Characterizing the first Alexander Z[Z]-modules of surface-links and of virtual links

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ABSTRACT

We characterize the first Alexander Z[Z]-modules of ribbon surface-links in the 4-sphere fixing the number of components and the total genus, and then the first Alexander Z[Z]-modules of surface-links in the 4-sphere fixing the number of components. Using the result of ribbon torus-links, we also characterize the first Alexander Z[Z]-modules of virtual links fixing the number of components. For a general surface-link, an estimate of the total genus is given in terms of the first Alexander Z[Z]-module. We show a graded structure on the first Alexander Z[Z]-modules of all surface-links and then a graded structure on the first Alexander Z[Z]-modules of classical links, surface-links and higher-dimensional manifold-links.

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1. The first Alexander Z[Z]-module of a surface-link

For every non-negative partition $g = g_1 + g_2 + ... + g_r$ of a non-negative integer g, we consider a closed oriented 2-manifold $F = F^r_g = F^r_{g_1,g_2,...,g_r}$ with r components F_i $(i = 1, 2, ..., r)$ such that the genus $g(F_i)$ of F_i is g_i. The integer g is called the *total genus* of F and denoted by $g(F)$. An *F-link* L is the ambient isotopy class of a locally-flatly embedded image of F into S^4, and for $r = 1$ it is also called an *F-knot*. The *exterior* of L is the compact 4-manifold $E = S^4\backslash\mathrm{int}N(L)$, where $N(L)$ denotes the tubular neighborhood of L in S^4. Let $p : \tilde{E} \to E$ be the infinite cyclic covering associated with the epimorphism $\gamma : H_1(E) \to Z$ sending every oriented meridian of L in $H_1(E)$ to $1 \in Z$. An F-link L is *trivial* if L is the boundary of the union of disjoint handlebodies embedded locally-flatly in S^4. A *ribbon F-link* is an F-link obtained from a trivial F_0^r-link by surgeries along embedded 1-handles in S^4(see [9, p.52]). When we put the trivial F_0^r-link in the equatorial 3-sphere $S^3 \subset S^4$, we can replace the 1-handles by mutually disjoint 1-handles embedded in the 3-sphere S^3 without changing the ambient isotopy class of the ribbon F-link by an argument of [9, Lemma 4.11] using a result of [2, Lemma 1.4]. Thus, every ribbon F-link is described by a *disk-arc presentation* consisting of oriented disks and arcs intersecting the interiors of the disks transversely in S^3 (see Fig. 1 for an illustration), where the oriented disks and the arcs represent the oriented trivial 2-spheres and the 1-handles, respectively. Let $\Lambda = Z[Z] = Z[t, t^{-1}]$ be the integral Laurent polynomial ring. The homology

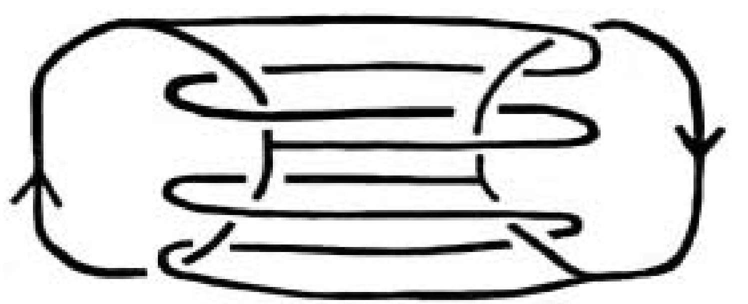

Figure 1: A ribbon $F_{1,1}^2$-link

$H_*(\tilde{E})$ is a finitely generated Λ-module. Specially, the first homology $H_1(\tilde{E})$ is called the *first Alexander $Z[Z]$-module*, or simply the *module* of an F-link L and denoted by $M(L)$. In this paper, we discuss the following problem:

Problem 1.1. Characterize the modules $M(L)$ of F_g^r-links L in a topologically meaningful class.

In §2, we discuss some homological properties of F_g^r-links. Fixing r and g, we shall solve Problem 1.1 for the class of ribbon F_g^r-links in §3. We also solve Problem 1.1 for the class of all F_g^r-links not fixing g as a collorary of the ribbon case in §3. In §4, we characterize the first Alexander $Z[Z]$-modules of virtual links by using the characterization of ribbon $F_{1,1,\dots,1}^r$-links. In §5, we show a graded structure on the first Alexander $Z[Z]$-modules of all F_g^r-links by establishing an estimate of the total genus g in terms of the first Alexander $Z[Z]$-module of an F_g^r-link. In fact, we show that there is the first Alexander $Z[Z]$-module of an F_{g+1}^r-link which is not the first Alexander $Z[Z]$-module of any F_g^r-link for every r and g. In §6, we show a graded structure on the first Alexander $Z[Z]$-modules of classical links, surface-links and higher-dimensional manifold-links.

This paper is a research announcement of the author's paper "The first Alexander Z[Z]-modules of surface-links and of virtual links"(cf. http://www.sci.osaka-cu.ac.jp/ kawauchi/index.html), which will appear elsewhere.

2. Some homological properties on surface-links

The following computation on the homology $H_*(E)$ of the exterior E of an F_g^r-link L is done by using the Alexander duality for (S^4, L):

Lemma 2.1.

$$H_d(E) = \begin{cases} Z^{r-1} & (d=3) \\ Z^{2g} & (d=2) \\ Z^r & (d=1) \\ Z & (d=0) \\ 0 & (d \neq 0,1,2,3). \end{cases}$$

For a finitely generated Λ-module M, let TM be the Λ-torsion part, and $BM = M/TM$ the Λ-torsion-free part. Let

$$DM = \{x \in M | \exists f_i \in \Lambda (i=1,2,...,s(\geqq 2)) \text{ with } \quad (f_1,...,f_s)=1 \text{ and } \quad f_i x = 0\},$$

which is the maximal finite Λ-submodule of M (cf. [4]). Let $TM = \mathrm{Tor}_\Lambda M$ and $T_D M = TM/DM$. Let $E^q M = Ext^q_\Lambda(M,\Lambda)$. The following proposition is more or less known (see J. Levine [11] for S^n-knot modules and [4] in the general):

Proposition 2.2. We have the following properties (1)-(5) on a finitely generated Λ-module.
(1) $E^0 M = \hom_\Lambda(M,\Lambda) = \Lambda^{\beta(M)}$,
(2) $E^1 M = E^2 M = 0$ if and only if M is Λ-free,
(3) there are natural Λ-exact sequences $0 \to E^1 BM \to E^1 M \to E^1 TM \to 0$ and $0 \to BM \to E^0 E^0 BM \to E^2 E^1 BM \to 0$,
(4) $E^1 BM = DE^1 M$,
(5) $E^1 TM = \hom_\Lambda(TM, Q(\Lambda)/\Lambda)$ and $E^2 M = E^2 DM = \hom_Z(DM, Q/Z)$.

Let $\beta(M)$ be the Λ-rank of the module M, namely the $Q(\Lambda)$-dimension of the $Q(\Lambda)$-vector space $M \otimes_\Lambda Q(\Lambda)$, where $Q(\Lambda)$ denotes the quotient field of Λ. The d^{th} *Λ-rank* of an F_g^r-link L is the number $\beta_d(L) = \beta(H_d(\tilde{E}))$. We call the integer $\tau(L) = r - 1 - \beta_1(L)$ the *torsion-corank* of L, which shown to be non-negative in Lemma 2.5. We use the following notion:

Definition 2.3. A finitely generated Λ-module M is a *cokernel-free* Λ-module of *corank* n if there is an isomorphism $M/(t-1)M \cong Z^n$ as abelian groups.

The corank n of M is denoted by $cr(M)$. We shall show in Corollary 3.3 that a Λ-module M is a cokernel-free Λ-module of corank n if and only if there is an F_g^{n+1}-link L for some g such that $M(L) = M$. The following lemma implies that the cokernel-free Λ-modules appear naturally in the homology of an infinite cyclic covering:

Lemma 2.4. Let $p : \tilde{X} \to X$ be an infinite cyclic covering over a finite complex X. If $H_d(X)$ is free abelian, then the Λ-modules $H_d(\tilde{X})$, $TH_d(\tilde{X})$ and $T_D H_d(\tilde{X})$ are cokernel-free Λ-modules.

From Lemmas 2.1 and 2.4, we see that the Λ-modules $H_*(\tilde{E})$, $TH_*(\tilde{E})$ and $T_D H_*(\tilde{E})$ are all cokernel-free Λ-modules for every F_g^r-link L. On these Λ-modules, we make the following calculations by using the dualities on the homology $H_*(\tilde{E})$ in [4]:

Lemma 2.5.
(1) $\beta_1(L) = \beta_3(L) \leqq r - 1$ and $\beta_2(L) = 2(g - \tau(L))$,
(2) $H_d(\tilde{E}) = 0$ for $d \neq 0, 1, 2, 3$, $H_0(\tilde{E}) \cong \Lambda/(t-1)\Lambda$ and $H_3(\tilde{E}) \cong \Lambda^{\beta_1(L)}$,
(3) $cr(M(L)) = r - 1$ and $cr(TM(L)) = cr(T_D M(L)) = \tau(L)$,
(4) $cr(H_2(\tilde{E})) = 2g - \tau(L)$ and $cr(TH_2(\tilde{E})) = cr(T_D H_2(\tilde{E})) = \tau(L)$.

The following corollary is direct from Lemma 2.5.

Corollary 2.6. An F_g^r-link L has $\beta_*(L) = 0$ if and only if $\beta_1(L) = 0$ and $g = r - 1$.

3. Characterizing the first Alexander Z[Z]-modules of ribbon surface-links

For a finitely generated Λ-module M, let $e(M)$ be the minimal number of Λ-generators of M. The following estimate is given by [14] and [6] for the case $r = 1$ where we have $\tau(L) = 0$:

Lemma 3.1. If L is a ribbon F_g^r-link, then we have

$$g \geqq e(E^2 M(L)) + \tau(L).$$

For proof, we use a standard Seifert hypersurface for a ribbon F_g^r-link in [9]. The following theorem is our first theorem, showing that the estimate of Lemma 3.1 is best possible.

Theorem 3.2. A finitely generated Λ-module M is the module $M(L)$ of a ribbon F_g^r-link L if and only if M is a cokernel-free Λ-module of corank $r-1$ and $g \geqq e(E^2 M) + \tau(M)$. Further, if a non-negative partition $g = g_1 + g_2 + \ldots + g_r$ is arbitrarily given, then we can take a ribbon F_g^r-link L with $g(F_i) = g_i$ for all i.

For proof, we use an algorithm of A. Pizer [12] to produce a Wirtinger presentation of a group from a given Λ-matrix and T. Yajima's construction of a ribbon surface-link in [16] from a given Wirtinger presentation as well as Lemmas 2.5 and 3.1. The following corollary is direct from Lemmas 2.4, 2.5 and Theorem 3.2.

Corollary 3.3. A finitely generated Λ-module M is a cokernel-free Λ-module of corank n if and only if there is an F_g^{n+1}-link L with $M(L) = M$ for some g.

The following corollary gives a characterization of the modules $M(L)$ of ribbon F_g^{n+1}-links L with $\beta_*(L) = 0$.

Crollary 3.4. A cokernel-free Λ-module M of corank n is the module $M(L)$ of a ribbon F_g^{n+1}-link L with $\beta_*(L) = 0$ (in this case, we have necessarily $g = n$) if and only if $\beta(M) = 0$ and $DM = 0$.

Here are two examples which are not covered by Corollary 3.4.

Example 3.5. For a cokernel-free Λ-module M of corank n with $\beta(M) = 0$ (so that $\tau(M) = n$) and $DM = 0$, we have the following examples (1) and (2).
(1) Let $M' = M \oplus \Lambda/(t+1, a)$ for an odd $a \geqq 3$. Since $E^2M' \cong \Lambda/(t+1, a) \neq 0$, the Λ-module M' is not the module $M(L)$ of a ribbon F_g^{n+1}-link L with $\beta_*(L) = 0$. On the other hand, $\Lambda/(t+1, a)$ is wel-known to be the module of a non-ribbon F_0^1-knot K (for example, the 2-twist-spun knot of the 2-bridge knot of type $(a, 1)$) and M is the module $M(L)$ of a ribbon F_n^{n+1}-link L with $\beta_*(L) = 0$ by Corollary 3.4. Hence M' is the module $M(L')$ of a non-ribbon F_n^{n+1}-link L' (taking a connected sum $L \# K$) with $\beta_*(L') = 0$.
(2) Let $M'' = M \oplus \Lambda/(2t-1, a)$ for an odd $a \geqq 5$. Although M'' is cokernel-free of corank n and $\beta(M'') = 0$, we can show that M'' is not the module $M(L)$ of any F_g^{n+1}-link L with $\beta_*(L) = 0$ by the second duality of [4]. On the other hand, there is a ribbon F_{n+1}^{n+1}-link L'' with $M(L'') = M''$ by Theorem 3.2, because $e(E^2M'') = e(\Lambda/(2t-1, a)) = 1$ and hence $e(E^2M'') + \tau(M'') = 1 + n$. In this case, we have $\beta_2(L'') = 2$ by Lemma 2.5.

4. A characterization of the first Alexander Z[Z]-modules of virtual links

The notion of virtual links was introduced by L. H. Kauffman [3]. A *virtual r-link diagram* is a diagram D of immersed oriented r loops in S^2 with two kinds of crossing points given in Fig. 2, where the left or right crossing point is called a *real* or *virtual* crossing point, respectively. A *virtual r-link* ℓ is the equivalence class of virtual r-link diagrams D under the local moves given in Fig. 3 which are called *R-moves* for the first three local moves and *virtual R-moves* for the other local moves. A virtual r-link is called a *classical r-link* if it is represented by a virtual link diagram without virtual crossing points. The group $\pi(\ell)$ of a virtual r-link ℓ is the group with finite presentation whose generators consist of the edges of a virtual link diagram D of ℓ and whose relations are obtained from D as they are indicated in Fig. 4. It is

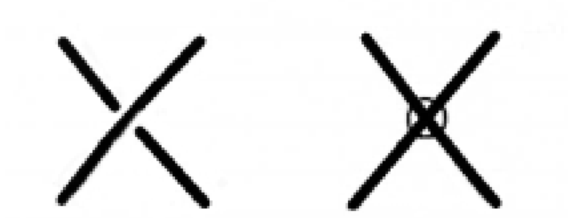

Figure 2: A real or virtual crossing point

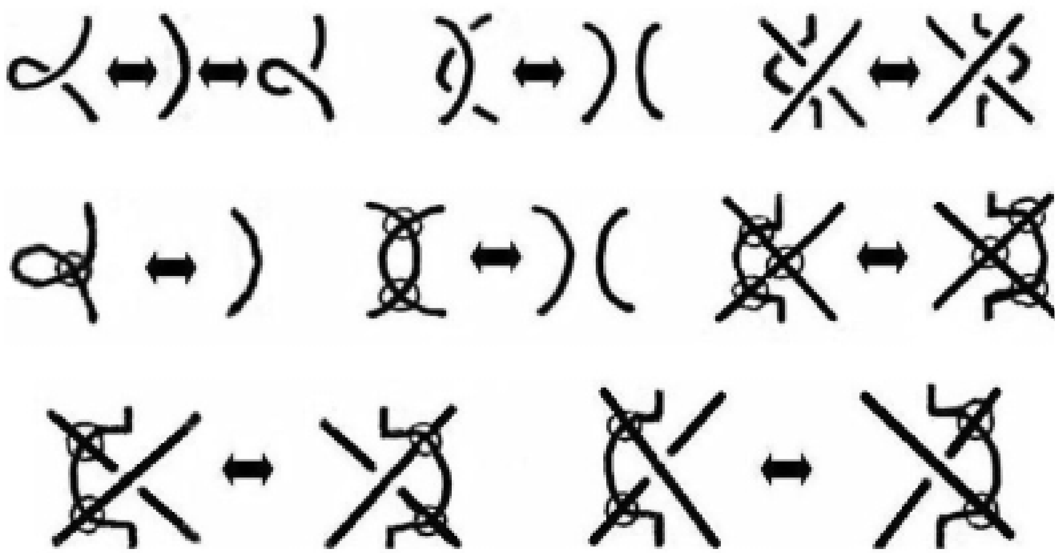

Figure 3: R-moves and Virtual R-moves

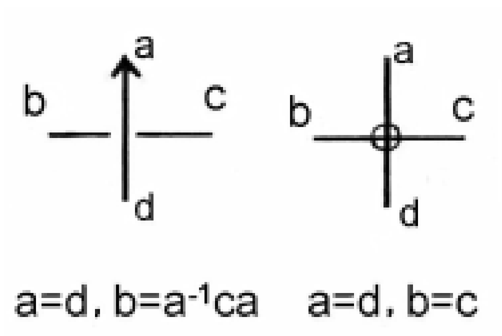

Figure 4: Relations

easily checked that the Wirtinger group $\pi(\ell)$ up to Tietze equivalences is unchanged under the R-moves and virtual R-moves. Fig. 5 defines a map σ' from a virtual r-link diagram to a disk-arc presentation of a ribbon $F^r_{1,1,\ldots,1}$-link. S. Satoh proved in [13]

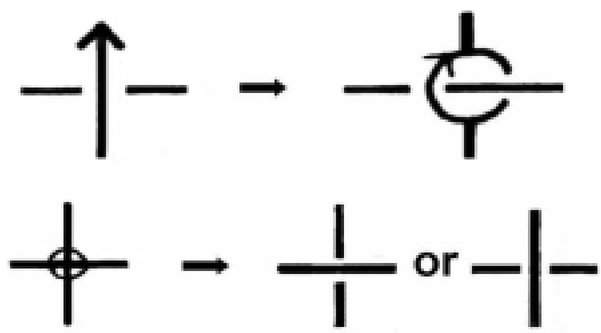

Figure 5: Definition of the map σ'

that this map σ' induces a (non-injective) surjective map σ from the set of virtual r-links onto the set of ribbon $F^r_{1,1,\ldots,1}$-links. For example, the map σ sends a nontrivial virtual knot into a trivial F^1_1-knot in Fig. 6, where non-triviality of the virtual knot is shown by the Jones polynomial (see [3]) and triviality of the F^1_1-knot is shown by an argument of [2] on deforming a 1-handle. T. Yajima in [16] gives a Wirtinger

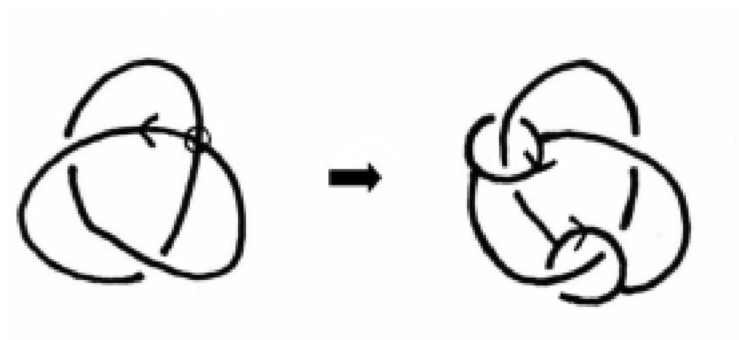

Figure 6: A non-trivial virtual knot sent to the trivial F^1_1-knot

presentation of the group $\pi_1(S^4 \backslash L)$ of a ribbon F^r_g-link L. From an analogy of the constructions, we see that the map σ induces the same Wirtinger presentation of a virtual r-link diagram D and the disk-arc presentation $\sigma'(D)$. Thus, we have the following proposition which has been independently observed by S. G. Kim [10], S. Satoh [13] and D. Silver-S. Williams [15] in the case of virtual knots:

Proposition 4.1. The set of the groups of virtual r-links is the same as the set of the groups of ribbon $F^r_{1,1,\ldots,1}$-links.

For a virtual r-link ℓ, let $\gamma : \pi(\ell) \to Z$ be the epimorphism sending every generator of a Wirtinger presentation to 1, which is independent of a choice of Wirtinger presentations. The *first Alexander* $Z[Z]$*-module*, or simply the *module* of a virtual r-link ℓ is the Λ-module $M(\ell) = \mathrm{Ker}\gamma/[\mathrm{Ker}\gamma, \mathrm{Ker}\gamma]$. The following corollary is direct from Proposition 4.1.

Corollary 4.2. The set of the modules of virtual r-links is the same as the set of the modules of ribbon $F^r_{1,1,\dots,1}$-links.

The following theorem giving a characterization of the modules of virtual r-links is direct from Theorem 3.2 and Corollary 4.2.

Theorem 4.3. A finitely generated Λ-module M is the module $M(\ell)$ of a virtual r-link ℓ if and only if M is a cokernel-free Λ-module of corank $r-1$ and has $e(E^2M) \leqq 1 + \beta(M)$.

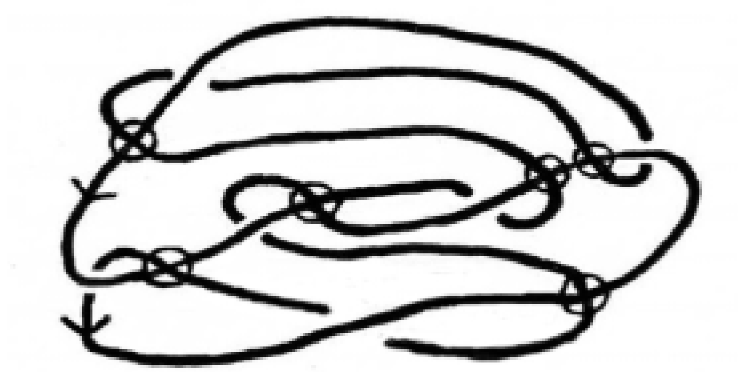

Figure 7: A virtual 2-link sent to the ribbon $F^2_{1,1}$-link in Fig. 1

Here is one example.

Example 4.4. The ribbon $F^2_{1,1}$-link in Fig. 1 is the σ-image of a virtual 2-link ℓ illustrated in Fig. 7 whose group has the Wirtinger presentation

$$\pi(\ell) = (x, y|x = (yx^{-1}y^{-1})x(yx^{-1}y^{-1})^{-1}, y = (x^{-1}yx^{-1})y(x^{-1}yx^{-1})^{-1})$$

and whose module is calculated as $M(\ell) = \Lambda/((t-1)^2, 2(t-1))$. Since $DM(\ell) = \Lambda/((t-1), 2) \neq 0$, the virtual 2-link ℓ is not any classical 2-link, because for any classical r-link ℓ' with $M(\ell')$ a torsion Λ-module, we must have $DM(\ell') = 0$ by the second duality of [4] (cf. [5]).

We see from Theorem 4.3 that M is the module of a virtual knot (i.e., a virtual 1-link) if and only if M is a cokernel-free Λ-module of corank 0 and has $e(E^2M) \leqq 1$, for we have $\beta(M) = 0$ for every cokernel-free Λ-module of corank 0. For a direct sum on the modules of virtual knots, we obtain the following observations.

Corollary 4.5.
(1) For the module M of every virtual knot with $e(E^2M) = 1$, the $n(> 1)$-fold direct sum M^n of M is a cokernel-free Λ-module of corank 0, but not the module of any virtual knot.
(2) For the module M of every virtual knot and the module M' with $e(E^2M') = 0$, the direct sum $M \oplus M'$ is the module of a virtual knot.

5. A graded structure on the first Alexander $Z[Z]$-modules of surface-links

Let $\mathcal{A}^r_g$ be the set of the modules $M(L)$ of all F^r_g-links L, and $\mathcal{A}^r_* = \cup_{g=0}^{+\infty}\mathcal{A}^r_g$. In this section, we show properness of the inclusions

$$\mathcal{A}^r_0 \subset \mathcal{A}^r_1 \subset \mathcal{A}^r_2 \subset \cdots \subset \mathcal{A}^r_*.$$

To see this, we establish an estimate of the total genus g by the module of a general F^r_g-link. To state this estimate, we need some notions on a finite Λ-module. A finite Λ-module D is *symmetric* if there is a t-anti isomorphism $D \cong E^2D = \hom_Z(D, Q/Z)$, and *nearly symmetric* if there a Λ-exact sequence

$$0 \to D_1 \to D \to D^* \to D_0 \to 0$$

such that $D_i (i = 0, 1)$ are finite Λ-modules with $(t - 1)D_i = 0$ and D^* is a finite symmetric Λ-module. For a general F^r_g-link L, we shall show the following theorem:

Theorem 5.1. If M is the module $M(L)$ of an F^r_g-link L, then we have a nearly symmetric finite Λ-submodule $D \subset DM$ such that $g \geqq e(E^2(M/D))/2 + \tau(M)$.

For proof, we use the second duality in [4]. For an application of this theorem, it is useful to note that every finite Λ-module D has a unique splitting $D_{t-1} \oplus D_c$ (see[8, Lemma 2.7]), where D_{t-1} is the Λ-submodule consisting of an element annihilated by the multiplication of some power of $t - 1$ and D_c is a cokernel-free Λ-submodule of corank 0. As a direct consequence of this property, we see that if D is nearly symmetric, then D_c is symmetric. Then we can obtain the following result from Theorem 5.1.

Corollary 5.2. For every $r \geqq 1$, we have

$$\mathcal{A}^r_0 \subsetneqq \mathcal{A}^r_1 \subsetneqq \mathcal{A}^r_2 \subsetneqq \mathcal{A}^r_3 \subsetneqq \cdots \subsetneqq \mathcal{A}^r_*$$

and the set $\mathcal{A}^r_*$ is equal to the set of finitely generated cokernel-free Λ-modules of corank $r - 1$, so that $\mathcal{A}^r_* \cap \mathcal{A}^{r'}_* = \emptyset$ if $r \neq r'$.

In this corollary, the characterization of $\mathcal{A}^r_*$ is direct from Corollary 3.3.

6. A graded structure on the first Alexander $Z[Z]$-modules of classical links, surface-links and higher-dimensional manifold-links

An *n-dimensional manifold-link with r components* is the ambient isotopy class of a closed oriented n-manifold with r components embedded in the $(n+2)$-sphere S^{n+2} by a locally-flat embedding. A 1-dimensional manifold-link with r components is the same as a classical r-link (as a virtual link) by a result of M. Goussarov, M. Polyak and O. Viro [1]. Let $E_Y = S^{n+2} \backslash \mathrm{int} N(Y)$ for a tubular neighborhood $N(Y)$ of Y in S^{n+2}. Since $H_1(E_Y) \cong Z^r$ has a unique oriented meridian basis, we have a unique infinite cyclic covering $p : \tilde{E}_Y \to E_Y$ associated with the epimorphism $\gamma : H_1(E_Y) \to Z$ sending every oriented meridian to 1. The *first Alexander $Z[Z]$-module*, or simply the *module* of the manifold-link Y is Λ-module $M(Y) = H_1(\tilde{E}_Y)$. Let $\mathcal{A}(n)^r$ be the set of the modules of n-dimensional manifold-links with r components. Since $\mathcal{A}(2)^r = \mathcal{A}^r_*$, it is suitable here to denote the set $\mathcal{A}^r_g$ by $\mathcal{A}(2)^r_g$. For the set $\mathcal{A}(1)^r$, we further consider the subset $\mathcal{A}(1)^r_g = \mathcal{A}(1)^r \cap \mathcal{A}(2)^r_g$. We have $\mathcal{A}(1)^r_g \subset \mathcal{A}(1)^r_{g+1} \subset \mathcal{A}(1)^r$ for every $g \geqq 0$. Taking a split union of classical knots with non-trivial Alexander polynomials, we see that the set $\mathcal{A}(1)^r_0$ is infinite.

We have the following theorem giving a graded structure on the modules of classical r-links, F^r_*-links and higher-dimensional manifold-links with r components:

Theorem 6.1. For every $r \geqq 1$ and $s \geqq 0$, we have $\mathcal{A}(1)^r_s \cup \mathcal{A}(2)^r_{s-1} \subsetneqq \mathcal{A}(2)^r_s$ where we take $\mathcal{A}(2)^r_{-1} = \emptyset$, and

$$\mathcal{A}(1)^r_0 \subsetneqq \mathcal{A}(1)^r_1 \subsetneqq \cdots \subsetneqq \mathcal{A}(1)^r_{r-1} = \mathcal{A}(1)^r \subsetneqq \mathcal{A}(2)^r_{r-1} \subsetneqq \cdots \subsetneqq \mathcal{A}(2)^r = \mathcal{A}(3)^r = \cdots .$$

On the inclusion $\mathcal{A}(1)^r \subset \mathcal{A}(2)^r$, we note that the invariant $\kappa_1(\ell)$ in [7] is equal to the torsion-corank $\tau(L)$ for every classical r-link ℓ and every F^r_g-link L with $M(\ell) = M(L)$.

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