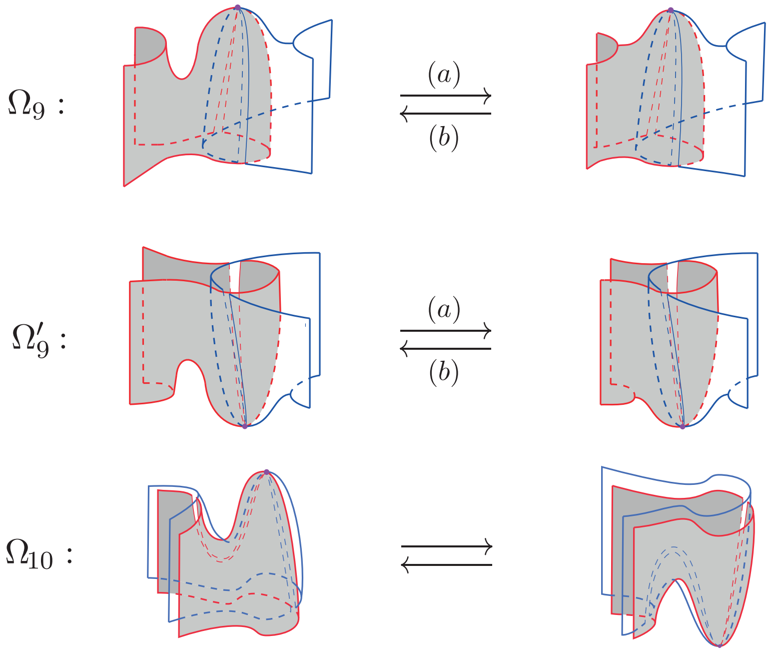

FIGURE 9. Immersed surface-links presented by Ω_9, Ω'_9, and Ω_{10}

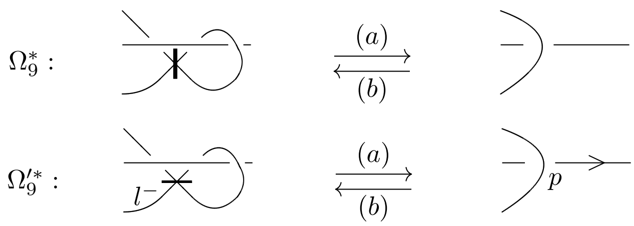

FIGURE 10. Moves Ω_9^* and $\Omega_9'^*$

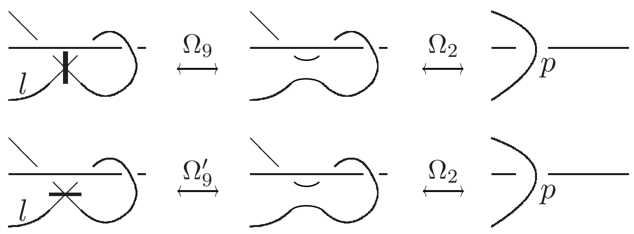

FIGURE 11. Moves Ω_9^*, $\Omega_9'^*$ are equivalent to Ω_9, Ω'_9.

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