On the orientations of monotone knot diagrams

Akio Kawauchi *　　Ayaka Shimizu †

April 25, 2017

Abstract

An oriented monotone knot diagram is a knot diagram such that one meets each crossing as an over-crossing first as one travels the diagram with the orientation by starting at a point on the diagram. In this paper, unoriented knot projections which are monotone with an orientation and any over/under information are characterized. Also, monotone diagrams which are monotone with exactly one orientation and unique basepoint are characterized. As an application, a necessary condition for a knot projection with reductiviy four is given.

1 Introduction

A *knot* is an embedded circle in S^3. A knot K is said to be *oriented* if K has a one-way orientation. For an oriented knot K, the *inverse of* K, denoted by $-K$, is the oriented knot which is K with orientation reversed. Oriented knots K and K' are *equivalent* if there is an orientation-preserving homeomorphism $h : S^3 \to S^3$ such that $h(K) = K'$. A knot K is *invertible* if K and $-K$ are equivalent. The invertibility of knots is one of the most important studies in knot theory and has a long history (see, for example, [9]). Looking toward the study of invetibility, this paper attempts to study

*Osaka City University Advanced Mathematical Institute, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan. Email: kawauchi@sci.osaka-cu.ac.jp

†Department of Mathematics, National Institute of Technology, Gunma College, 580 Toriba-cho, Maebashi-shi, Gunma 371-8530, Japan. Email: shimizu@nat.gunma-ct.ac.jp

canonical orientations of knot diagrams and knot projections on S^2 in a sense of monotonicity. Note that it is shown in [11] by the authors that every knot projection on $\mathbb{R}^2$ with even crossing points has two independent canonical orientations and every based knot projection on $\mathbb{R}^2$ with odd crossing points has two independent canonical orientations due to warping degree.

A *knot projection* P on S^2 is an image of a regular projection of a knot on S^2 with double point singularities. Each double point is called a *crossing*. A *knot diagram* on S^2 is a knot projection with over/under information at each crossing. For a knot projection and a knot diagram, an *edge* is a path bounded by two crossings which has no crossings in the interior. A knot diagram is said to be *oriented* if the diagram has a one-way orientation. An orientation of a knot diagram is shown by an arrow as depicted in Figure 1. Let D be an oriented knot diagram. The *inverse* of D, denoted by $-D$, is the oriented knot diagram D with orientation reversed. The unoriented knot diagram D without orientation is denoted by $|D|$, and the knot projection D without over/under information is denoted by $\bar{D}$ (see Figure 1). For D and

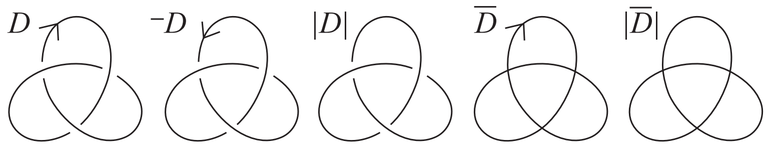

Figure 1: Knot diagrams and projections.

$\bar{D}$, it is said that D *is constructed from* $\bar{D}$. In this paper, assume that every knot diagram and knot projection has at least one crossing. An oriented knot diagram D is *monotone* if one can travel D starting at a point in the interior of an edge of D with the orientation so that one goes through each crossing as an over-crossing at the first time. An unoriented knot diagram $|D|$ is *monotone* if it is monotone with an orientation. It is well-known that any knot projection $\bar{D}$ (and also $|\bar{D}|$) can be monotone with some suitable over/under information. In this paper, the following is shown:

Theorem 1.1. *Any unoriented knot diagram constructed from a knot projection P is monotone if and only if P is one of the five knot projections illustrated in Figure 2.*

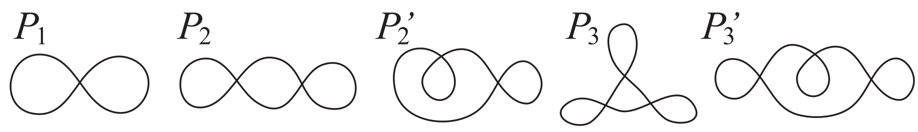

Figure 2: The knot projections P_1, P_2, $P_2{}'$, P_3 and $P_3{}'$.

In particular, the following holds:

Theorem 1.2. *Any knot diagram constructed from a knot projection P is monotone with exactly one orientation if and only if P is P_3 or $P_3{}'$ in Figure 2.*

Theorem 1.2 implies that any knot diagram constructed from P_3 or $P_3{}'$ is monotone, and a unique orientation is determined such that it is monotone. Hence, for P_3 and $P_3{}'$, there exists the unique orientation for each over/under information. The following theorem characterizes unoriented monotone diagrams such that the basepoint and orientation are determined uniquely with respect to monotonicity.

Theorem 1.3. *Let $|D|$ be an unoriented monotone knot diagram. Then $|D|$ is monotone with the unique orientation and basepoint if and only if $|D|$ is neither a one-bridge diagram nor a diagram obtained by connecting two or more monotone 1-tangles with crossings.*

The definitions of a one-bridge diagram and a monotone 1-tangle are given in Section 4. See also Figure 3 for a knot diagram which is obtained by connecting monotone 1-tangles.

The rest of the paper is organized as follows: In Section 2, the review of warping degree and warping degree labeling are provided. In Section 3, proofs of Theorems 1.1 and 1.2 are given. In Section 4, a proof of Theorem 1.3 is given. In Section 5, the relation between reductivity and warping degree labeling is discussed as an application.

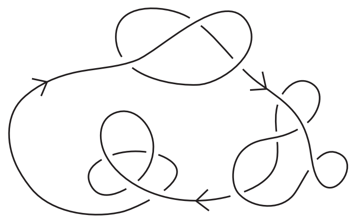

Figure 3: A knot diagram obtained by connecting monotone 1-tangles.

2 Warping degree

In this section, the review of the warping degree of knot diagrams and related topics are presented. Let D be an oriented knot diagram and b a basepoint of D which is not on any crossing. The pair of D and b is denoted by D_b. A crossing p of D is a *warping crossing point of* D_b if one meets p as an under-crossing first when one travels D from b with the orientation. The *warping degree* $d(D_b)$ *of a based oriented knot diagram* D_b is the number of warping crossing points of D_b. The *warping degree* $d(D)$ *of an oriented knot diagram* D is the minimal $d(D_b)$ for all basepoints b of D (see Figure 4). Remark that the choice of a basepoint of D is finite; it depends only on

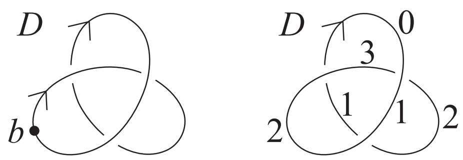

Figure 4: The warping degree of D_b is two, and the warping degree of D is zero. The warping degree labeling is given on the right-hand side.

which edge one takes a basepoint on. In this paper, two basepoints on the same edge are assumed as the same basepoint. By definition, an oriented knot diagram D is monotone if and only if $d(D) = 0$. Hence it is also said that the warping degree represents how far from a monotone diagram. Note that warping degree was defined for oriented knot and link diagrams in [10] and the similar notions and their expansions are also studied ([8], [12], [14], [17], etc.).

Let D be an oriented knot diagram. The *warping degree labeling* for D is the labeling such that each edge has the integer which is the warping degree with a basepoint at the edge as depicted in Figure 4. By the definition, warping degree labeling has the rule shown in Figure 5 (Lemma 2.5 in [16]).

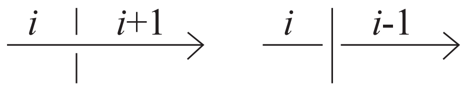

Figure 5: The labeling of the warping degree labeling increases by one by passing an over-crossing because the crossing becomes to be counted as a warping crossing point. Similarly, the labeling decreases by one by passing an under-crossing.

The *chord diagram of an oriented knot projection* P is a preimage of P with each pair of points corresponding to the same double point connected by a segment when P is assumed as an image of an immersion of a circle to S^2. A chord diagram represents the cyclic order of crossings which one passes as one travels along a knot projection with the orientation. By giving an orientation to each segment from over-crossing to under-crossing, a chord diagram is enhanced with respect to over/under information, and is called the *chord diagram of an oriented knot diagram*. Moreover, by giving the sign of the corresponding crossing to each segment, a chord diagram is still enhanced and called the *Gauss diagram*. For more details, see [4] and [7].

A *connected sum* of two oriented knot projections P and Q is a knot projection obtained from P and Q by deleting a subarc of an edge of each projection and connecting them so that the orientation is preserved (see Figure 6). Note that it depends on the choice of edges to connect. A *prime knot*

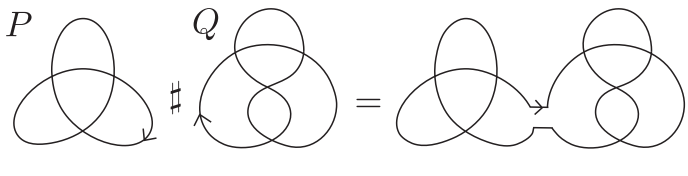

Figure 6: A connected sum of P and Q.

projection is a knot projection which is not any connected sum of (nontrivial) knot projections. Remark that any prime knot diagram can be restored from its chord diagram (see, for example, [3]), and any knot diagram can be restored from its Gauss diagram ([4, 7]). In [2], all the chord diagrams of the prime knot projections up to ten crossings are listed.

3 Proofs of Theorems 1.1 and 1.2

In this section Theorems 1.1 and 1.2 are proved. For knot projections with four crossings, the following lemma holds:

Lemma 3.1. *Any unoriented knot projection with four crossings can be non-monotone with an over/under information.*

Proof. All the chord diagrams of knot projections with four crossings are listed in Figure 7, and they can be non-monotone with some suitable over/under information as shown in Figure 8.

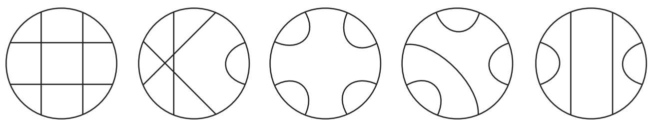

Figure 7: All the chord diagrams of knot projections with four crossings.

□

For knot projections with five crossings, the following holds:

Lemma 3.2. *Any unoriented knot projection with five crossings can be non-monotone with an over/under information.*

Proof. All the knot projections with five crossings are the two prime knot projections which have the chord diagrams depicted in Figure 9 and the knot projections which are obtained from a knot projection with four crossings

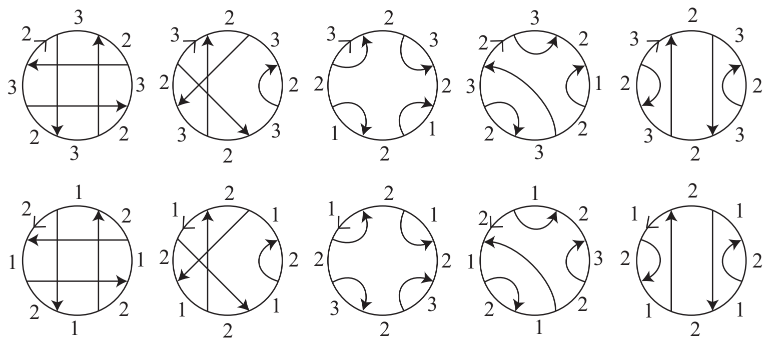

Figure 8: Non-monotone chord diagrams. The digits represent the warping degree labeling.

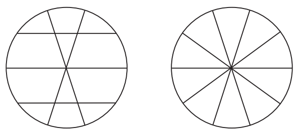

Figure 9: All the two chord diagrams of prime knot projections with five crossings.

by a single RI+ move (see Figure 10). The knot projections which have the chord diagrams in Figure 9 can be non-monotone with the over/under information shown in Figure 11.

Next, let P be a knot projection which is obtained from a knot projection P' with four crossings by an RI+. Let D' be a knot diagram constructed from P' such that D' is not monotone. Remark that such a diagram D' exists by Lemma 3.1. Let D be a knot diagram constructed from P such that the crossings of D which correspond to the crossings of D' have the same over/under information as D'. Since D' is non-monotone, D' has a warping crossing point with every basepoint and orientation, and so does D. Hence D is also a non-monotone diagram. □

For knot projections with four or more crossings, the following holds:

Lemma 3.3. *Any unoriented knot projection with four or more crossings can be non-monotone with an over/under information.*

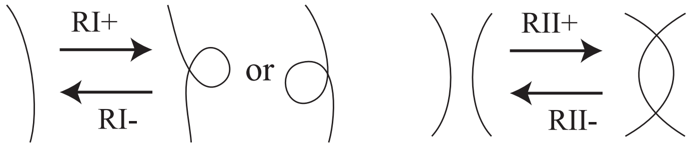

Figure 10: The moves RI+, RI-, RII+ and RII-.

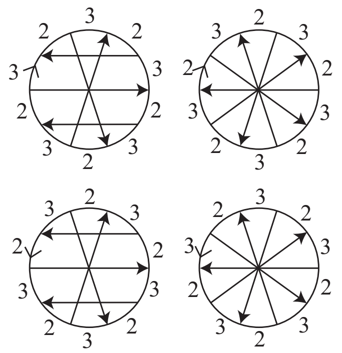

Figure 11: Non-monotone chord diagrams. The digits represent the warping degree labeling.

Proof. The proof is an induction on the crossing number c of an unoriented knot projection. When $c = 4$ or 5, it holds by Lemmas 3.1 and 3.2. Assume the lemma holds for $c \leq k$ for an integer $k \geq 5$. Let P be a knot projection with $k + 1$ crossings, and C_P the chord diagram of P.

According to [6], if C_P does not include the three segments depicted in the left-hand side of Figure 12, then P must have a 1-gon or 2-gon (a 1-gon and 2-gon are shown in Figure 13). If P has the three segments in Figure 12

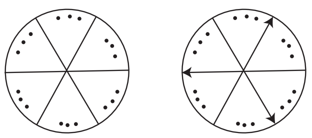

Figure 12: If a knot projection P does not have the three segments as the left-hand side in the chord diagram, P has a 1-gon or 2-gon.

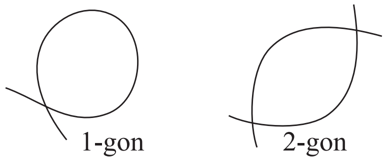

Figure 13: 1-gon and 2-gon.

in the chord diagram, then by giving the over/under information as depicted in the right-hand side of Figure 12, a knot diagram such that it has a warping crossing point with any basepoint and any orientation can be constructed.

If P has a 1-gon, apply an RI- at a 1-gon. Then a knot projection P' with k crossings is obtained. By assumption, there exists a non-monotone knot diagram D' constructed from P'. Let D be a knot diagram constructed from P such that the crossings corresponding to D' have the same over/under information as D'. Since D' is non-monotone, D' has a warping crossing point with every basepoint and orientation, and so does D. Hence D is non-monotone, too.

If P has a 2-gon, apply an RII- at a 2-gon. Then a knot projection P'' with $k-1$ crossings is obtained, and P'' has a non-monotone diagram D'' by assumption. Let D be a knot diagram constructed from P such that the crossings corresponding to D'' have the same over/under information as D''. Then D'' has a warping crossing point with every basepoint and orientation, so does D. Hence D is non-monotone. Therefore, the lemma holds for all $c \geq 4$. □

Furthermore, the following holds:

Lemma 3.4. *Any unoriented knot projection P except for the five knot projections illustrated in Figure 2 can be non-monotone with an over/under information.*

Proof. As Lemma 3.3 has been shown, it is sufficient to check the knot projections with three or less crossings. There are exactly five chord diagrams of knot projections with three or less crossings as depicted in Figure 14. The

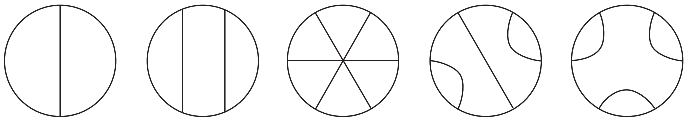

Figure 14: All the chord diagrams of knot projections with three or less crossings.

left two chord diagrams is the chord diagrams of P_1 and P_2 and P_2' in Figure 2. The next two chord diagrams can be non-monotone with the over/under information depicted in Figure 15. The chord diagram on the right-hand side

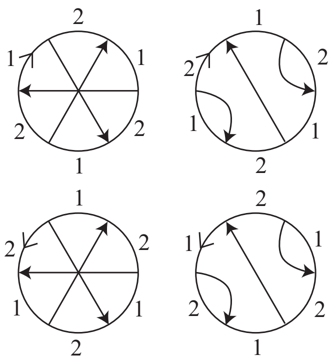

Figure 15: Non-monotone chord diagrams. The digits represent the warping degree labeling.

in Figure 14 is the chord diagram of P_3 and P_3' in Figure 2. □

Now the proof of Theorem 1.1 is given.

Proof of Theorem 1.1. By Lemma 3.4, it is sufficient to check that the knot projections in Figure 2 are monotone with any over/under information. Figure 16 shows that P_1, P_2 and P_2' are monotone with any over/under information. For P_3 and P_3', which have the chord diagram at the right-hand side in Figure 14, the chord diagrams with all the over/under information are listed in Figure 17, and they are monotone.

□

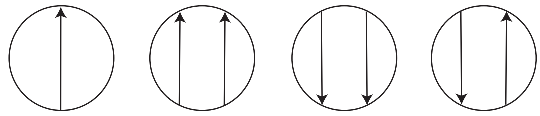

Figure 16: The left chord diagram corresponds to P_1, and the other ones correspond to P_2 and P_2'. A monotone diagram is obtained from them with any over/under information. In particular, the left three chord diagrams correspond to a monotone diagram with both orientations.

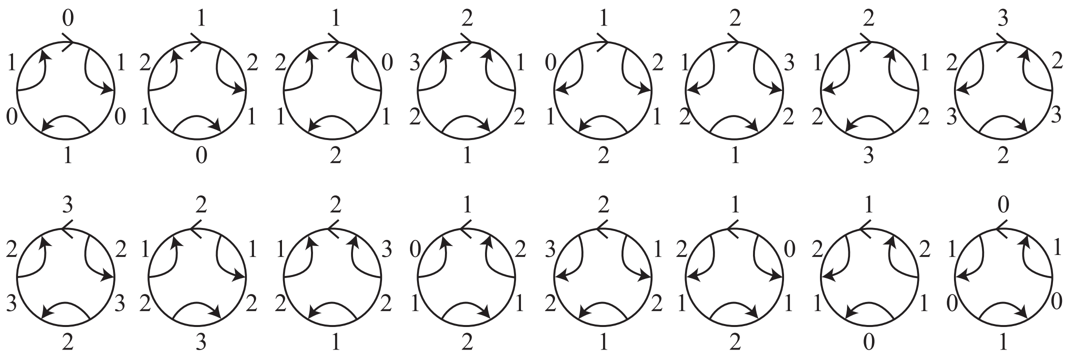

Figure 17: Chord diagrams of P_3 and P_3' with all the over/under information and orientations.

Next the proof of Theorem 1.2 is given:

Proof of Theorem 1.2. By Theorem 1.1, the five knot projections P_1, P_2, P_2', P_3 and P_3' are monotone with any over/under information. The knot projections P_1, P_2 and P_2' have knot diagrams which are monotone with both orientations as shown in Figure 16. On the other hand, P_3 and P_3' do not have any knot diagram which is monotone with both orientations as shown in Figure 17. □

From the proof of Theorem 1.2, the following corollary is obtained:

Corollary 3.5. *Any oriented knot diagram constructed from a knot projection P is monotone if and only if P is P_1 in Figure 2.*

4 Proof of Theorem 1.3

In this section, the uniqueness of the orientation and basepoint for some monotone diagrams is discussed. A knot diagram is said to be a *one-bridge diagram* if the circle S^1 of the chord diagram can be divided into two sides ${S^1}_+$ and ${S^1}_-$ such that ${S^1}_+ \cup {S^1}_- = S^1$, there are no segments on the boundaries of ${S^1}_+$ and ${S^1}_-$, and every segment belongs to the two sides of ${S^1}_+$ and ${S^1}_-$ with the orientation from ${S^1}_+$ to ${S^1}_-$. In other words, a knot diagram D is a one-bridge diagram if D has an arc which has all the over-crossings as depicted in Figure 18, where an *arc* of a knot diagram D is a path of D which has under-crossings at the endpoints and has no under-crossings in the interior. In [18], it is shown that a knot diagram D with c crossings is

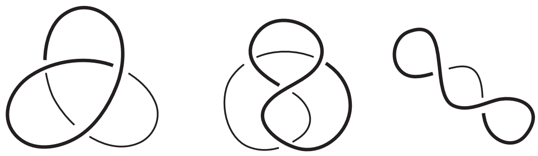

Figure 18: One-bridge diagrams.

a one-bridge diagram if and only if there are exactly one "0" and "c" and two "i"s for $i = 1, 2, \ldots, c-1$ on the warping degree labeling of D. Remark that there are knot projections which has no one-bridge diagram with any over/under information. See, for example, the knot projections in Figure 19 and P_3 and $P_3{}'$ in Figure 2. By definition, the following lemma holds:

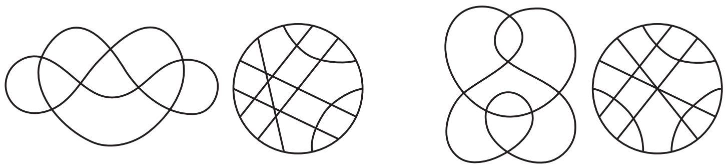

Figure 19: Knot projections such that any knot diagram constructed from the knot projection is not one-bridge.

Lemma 4.1. *If the chord diagram of a knot projection P includes the three segments illustrated in Figure 20, then P can not be a one-bridge diagram with any over/under information.*

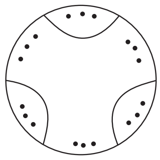

Figure 20: If a knot projection P has the three segments in its chord diagram, then P can not be a one-bridge diagram with any over/under information.

For monotonicity, the following holds:

Lemma 4.2. *An oriented knot diagram D and its inverse $-D$ are both monotone if and only if D is a one-bridge diagram.*

Proof. Let D be an oriented knot diagram with c crossings such that D and $-D$ are monotone. Then D and $-D$ have $d(D_a) = 0$ and $d(-D_b) = 0$ with some basepoints a and b, respectively. Since $d(D_b) + d(-D_b) = c$ by the definition of warping degree, D has $d(D_b) = c - d(-D_b) = c$ with the basepoint b. Hence both "0" and "c" appear in the warping degree labeling of D. This means all the c over-crossings are placed in a row by the rule of Figure 5. Hence D is a one-bridge diagram.

On the other hand, if D is a one-bridge diagram, D and $-D$ are monotone by taking basepoints near the end points on the arc which has all the over-crossings.

□

An *oriented 1-tangle* is a tangle obtained from an oriented knot diagram by deleting a subarc of an edge. An oriented 1-tangle is said to be *monotone* if one goes through each crossing as an over-crossing at the first time when one travels the 1-tangle from the initial point to the terminal point. Also, the warping degree and warping degree labeling of 1-tangles are defined in the same way ([11]). An *e-connected sum of two split 1-tangles T and U* is a 1-tangle obtained from T and U by connecting the terminal point of T and the initial point of U. Remark that it depends on an order of T and U. A 1-tangle is said to be *e-prime* if it is not any e-connected sum of 1-tangles which have crossing points. The *closure of a 1-tangle* T is a knot diagram which is obtained from T by closing the terminal point and initial point of T in S^2 without introducing a crossing point. Remark that it does not necessary hold that the closure of an e-prime 1-tangle is a prime knot diagram (see Figure 21) whereas it holds that a non-prime knot diagram is a closure of an e-connected sum of some two 1-tangles.

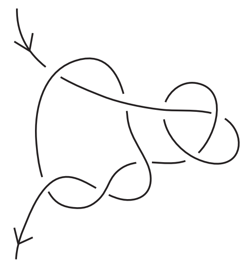

Figure 21: The closure of an e-prime 1-tangle is not always prime.

The following lemma holds:

Lemma 4.3. *A monotone 1-tangle T is not e-prime if and only if there are three or more "0"s on the warping degree labeling of T.*

Note that the warping degree labeling at the initial and terminal edges of a monotone 1-tangle are always 0. Hence Lemma 4.3 implies that a monotone 1-tangle is e-prime if and only if there are no "0"s on the warping degree labeling except at the initial and terminal points.

Proof of Lemma 4.3. If a monotone 1-tangle T is an e-connected sum of 1-tangles T_1 and T_2 which have crossings, then T_1 and T_2 are also monotone because T has no warping crossing points with a basepoint at the initial edge. Then the warping degree of T with the base point at the connecting point is zero. Hence T has three or more "0"s.

Next, let T be a monotone 1-tangle which has the warping degree zero with a basepoint c on an edge which is neither the initial edge nor the terminal edge. Let a and b be basepoints on the initial and terminal edges, respectively. Since $d(T_a) = 0$, all the crossings one meets from a to c are over-crossing at the first time one meets. Since $d(T_a) = d(T_c)$, there are under-crossings as many as over-crossings from a to c. Hence one meets each crossing from a to c twice. Therefore the part from a to c is a monotone 1-tangle which has a crossing point. Similarly, the part from c to b is also a monotone 1-tangle which has a crossing point. Hence T is not e-prime.

□

More precisely, the following holds:

Lemma 4.4. *A monotone 1-tangle T is an e-connected sum of n e-prime 1-tangles with crossings if and only if there are $n+1$ "0"s on the warping degree labeling of T.*

Proof. If T is a monotone 1-tangle which is an e-connected sum of n e-prime 1-tangles, i.e., T is an e-connected sum of n e-prime monotone 1-tangles, only connecting points have the warping degree zero because the warping degree labeling is preserved by connecting monotone 1-tangles, and they have "0" at just their initial and terminal edges by Lemma 4.3. Hence there are exactly $n+1$ "0"s in the warping degree labeling of T.

Next, let T be a monotone 1-tangle which has $n+1$ "0"s in the warping degree labeling of T. Take the basepoints b_0, b_1, b_2, ..., b_n at the different edges such that $d(T_{b_i}) = 0$ $(i = 0, 1, 2, \dots, n)$, where the order is due to the orientation and hence b_0 is on the initial edge and b_n is on the terminal edge. As discussed in Lemma 4.3, each parts from b_i to b_{i+1} $(i = 0, 1, 2, \dots, n-1)$ is a monotone 1-tangle. Moreover, since the monotone 1-tangles do not have the warping degree zero except at the initial and terminal edges, they are all e-prime. □

For oriented knot diagrams, the following lemma holds:

Lemma 4.5. *Let D be an oriented knot diagram, and n be a positive integer. Then D has exactly n edges with warping degree zero if and only if D is a* *closure of a 1-tangle which is an e-connected sum of n e-prime monotone 1-tangles with crossings.*

Proof. Let b_1, b_2, ..., b_n be the basepoints of D on the different edges such that $d(D_{b_i}) = 0$ for $i = 1, 2, \ldots, n$ with the cyclic order according to the orientation. As discussed in Lemma 4.4, each part from b_i to b_{i+1} ($i = 1, 2, \ldots, n-1$) and from b_n to b_1 is an e-prime monotone 1-tangle with a crossing point.

On the other hand, when n e-prime monotone 1-tangles are connected to obtain a knot diagram, the warping degree labeling is preserved and "0" appears only at the n connecting points. □

For example, the knot diagram in Figure 3 that is a closure of a 1-tangle which is an e-connected sum of three e-prime monotone 1-tangles has exactly three edges with warping degree zero. Since every prime knot diagram is a closure of a single e-prime 1-tangle, the following corollary is obtained from Lemma 4.5:

Corollary 4.6. *Every oriented prime knot diagram has at most one edge such that the warping degree is zero with a basepoint.*

Now the proof of Theorem 1.3 is given:

Proof of Theorem 1.3. It follows from Lemmas 4.2 and 4.5.

□

5 Application to reductivity

In this section, the studies of monotone diagrams and warping degree labeling are applied to reductivity. Let P be an unoriented knot projection, and p a crossing of P. An *inverse-half-twisted splice*, denoted by HS^{-1}, at p is defined in [5] (see also [1]) to be a splice at p so that one obtains another knot (not link) projection. The *reductivity* $r(P)$ of P is the minimal number of HS^{-1}s needed to obtain a reducible knot projection from P (see Figure 22), where a *reducible knot projection* is a knot projection such that one can put a circle

on S^2 so that it intersects only one crossing of P transversely (see Figure 23).
In [15], it is shown that every knot projection has the reductivity four or

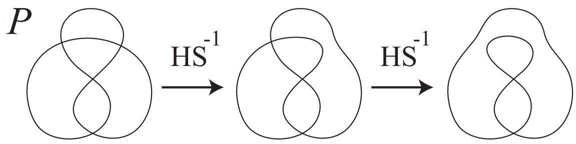

Figure 22: The reductivity of the knot projection P is two.

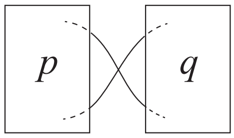

Figure 23: A reducible knot projection, where p and q are 1-tangles.

less. However, at the moment it is unknown if there exists a knot projection with reductivity four. Trigons are divided into the four types with respect to outer connections as shown in Figure 24. It is also shown in [15] that if

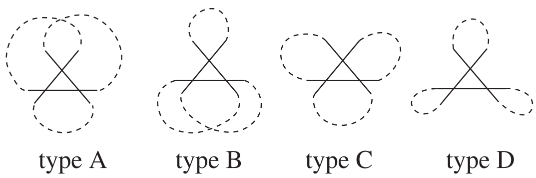

Figure 24: The trigons of type A, B, C and D. The broken curves represent the outer connection.

a knot projection P with $r(P) = 4$ exists, then P has a trigon of type D. More detailed necessary conditions are discussed in [13]. From Lemma 4.1, the following corollary is obtained:

Corollary 5.1. *Let P be a knot projection. If $r(P) = 4$, then P can not be one-bridge with any over/under information.*

From the contraposition of Corollary 5.1 and the proof of Lemma 4.2, the following corollary is obtained:

Corollary 5.2. *Let P be a knot projection with c crossings. If P has a knot diagram which has both "0" and "c" on the warping degree labeling with an over/under information, then $r(P)$ is three or less.*

Corollary 5.2 would be helpful to detect reductivity. Note that a knot projection P with c crossings has 2^c knot diagrams, and all the warping degree labelings of the 2^c diagrams with an orientation can be seen from the "warping matrix" defined in [17].

Acknowledgment.

The first author was supported by JSPS KAKENHI Grant Number 24244005. The second author was supported by Grant for Basic Science Research Projects from The Sumitomo Foundation (160154).

References

[1] J. A. Calvo, *Knot enumeration through flypes and twisted splices*, J. Knot Theory Ramifications **6** (1997), 785–798.

[2] M. Chmutov, T. Hulse, A. Lum, P. Rowell, Plane and spherical curves: an investigation of their invariants, Proceedings of the Research Experiences for Undergraduates Program in Mathematics, Oregon State University, 2006, 1–92.

[3] C. H. Dowker and M. B. Thistlethwaite, Classification of knot projections, Topol. Appl. **16** (1983), 19–31.

[4] M. Goussarov, M. Polyak and O. Viro, Finite type invariants of classical and virtual knots, Topology **39** (2000), 1045–1068.

[5] N. Ito and A. Shimizu: *The half-twisted splice operation on reduced knot projections*, J. Knot Theory Ramifications **21**, 1250112 (2012) [10 pages].

[6] N. Ito and Takimura, Any nontrivial knot projection with no triple chords has a monogon or a bigon, preprint.

[7] L. H. Kauffman, Virtual knot theory, European J. Combin. **20** (1999), 663–690.

[8] A. Kawauchi, Knot theory for spatial graphs attached to a surface, Contemporary Mathematics **670** (2016), 141–169.

[9] A. Kawauchi, The invertibility problem on amphicheiral excellent knots, Proc. Japan Acad., Ser. A, Math. Sci. **55** (1979), 399–402.

[10] A. Kawauchi, Lectures on Knot Theory (in Japanese), Kyoritsu shuppan Co. Ltd (2007).

[11] A. Kawauchi and A. Shimizu, Quantizations of the crossing number of a knot diagram, Kyungpook Math. J. **55** (2015), 741–752.

[12] W. B. R. Lickorish and K. C. Millett, A polynomial invariant of oriented links, Topology **26** (1987), 107–141.

[13] Y. Onoda and A. Shimizu, The reductivity of spherical curves Part II: 4-gons, to appear in Tokyo J. Math.

[14] M. Ozawa, Ascending number of knots and links, J. Knot Theory Ramifications **19** (2010), 15–25.

[15] A. Shimizu, The reductivity of spherical curves, Topology Appl. **196** (2015), 860–867.

[16] A. Shimizu, The warping degree of a knot diagram, J. Knot Theory Ramifications **19** (2010), 849–857.

[17] A. Shimizu, The warping matrix of a knot diagram, Contemporary Mathematics **670** (2016), 337–344.

[18] A. Shimizu, The warping polynomial of a knot diagram, J. Knot Theory Ramifications **21** (2012), 1250124 [15 pages].