ON PSEUDO-RIBBON SURFACE-LINKS

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ABSTRACT

We first introduce the null-homotopically peripheral quadratic function of a surface-link to obtain a lot of pseudo-ribbon, non-ribbon surface-links, generalizing a known property of the turned spun torus-knot of a non-trivial knot. Next, we study the torsion linking of a surface-link to show that the torsion linking of every pseudo-ribbon surface-link is the zero form, generalizing a known property of a ribbon surface-link. Further, we introduce and algebraically estimate the triple point cancelling number of a surface-link.

Keywords: Surface-knot, Surface-link, Pseudo-ribbon, Ribbon, Torsion linking, Triple point cancelling number, Triple point number

0. Introduction

A surface-knot in the 4-space \(\mathbb{R}^4\) is a closed connected oriented surface embedded in \(\mathbb{R}^4\) by a locally flat PL embedding. A surface-link \(F\) in \(\mathbb{R}^4\) (with components \(F_i\) \((i = 1, 2, \ldots, r)\)) is the union \(\bigcup_{i=1}^{r} F_i\) where \(F_i\) \((i = 1, 2, \ldots, r)\) are mutually disjoint surface-knots in \(\mathbb{R}^4\). Let \(\rho : \mathbb{R}^4 \to \mathbb{R}^3\) be the projection sending every point \((x, t)\) (where \(x \in \mathbb{R}^3\) and \(t \in \mathbb{R}\)) to the point \(x\). The singularity of a surface-link \(F\) in \(\mathbb{R}^4\) is the set

\[ S(F) = \{x \in F \mid |F \cap \rho^{-1}(\rho(x))| \geq 2\}. \]

A surface-link \(F\) in \(\mathbb{R}^4\) is generic if for every point \(x \in S(F)\), we have either

1. \(\rho(x)\) is a double point, that is, there is a 3-ball neighborhood \(V_{\rho(x)}\) of \(\rho(x)\) in \(\mathbb{R}^3\) such that \(F \cap \rho^{-1}(V_{\rho(x)})\) consists of two disjoint disks whose images by \(\rho\) meet transversely in a line containing \(\rho(x)\), or
2. \(\rho(x)\) is a triple point, that is, there is a 3-ball neighborhood \(V_{\rho(x)}\) of \(\rho(x)\) in \(\mathbb{R}^3\) such that \(F \cap \rho^{-1}(V_{\rho(x)})\) consists of three disjoint disks every pair of whose images by \(\rho\) meet transversely in a line and the resulting three lines meet transversely only at \(\rho(x)\).

\(^1\)Dedicating this paper to Professor Jerome Levine on his 65th birthday.
It is known that every surface-link in $R^4$ is ambient isotopic to a generic surface-link. In this paper, we concern the following type of surface-link:

**Definition 0.1.** A surface-link $F$ in $R^4$ is a pseudo-ribbon surface-link if $F$ is ambient isotopic to a generic surface-link $F'$ in $R^4$ such that $\rho(F')$ has no triple points, namely $|F' \cap \rho^{-1}(\rho(x))| = 2$ for every point $x \in S(F')$.

In Definition 0.1, we can see further that $F'$ is ambient isotopic to a generic surface $F''$ such that $S(F'')$ is $\emptyset$ or a closed 1-manifold, namely the singular surface $\rho(F'')$ has no triple points and no branch points (see J. S. Carter-M. Saito [3], D. Roseman [15]). A ribbon surface-link is a surface-link in $R^4$ obtained from a trivial $S^2$-link by a surgery along some embedded 1-handles. Every ribbon surface-link is a pseudo-ribbon surface-link, and conversely every pseudo-ribbon $S^2$-link is a ribbon $S^2$-link (see T. Yajima [22]). On the other hand, the turned spun $T^2$-knot $T(k)$ of a non-trivial knot $k$ (see J. Boyle [1], Z. Iwase [4], C. Livingston [13]) is pseudo-ribbon and non-ribbon (see A. Shima [19]), although the spun $T^2$-knot $T^0(k)$ is ribbon. In §1, we introduce the null-homotopically peripheral quadratic function of a surface-link which is useful in identifying a non-ribbon surface-link. Using this invariant, we can show in §1 that any connected sum of the turned spun $T^2$-knot of a non-trivial knot and any ribbon (or more generally, any pseudo-ribbon) surface-link is a pseudo-ribbon, non-ribbon surface-link. Next, we discuss the torsion linking $\ell_F$ of a surface-link $F$, which is a generalization of the Farber-Levine pairing on $S^2$-knots as it is mentioned in [7] (see also [8,9,10]). We show that the torsion linking $\ell_F$ of every pseudo-ribbon surface-link $F$ in $S^4 = R^4 \cup \{\infty\}$ vanishes by studying a canonical Seifert hypersurface of $F$ which is constructed by a method analogous to Seifert’s algorithm on constructing a Seifert surface of a knot. This result is stated in §2 together with an explanation of the torsion linking $\ell_F$ and proved in §3. We note that if $F$ is a ribbon surface-link, then this result is known. In fact, the ribbon surface-link $F$ bounds a Seifert hypersurface $V$ such that the torsion part $tH_1(V; Z) = 0$ (see [11]). Then the vanishing $\ell_F = 0$ follows from a result of M. Sekine [18] showing that the torsion linking $\ell_F$ is induced from a singular sublinking of the linking $\ell_V : tH_1(V, \partial V; Z) \times tH_1(V; Z) \to Q/Z$ defined by the Poincaré duality for every Seifert hypersurface $V$ of $F$. In §4, we introduce the triple point cancellation number $T(F)$ of a surface-link $F$ which measures a distance to the pseudo-ribbon surface-links. Using the vanishing of the torsion linking of a pseudo-ribbon surface-knot, we shall make an algebraic estimate of $T(F)$. This method is similar to S. Kamada’s argument in [5]. As an application of this estimate, we shall show in §4 that every surface-link $F$ is concordant to a surface-link $F_*$ with $T(F_*) = n$ for every previously given integer $n > T(F)$.

1. The null-homotopically peripheral quadratic function of a surface-link

By a 2-chain, we mean a simplicial 2-chain $C$ with $Z_2$-coefficients in a 4-manifold $W$. This 2-chain $C$ is regular if $|\partial C|$ is a closed 1-manifold. The support of $C$,
denoted by $|C|$ is the union of all simplices in $C$ with non-zero coefficients. For a 2- or 3-submanifold $Y$ of a 4-manifold $W$, the 2-chain $C$ in $W$ is $Y$-proper if $|C| \cap Y = |\partial C|$. A disk-chain is a 2-chain which is obtained from a simplicial map $f: B^2 \to W$ where $B^2$ is a triangulated disk. Let $F$ be a surface-link in $S^4 = R^4 \cup \{\infty\}$ with components $F_i (i = 1, 2, \ldots, r)$. Let $K(F; Z_2)$ be the subgroup of $H_1 (F; Z_2)$ consisting of an element represented by the boundary of an $F$-proper 2-chain in $S^4$. We note that every element of $K(F; Z_2)$ is represented by the boundary of an $F$-proper regular 2-chain in $S^4$. The peripheral quadratic function of a surface-link $F$ is the function

$$\Xi: K(F; Z_2) \to Z_2$$

defined by putting $\Xi(x)$ to be the $Z_2$-intersection number $\text{Int}_{S^4}(C, \tilde{C})$ in $S^4$ where $C$ and $\tilde{C}$ are $F$-proper regular 2-chains in $S^4$ with $x = [\partial C] = [\partial \tilde{C}]$ and $|\partial C| \cap |\partial \tilde{C}| = \emptyset$ such that $\tilde{C}$ is obtained from $C$ by sliding $\partial \tilde{C}$ along $F$. The function $\Xi$ is well-defined and is a quadratic function with respect to the $Z_2$-intersection form $\text{Int}_F(\ , \ )_2$ on $F$, that is, $\Xi$ has the identity

$$\Xi(x + y) = \Xi(x) + \Xi(y) + \text{Int}_F(x, y)_2$$

for all $x, y \in K(F; Z_2)$. We note that $\Xi$ may be a singular quadratic function for a general surface-link $F$, although it is always non-singular when $F$ is a surface-knot. Let $\Delta(F; Z_2)$ be the subgroup of $K(F; Z_2)$ generated by the elements represented by the boundaries of all $F$-proper disk-chains in $S^4$. For our purpose, we are interested in the restricted quadratic function

$$\xi = \Xi|_{\Delta(F; Z_2)}: \Delta(F; Z_2) \to Z_2,$$

which we call the null-homotopically peripheral quadratic function of the surface-link $F$. For a surface-link $F$ in $S^4$, let $N$ be a tubular neighborhood of $F$ in $S^4$, and $E = \text{cl}(S^4 - N)$ be the link-exterior of $F$. Let $D^2$ be the unit disk in the complex plane. A canonical trivialization of $N$ is an identification $(N, F) = (F \times D^2, F \times 0)$ such that the natural injection $F \times 1 \subset F \times S^1 = \partial N = \partial E \subset E$ induces the trivial composite homomorphism

$$H_1(F \times 1; Z) \to H_1(\partial E; Z) \to H_1(E; Z) \xrightarrow{\gamma} Z,$$

where $\gamma \in \text{hom}(H_1(E; Z), Z) = H^1(E; Z)$ is the epimorphism sending every oriented meridian to $1 \in Z$. The $\gamma$-structure on the link-exterior $E$ of a surface-link $F$ is the subset of $H^1(E; Z)$ consisting of the $\gamma$’s under all orientation changes of components of $F$, whose cardinal number is seen to be $2^r$ for the component number $r$ of $F$.

Let $\Delta_E(\partial E; Z_2)$ be the subgroup of $H_1(\partial E; Z_2)$ generated by the elements represented by the boundaries of all $\partial E$-proper disk-chains in $E$. Similarly, let
\( \Delta_E(F \times 1; Z_2) \) be the subgroup of \( H_1(F \times 1; Z_2) \) generated the elements represented by the boundaries of all \( F \times 1 \)-proper disk-chains in \( E \). The quadratic function

\[
\xi_E : \Delta_E(F \times 1; Z_2) \to Z_2
\]

is defined by a method analogous to the definition of \( \xi \), namely by

\[
\xi_E(x) = \text{Int}_E(C, \tilde{C})_2
\]

where \( C \) and \( \tilde{C} \) are \( F \times 1 \)-proper regular 2-chains in \( E \) with \( x = [\partial C] = [\partial \tilde{C}] \) and \( [\partial C] \cap [\partial \tilde{C}] = \emptyset \) such that \( \tilde{C} \) is obtained from \( C \) by sliding \( \partial C \) along \( F \times 1 \). The following lemma concerns the arguments of J. Boyle [1] and Z. Iwase [4] on \( T^2 \)-knots.

**Lemma 1.1.**

1. The natural composite map

\[
k_* : H_1(F \times 1; Z_2) \to H_1(\partial E; Z_2) \to H_1(N; Z_2) \xrightarrow{\cong} H_1(F \times 0; Z_2) = H_1(F; Z_2)
\]

induces an isomorphism

\[
k_* : \Delta_E(F \times 1; Z_2) \xrightarrow{\cong} \Delta_E(\partial E; Z_2) \xrightarrow{\cong} \Delta(F; Z_2).
\]

2. The isomorphism \( k_* \) induces an isomorphism from the quadratic function \( \xi_E : \Delta_E(F \times 1; Z_2) \to Z_2 \) to the quadratic function \( \xi : \Delta(F; Z_2) \to Z_2 \).

3. The quadratic function \( \xi_E : \Delta_E(F \times 1; Z_2) \to Z_2 \) is invariant (up to isomorphisms) under any \( \gamma \)-structure-preserving homeomorphism between the link-exteriors \( E \) for all surface-links \( F \). In particular, \( \xi_E \) is invariant (up to isomorphisms) under any homeomorphism between the knot-exteriors \( E \) for all surface-knots \( F \).

**Proof.** To see (1), we take a simplicial map \( f : B^2 \to S^4 \) giving an \( F \)-proper disk-chain \( C \) in \( S^4 \). By a general position argument on \( f \) and the uniqueness of a regular neighborhood, we may consider that the regular neighborhood \( N = F \times D^2 \) of \( F \) in \( S^4 \) meets \( |f(B^2)| \) in a singular annulus \( A \) such that \( L = (\partial N) \cap |f(D^2)| \) is a simple loop bounding a singular disk \( \text{cl}(|f(B^2)| - A) \) in \( E \). For the infinite cyclic covering \( p : \tilde{E} \to E \) associated with \( \gamma \), the boundary \( \partial E = F \times S^1 \) of \( E \) lifts to \( \partial \tilde{E} = F \times R \) and the loop \( L \) lifts to \( F \times R \) trivially. Since any component of \( p^{-1}(L) \) is homotopic in \( \partial \tilde{E} \) to a loop in \( F \times 1 \subset F \times R = \partial \tilde{E} \), we see that \( L \) is homotopic in \( \partial E \) to a loop \( L' \) in \( F \times 1 \subset F \times S^1 = \partial E \). Since the inclusion \( F \times 1 \subset N \) is a homotopy equivalence, we see that \( L' \) is homotopic to the loop \( [\partial C] \times 1 \) in \( F \times 1 \). Thus, we have a simplicial map \( f' : B^2 \to E \) giving an \( F \times 1 \)-proper disk-chain \( C' \) in \( E \) such that \( \partial C' = \partial C \times 1 \in F \times 1 \). Conversely, if we are given an \( F \times 1 \)-proper disk-chain \( C' \) in \( E \), then we can construct an \( F \)-proper disk-chain \( C \) in \( S^4 \) with \( \partial C' = \partial C \times 1 \)
by sliding $\partial C'$ along $F \times [0, 1] \subset F \times D^2$ = $N$. This implies that the map $k_*$ is an isomorphism from $\Delta_E(F \times 1; Z_2)$ onto $\Delta(F; Z_2)$ proving (1), and further that $k_*$ is an isomorphism from $\xi_E$ to $\xi$ proving (2). To see (3), let $E' = \text{cl}(S^4 - N')$ be the exterior of a surface-link $F'$ in $S^4$ where $N' = F' \times D^2$ is a tubular neighborhood of $F'$ in $S^4$ with the specified trivialization. Let $\gamma' \in H^1(E'; Z)$ be the cohomology class sending each oriented meridian of $F'$ to 1. Assume that there is a $\gamma$-structure preserving homeomorphism $h : E' \cong E$. Since by definition, $\xi$ is invariant under all the choices of the orientations on $S^4$ and the components of $F$, we can assume from (2) that $h$ is orientation-preserving and $h^*(\gamma) = \gamma'$.

We need the following sublemma:

**Sublemma 1.1.1.** Let $F_*$ be a closed oriented surface of a positive genus. Let $h$ be an orientation-preserving auto-homeomorphism of $F_* \times S^1$ such that $h^*(\gamma) = \gamma$ for the Poincaré dual $\gamma \in H^1(F_* \times S^1; Z)$ of the homology class $[F_* \times 1] \in H_2(F_* \times S^1; Z)$. Then $h$ is isotopic to an auto-homeomorphism $h'$ with $h'(F_* \times 1) = F_* \times 1$.

Let $F'_*$ and $F_*$ be any positive genus surface components of $F'$ and $F$ respectively such that $h(F'_* \times S^1) = F_* \times S^1$. Then we see from Sublemma 1.1.1 and a property of $\gamma'$, $\gamma$ that the homeomorphism $h$ is isotopic to a homeomorphism $h' : E' \cong E$ such that $h'(F'_* \times 1) = F_* \times 1$ for all $F'_*$ and $F_*$ with $h(F'_* \times S^1) = F_* \times S^1$. The homeomorphism $h'$ induces an isomorphism from the quadratic function $\xi_{E'} : \Delta_{E'}(F' \times 1; Z_2) \to Z_2$ to the quadratic function $\xi_E : \Delta_E(F \times 1; Z_2) \to Z_2$. we have (3) except the proof of Sublemma 1.1.1. □

**Proof of Sublemma 1.1.1.** Using that the intersection number of $F_* \times 1$ and every 1-cycle in $F_* \times 1$ in $F_* \times S^1$ is 0, we see that $\gamma|_{F_* \times 1} = 0$. For a point $x \in F_*$, we assume that $h(x, 1) = (x, 1)$ by an isotopic deformation of $h$. Since $h^*(\gamma) = \gamma$, the automorphism $h_\#$ of $\pi_1(F_* \times S^1, (x, 1))$ preserves the subgroup $\pi_1(F_* \times 1, (x, 1))$. Let $f$ be an auto-homeomorphism of $(F_* \times 1, (x, 1))$ inducing the automorphism $h_\#|_{\pi_1(F_* \times 1, (x, 1))}$ up to an conjugation. Then the auto-homeomorphism $h' = f \times 1$ of $F_* \times S^1$ is homotopic to $h$ since $F_* \times S^1$ is a Haken manifold and $h'_\#$ coincides with $h_\#$ up to a conjugation. By F. Waldhausen’s result in [21], $h$ is isotopic to $h'$ and $h'(F_* \times 1) = F_* \times 1$. □

Let $\zeta(F)$ be the Gauss sum

$$GS(\xi) = \sum_{x \in \Delta(F; Z_2)} \exp(2\pi \sqrt{-1} \xi(x) / 2).$$

The following theorem is useful to obtain a pseudo-ribbon non-ribbon surface-link:

**Theorem 1.2.**

(1) For every surface-link $F$ of total genus $g$, the invariant $\zeta(F)$ is 0, 1 or $\pm 2^s$ for an integer $s$ with $1 \leq s \leq g$.  

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(2) If \( F \) is a ribbon surface-link of total genus \( g \), then we have \( \zeta(F) = 2^g \).

(3) For any connected sum \( F = F_1 \# F_2 \) of any surface-links \( F_1 \) and \( F_2 \), we have

\[
\zeta(F) = \zeta(F_1)\zeta(F_2).
\]

It appears unknown whether there is a surface-link \( F \) with \( \zeta(F) = -2^g \).

**Proof.** If \( \Delta(F; Z_2) = 0 \), then we have \( \zeta(F) = 1 \) since \( \xi(0) = 0 \). Assume \( \Delta(F; Z_2) \neq 0 \). Let \( x_i \) (\( i = 1, 2, \ldots, s \)) and \( y_j \) (\( j = 1, 2, \ldots, u \)) be a \( Z_2 \)-basis for \( \Delta(F; Z_2) \) with \( 0 \leq u \leq s \leq g \) such that \( \Int_2(x_i, x_i') = \Int_2(y_j, y_j') = 0 \) and \( \Int_2(x_i, y_j) = \delta_{ij} \) for all \( i, i', j, j' \). Let \( \Delta_i \) (\( i = 1, 2, \ldots, s \)) be the direct summand of \( \Delta(F; Z_2) \) with basis \( x_i, y_i \) for \( i \leq u \) or \( x_i \) for \( i > u \). Let \( \xi_i \) be the restriction of \( \xi \) to \( \Delta_i \). For every \( i \leq u \), we have either \( \xi_i(x_i) = 0 \) and \( \xi_i(x_i + y_i) = \xi_i(y_i) + 1 \) or \( \xi_i(x_i) = \xi_i(y_i) = \xi_i(x_i + y_i) = 1 \), so that \( GS(\xi_i) = \pm 2 \) by noting \( \exp(\pi\sqrt{-1}) = -1 \). For every \( i > s \), it is direct to see that \( GS(\xi_i) \) is 2 or 0 according to whether \( \xi_i(x_i) \) is 0 or 1. Since \( \Delta(F; Z_2) \) splits into \( \Delta_i \) (\( i = 1, 2, \ldots, s \)) orthogonally with respect to the \( Z_2 \)-intersection form on \( F \), we have

\[
\zeta(F) = GS(\xi) = GS(\xi_1)GS(\xi_2) \cdots GS(\xi_s) = 0, 1 \text{ or } \pm 2^s,
\]

showing (1). To show (2), let \( F \) be a ribbon surface-link of total genus \( g \). By [11], \( F \) admits a Seifert hypersurface \( V \) which is homeomorphic to the connected sum of handlebodies \( V_i \) (\( i = 1, 2, \ldots, r \)) of total genus \( g \) and some copies of \( S^1 \times S^2 \). Let \( O(F; Z_2) \) be the subgroup of \( \Delta(F; Z_2) \) generated by a half \( Z_2 \)-basis of \( H_1(F; Z_2) \) which are represented by meridian loops of \( V_i \) (\( i = 1, 2, \ldots, r \)). It is direct to see that \( \xi(x) = 0 \) for all \( x \in O(F; Z_2) \). We take a \( Z_2 \)-basis \( x_i \) (\( i = 1, 2, \ldots, g \)), \( y_j \) (\( j = 1, 2, \ldots, u \)) of \( \Delta(F; Z_2) \) with \( 0 \leq u \leq g \) such that \( x_i \) (\( i = 1, 2, \ldots, g \)) are a \( Z_2 \)-basis of \( O \), and \( \Int_2(y_j, y_j') = 0 \) and \( \Int_2(x_i, y_j) = \delta_{ij} \) for all \( i, j, j' \). As in the argument of the first half, we denote by \( \xi_i \) (\( i = 1, 2, \ldots, g \)) the restriction of \( \xi \) to the direct summand with basis \( x_i, y_i \) for \( i \leq u \) or \( x_i \) for \( i > u \). For every \( i \leq u \), the identity \( \xi_i(x_i) = 0 \) implies \( \xi_i(x_i + y_j) = \xi_i(y_j) + 1 \), so that \( GS(\xi_i) = 2 \). For every \( i > u \), it is direct to see that \( GS(\xi_i) \) is 2 since \( \xi_i(x_i) = 0 \). Thus, we have

\[
\zeta(F) = GS(\xi) = GS(\xi_1)GS(\xi_2) \cdots GS(\xi_g) = 2^g,
\]

showing (2). To show (3), we first show that \( \Delta(F; Z_2) = \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2) \) under the identification \( H_1(F; Z_2) = H_1(F_1; Z_2) \oplus H_1(F_2; Z_2) \). Since \( \Delta(F; Z_2) \supset \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2) \) is obvious, it suffices to show that \( \Delta(F; Z_2) \subset \Delta(F_1; Z_2) \oplus \Delta(F_2; Z_2) \). For the infinite cyclic covering \( p : \tilde{E} \to E \) associated with \( \gamma \), we consider an embedding

\[
e : F = F \times 1 \xrightarrow{\subseteq} F \times R^1 = \partial \tilde{E} \subseteq \tilde{E}.
\]

For \( F_i \) (\( i = 1, 2 \)) instead of \( F \), we have a similar embedding

\[
e_i : F_i \to \tilde{E}_i.
\]

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We regard $F$ as the union $F_1^0 \cup F_2^0$ where $F_i^0$ is a compact punctured surface of $F_i$ with $\partial F_1^0 = \partial F_2^0$. We note that every (possibly singular) loop $L \subset F$ is homotopic in $F$ to a bouquet

$$B_L = L_1^1 \lor L_2^1 \lor \cdots \lor L_1^r \lor L_2^r$$

such that $L_i^j$ is a (possibly singular) loop in $F_i^0$ and the base point is sent to a point $b \in e(\partial F_i^0) = e(\partial F_1^0)$ by $e$. Since $\pi_1(\tilde{E},b)$ is the free product $\pi_1(\tilde{E}_1,b) \ast \pi_1(\tilde{E}_2,b)$ and each loop $L_i^j$ represents an element of $\pi_1(\tilde{E}_i,b)$, we see from a result of W. Magnus-A. Karrass-D. Solitar [14;p.182] that if $L$ is null-homotopic in $\tilde{E}$, then some loop $L_i^j$ represents a trivial element of $\pi_1(\tilde{E}_i,b)$. The bouquet obtained from $B_L$ by removing this loop $L_i^j$ is homotopic in $F$ to a bouquet

$$B_{L'} = (L')_1^1 \lor (L')_2^1 \lor \cdots \lor (L')_1^{r-1} \lor (L')_2^{r-1}$$

such that $(L')_i^j$ is a loop in $F_i^0$ and the base point is sent to the point $b$ by $e$. Then we note that $L$ is homologous to $L_i^j + B_{L'}$ in $F$. Since $B_{L'}$ is null-homotopic in $\tilde{E}$, we can conclude by induction on $r$ that $L$ is homologous in $F$ to the sum $L_1 + L_2$ where $L_i$ is the sum of loops in $F_i^0$ which are null-homotopic in $\tilde{E}_i$. This implies that

$$\Delta(F;Z_2) \subset \Delta(F_1;Z_2) \oplus \Delta(F_2;Z_2)$$

and hence

$$\Delta(F;Z_2) = \Delta(F_1;Z_2) \oplus \Delta(F_2;Z_2).$$

For every element $x_i \in \Delta(F_i;Z_2)$ ($i = 1, 2$), we have $\text{Int}_2(x_1, x_2) = 0$, so that

$$\xi(x_1 + x_2) = \xi(x_1) + \xi(x_2).$$

By this identity, we have

$$\exp(2\pi \sqrt{-1} \frac{\xi(x_1 + x_2)}{2}) = \exp(2\pi \sqrt{-1} \frac{\xi(x_1)}{2}) \exp(2\pi \sqrt{-1} \frac{\xi(x_2)}{2}),$$

which implies the identity $\xi(F) = \xi(F_1)\xi(F_2)$. $\square$

Let $D^2$ be the disk, and $\rho_D : D^2 \times [0, 1] \rightarrow D^2$ the projection to the first factor. Let $k$ be a knot in $D^2 \times [0, 1]$ such that $\rho_D(k)$ be a transversely immersed loop in $D^2$. We consider an unknotted embedding $f^1 : D^2 \times S^1 \rightarrow R^3$ with +1-framing. Let

$$\tilde{f}^1 : (D^2 \times S^1) \times [0, 1] \xrightarrow{f^1 \times id} R^3 \times [0, 1] \subset R^4$$

be the associated embedding. Under the identification

$$(D^2 \times [0, 1]) \times S^1 = (D^2 \times S^1) \times [0, 1],$$

we obtain the torus $k \times S^1$ in $(D^2 \times S^1) \times [0, 1]$. The turned spun $T^2$-knot of the knot $k$ is the $T^2$-knot $T(k) = \tilde{f}^1(k \times S^1)$ in $R^4 \subset R^4 \cup \{\infty\} = S^4$ (see J. Boyle.
can see also that the knot-exteriors of an observation due to F. Gonzalez-Acuña found in J. Boyle’s paper [1], the knot-
On the other hand, we have lent. However, from Lemma 1.1(3), we see that the knot-exteriors of 2. The torsion linking of a pseudo-ribbon surface-link

**Corollary 1.3.** Any connected sum \(F \# T(k)\) of a surface-link \(F\) and the turned spun \(T^2\)-knot \(T(k)\) of a non-trivial knot \(k\) is a non-ribbon surface-link.

**Proof.** First, we show the Gauss sum invariant \(\zeta(T(k)) = 0\). By the unknotted embedding \(f^1 : D^2 \times S^1 \to R^3\) with 0-framing instead of \(f^1\), we obtain the spun \(T^2\)-knot \(T^0(k)\) of the knot \(k\), which is directly seen to be a ribbon \(T^2\)-knot. We have the following corollary to Theorem 1.2, generalizing a result of A. Shima [19]:

For any integer sequence \(g_1 \geq g_2 \geq \cdots \geq g_r \geq 0\) with \(g_1 > 0\), we take any pseudo-ribbon (e.g. trivial or ribbon) surface-link with components \(F_i\) \((i = 1, 2, \ldots, r)\) such that genus\((F_1) = g_1 - 1\) and genus\((F_i) = g_i\) \((i = 2, 3, \ldots, r)\). Then we see from Corollary 1.3 that the connected sum \(F_1 \# T(k) \cup F_2 \cup \ldots F_r\) is a pseudo-ribbon, non-ribbon surface-link of genera \(g_i\) \((i = 1, 2, \ldots, r)\). For a surface-knot, we have a result on the knot-exterior as follows:

**Example 1.4** Let \(F\) be a surface-knot \(F\) with \(\zeta(F) \neq 0\). Since the spun \(T^2\)-knot \(T^0(k)\) of a non-trivial knot \(k\) is a ribbon \(T^2\)-knot, we see from Theorem 1.2 that

\[
\zeta(F \# T^0(k)) = 2 \zeta(F) \neq 0.
\]

On the other hand, we have \(\zeta(F \# T(k)) = 0\), for \(\zeta(T(k)) = 0\) by Corollary 1.3. By an observation due to F. González-Acuña found in J. Boyle’s paper [1], the knot-exteriors of \(T(k)\) and \(T^0(k)\) are homotopy equivalent. Examining it carefully, we can see also that the knot-exteriors of \(F \# T^0(k)\) and \(F \# T(k)\) are homotopy equivalent. However, from Lemma 1.1(3), we see that the knot-exteriors of \(F \# T^0(k)\) and \(F \# T(k)\) are not homeomorphic, generalizing the property between the \(T^2\)-knots \(T(k)\) and \(T^0(k)\) known by J. Boyle [1] and Z. Iwase [4].

2. The torsion linking of a pseudo-ribbon surface-link

Let \(p : \tilde{W} \to W\) be the infinite cyclic covering of a compact oriented 4-manifold \(W\) belonging to an element \(\gamma \in H^1(W; Z)\). Let \(A\) and \(A'\) be \(0\) or compact 3-submanifolds of \(\partial W\) such that \(A' = \text{cl}(\partial W - A)\). Let \(\tilde{A} = p^{-1}(A)\) and \(\tilde{A}' = p^{-1}(A')\). We briefly explain the torsion linking of \((\tilde{W}, \tilde{A}, \tilde{A}')\) which has been done in [7]. For
a \Lambda\text{-module } H, \text{ let } TH \text{ be the } \Lambda\text{-torsion part of } H, \text{ and } BH = H/TH. \text{ (Unless otherwise stated, abelian groups are regarded as } \Lambda\text{-modules on which } t \text{ operates as the identity.) Let } tH \text{ be the } Z\text{-torsion part of } H, \text{ and } bH = H/tH. \text{ Let } E^q(H) = \text{Ext}^q_\Lambda(H, \Lambda). \text{ For a finitely generated } \Lambda\text{-module } H, \text{ we have a unique maximal finite } \Lambda\text{-submodule } DH \text{ of } H. \text{ Then we have a } t\text{-anti epimorphism }

\theta_{A,A'} : DH_1(\tilde{W}, \tilde{A}; Z) \rightarrow E^1(BH_2(\tilde{W}, \tilde{A}'; Z))

which is an invariant of \((\tilde{W}, \tilde{A}, \tilde{A}')\) or \((W, A, A', \gamma)\). We denote the kernels of \(\theta_{A,A'}\) and \(\theta_{A',A}\) by \(DH_1(\tilde{W}, \tilde{A}; Z)\) and \(DH_1(\tilde{W}, \tilde{A}'; Z)\), respectively. Let \(\mu \in TH_3(\tilde{W}, \partial W; Z)\) be the fundamental class of the covering \(p : \tilde{W} \rightarrow W\), which is characterized by \(t\mu = \mu\) and \(p_*(\mu) = \gamma \cap [W]\) for the fundamental class \([W]\) of \(W\). Let \(\tau H^2(\tilde{W}, \tilde{A}; Z)\) be the image of the Bockstein coboundary map

\(\delta_{Q/Z} : H^1(\tilde{W}, \tilde{A}; Z) \rightarrow H^2(\tilde{W}, \tilde{A}; Z).\)

The second duality in [7] is equivalent to the following lemma (see [7; Theorem 6.5]):

**Lemma 2.1.** The cap product map \(\cap \mu : \tau H^2(\tilde{W}, \tilde{A}; Z) \rightarrow tH_1(\tilde{W}, \tilde{A}'; Z)\) induces an isomorphism

\(\cap \mu : \text{hom}(DH_1(\tilde{W}, \tilde{A}; Z), Q/Z) \cong DH_1(\tilde{W}, \tilde{A}'; Z).\)

In fact, by Lemma 2.1 we have a \(t\)\text{-isometric non-singular bilinear form

\(\ell : DH_1(\tilde{W}, \tilde{A}; Z) \times DH_1(\tilde{W}, \tilde{A}'; Z) \rightarrow Q/Z\)

by taking \(\ell(x, y) = f_y(x) \in Q/Z\) for \(x \in DH_1(\tilde{W}, \tilde{A}; Z), y \in DH_1(\tilde{W}, \tilde{A}'; Z), f_y \in \text{hom}(DH_1(\tilde{W}, \tilde{A}; Z), Q/Z)\) with \(f_y \cap \mu = y\). This bilinear form \(\ell\) is an invariant of \((\tilde{W}, \tilde{A}, \tilde{A}')\) or \((W, A, A', \gamma)\) and called the **torsion linking** of \((\tilde{W}, \tilde{A}, \tilde{A}')\) or \((W, A, A', \gamma)\). Let \(F\) be a surface-link in \(S^4 = R^4 \cup \{\infty\}\), and \(E\) the compact exterior \(\text{cl}(S^4 - N)\) where \(N\) denotes a normal disk bundle of \(F\) in \(S^4\). Taking \(W = E, A = \partial E, A' = \emptyset\) and the element \(\gamma \in H^1(E; Z) = \text{hom}(H_1(E; Z), Z)\) sending each oriented meridian of \(F\) to 1 \(\in Z\), we have, as a surface-link type invariant, the **torsion linking**

\(\ell = \ell_F : DH_1(\tilde{E}, \partial \tilde{E}; Z) \times DH_1(\tilde{E}; Z) \rightarrow Q/Z\)

of the surface-link \(F\). The following theorem is proved in §3:

**Theorem 2.2.** If \(F\) is a pseudo-ribbon surface-link, then the torsion linking \(\ell_F\) vanishes. In other words,

\(DH_1(\tilde{E}, \partial \tilde{E}; Z) = DH_1(\tilde{E}; Z) = 0.\)
The linking signature $\sigma(F)$ and the local linking signatures $\sigma_i(F)$ are defined as modulo 4 integers in [10] by using the Gauss sum of the quadratic function associated with the torsion linking $\ell_F$. The following corollary is direct from Theorem 2.2:

**Corollary 2.3.** If $F$ is a pseudo-ribbon surface-link, then we have

$$\sigma(F) = \sigma_p^i(F) = 0$$

for all prime numbers $p$ and all positive integers $i$.

### 3. Constructing a canonical Seifert hypersurface for a pseudo-ribbon surface-link

We assume that the singularity image $\rho S(F)$ in $R^3$ of a pseudo-ribbon surface-link $F$ in $R^4$ consists of mutually disjoint simple loops $C_i$ ($i = 1, 2, \ldots, r$). Let $N_i$ be a regular neighborhood of $C_i$ in $\rho(F)$, so that we have a homeomorphism

$$h_i : (X, v) \times S^1 \cong (N_i, C_i) \quad (i = 1, 2, \ldots, r),$$

where $X$ denotes a cone over a four-point set with $v$ as the vertex. We regard $X$ as the subgraph of the 1-skeleton $J^{(1)}$ of a bouquet $J$ of two 2-simplices at a vertex such that the complement graph is the union of two disjoint 1-simplices $I$ and $I'$. In this case, $v$ is the vertex of the bouquet $J$. Then $h_i$ extends to an embedding $\tilde{h}_i : J \times S^1 \to R^3$. We note that there are two choices on regarding $X$ as such a subgraph of $J^{(1)}$. Our choice is made to satisfy the condition that

$$P = \text{cl}(\rho(F) - \cup_{i=1}^r N_i) \cup (\cup_{i=1}^r \tilde{h}_i((I \cup I') \times S^1))$$

is an orientable 2-manifold with an orientation induced from $\rho(F) - \cup_{i=1}^r N_i$. The 2-manifold $P$ is referred to as a 2-manifold obtained from $\rho(F)$ by orientation-preserving cut along the $C_i$’s. Let $P_j$ ($j = 1, 2, \ldots, s$) be the components of $P$ such that the compact 3-manifold $V_j$ in $R^3$ bounded by $P_j$ satisfies the condition that $P_j \subset V_{j'}$ implies $j < j'$.

Let $T$ be a four-sided disk, and $I_0$ a proper interval in $T$ splitting $T$ into two four-sided disks. We identify the quotient space $T/I_0$ with $J$ so that the quotient map $q : T \to J = T/I_0$ is a half-twist band projection with $q(I_0) = v$. For a subset $A$ of $R^3$ and an interval $[a, b]$ granting $a = b$, we denote the subset $\{(x, t) | x \in A, t \in [a, b]\}$ of $R^4$ by $A[a, b]$. We choose real numbers $t_j$ ($j = 1, 2, \ldots, s$) so that $t_1 < t_2 < \cdots < t_s$. After an ambient deformation of $F$, we have a Seifert hypersurface $V$ for $F$ so that

$$V = \left( \bigsqcup_{j=1}^s V_j[t_j] \right) \cup \left( \bigsqcup_{i=1}^r \tilde{h}_i^* (T \times S^1) \right).$$
where $h_i^*$ denotes an embedding $h_i^* : T \times S^1 \to R^4$ such that the composite

$$\rho h_i^* : T \times S^1 \to R^4 \xrightarrow{\rho} R^3$$

is equal to the composite

$$T \times S^1 \xrightarrow{q \times 1} J \times S^1 \to R^3.$$

By construction, we have

$$\rho(V) = \left( \prod_{j=1}^s V_j \right) \cup \left( \prod_{i=1}^r \bar{h}_i(J \times S^1) \right).$$

Further, we can assume that $V \subset \rho(V)[t_1, t_s]$. We call this hypersurface $V$ a *canonical Seifert hypersurface* of the pseudo-ribbon surface-link $F$. Let $E_V$ be the compact oriented 4-manifold obtained from the exterior $E$ of $F$ in $S^4 = R^4 \cup \{\infty\}$ by splitting along $V \cap E(\equiv V)$. Let $V^\pm(\equiv V)$ be the two copies of $V$ in $\partial E_V \subset E_V$. Let $i^\pm : V \cong V^\pm \subset E_V$ be the composite injections. The following theorem is a key to our argument:

**Theorem 3.1.** For a canonical Seifert hypersurface $V$ of a pseudo-ribbon surface-link $F$, the induced homomorphisms

$$i^\pm_* : tH_1(V; Z) \to tH_1(E_V; Z)$$

on the torsion part of the first integral homology are trivial.

To prove this theorem, we need some preliminaries. First we show the following lemma:

**Lemma 3.2.** For the inclusion $k : \coprod_{j=1}^s V_j[t_j] \subset V$, we have

$$tH_1(V; Z) \subset \text{image}[H_1(\coprod_{j=1}^s V_j[t_j]; Z) \xrightarrow{k_*} H_1(V; Z)].$$

**Proof.** Let $A_i = \bar{h}_i^*(I \times S^1) (i = 1, 2, \ldots, r)$ be proper annuli with any orientations in $V$. Then we have

$$\text{Int}([A_i], x) = 0$$

for all $x \in tH_1(V; Z)$ with respect to the intersection form

$$\text{Int} : H_2(V, \partial V; Z) \times H_1(V; Z) \to Z.$$
This means that every element \( x \in tH_1(V; Z) \) is represented by an embedded closed oriented 1-manifold \( L \) with \( L \cap A_i = \emptyset \) for all \( i \), so that \( L \) is isotopically deformed into \( \coprod_{j=1}^s V_j[t_j] \). \( \square \)

We also need the following lemma:

**Lemma 3.3.** For every polyhedron \( V \) in \( R^3 \), we have \( tH_1(V; Z) = 0 \).

**Proof.** We may assume that \( V \) is compact and connected. Further, we may assume that \( V \) is a compact connected 3-submanifold of \( R^3 \) by taking a regular neighborhood of \( V \) instead of \( V \). For any elements \( x \in tH_1(V, \partial V; Z) \) and \( y \in tH_1(V; Z) \), we can represent \( x \) and \( y \) by disjoint closed oriented 1-manifolds \( L \) and \( L_y \) in \( V \). Then there are a non-zero integer \( m \) and a 2-chain \( c_y \) in \( V \) such that \( \partial c_y = mL_y \) and the torsion linking

\[
\ell_V : tH_1(V, \partial V; Z) \times tH_1(V; Z) \longrightarrow Q/Z
\]

is computed by the identity \( \ell_V(x, y) = \text{Int}(L_x, c_y)/m \pmod{1} \). Since the linking number \( \text{Link}(L_x, L_y) \in Z \) in \( R^3 \) is defined, we have

\[
\text{Int}(L_x, c_y)/m = \text{Link}(L_x, mL_y)/m = \text{Link}(L_x, L_y) \in Z
\]

and hence \( \ell_V(x, y) = 0 \in Q/Z \). Using that the torsion linking \( \ell_V \) is non-singular, we obtain \( tH_1(V, \partial V; Z) = tH_1(V; Z) = 0 \). \( \square \)

By using Lemmas 3.2 and 3.3, Theorem 3.1 is proved as follows:

**Proof of Theorem 3.1.** We regard \([t_1, t_s] \subset R^1 \cup \{ \infty \} = S^1 \). Let \( f : \rho(V) \times S^1 \longrightarrow R^4 \subset R^4 \cup \{ \infty \} = S^4 \) be an embedding sending \( \rho(V) \times [t_1, t_s] \) to \( \rho(V)[t_1, t_s] \) identically. We represent any element \( x \in tH_1(V; Z) \) by a closed oriented 1-manifold \( L \) in \( \coprod_{j=1}^s V_j[t_j] \). Let \( L_j[t_j] = L \cap V_j[t_j] \) for a closed 1-manifold \( L \) or \( \emptyset \) in \( V \). We take a point \( t_0 \in S^1 - [t_1, t_s] \). For any \( j \) with \( L_j \neq \emptyset \), we further take a subarc \( \alpha_j^+ \subset S^1 \) with \( \partial \alpha_j^+ = \{ t_0, t_j \} \) so that \( L_j \times \alpha_j^+ \) meets \( V_j[t_j] \) from the positive side of \( V_j[t_j] \). Then the image \( f(\coprod_{j=1}^s L_j \times \alpha_j^+) \) is a disjoint union of annuli which is contained in \( E_V \) and whose boundary consists of \( L \) in \( V^+ \) and \( L'_x = f(\coprod_{j=1}^s L_j \times t_0) \) in \( f(\rho(V) \times t_0) \subset E_V \). Since

\[
H_1(\rho(V) \times S^1; Z) \cong H_1(\rho(V); Z) \oplus H_1(S^1; Z),
\]

we see from Lemma 3.3 that \( [L_x] = 0 \in H_1(\rho(V) \times S^1; Z) \). Using that the natural homomorphism \( H_1(\rho(V) \times t_0; Z) \longrightarrow H_1(\rho(V) \times S^1; Z) \) is injective, we see that \( L'_x \) bounds a 2-chain \( c' \) in \( f(\rho(V) \times t_0) \). Thus, \( L_x \subset V^+ \) bounds a 2-chain \( f(\coprod_{j=1}^s L_j \times \alpha_j^+) + c' \) in \( E_V \), which means that \( i^+_x(x) = 0 \). Similarly, \( i^-_x(x) = 0 \). \( \square \)
The following corollary is direct from Theorem 3.1 since the infinite cyclic covering space $\tilde{E}$ is constructed from the copies $((E_V)_i; (V^+_i), (V^-_i))$ of the triplet $(E_V; V^+, V^-)$ by pasting $(V^-)_i$ to $(V^+_i)$ for all $i$.

**Corollary 3.4.** For any canonical Seifert hypersurface $V$ of a pseudo-ribbon surface-link $F$, every lift $\tilde{i} : V \to \tilde{E}$ of the natural injection $i : V \to E$ induces the trivial homomorphism

$$\tilde{i}_* = 0 : tH_1(V; Z) \to tH_1(\tilde{E}; Z).$$

By using Corollary 3.4, Theorem 2.2 is proved as follows:

**Proof of Theorem 2.2.** We consider the following commutative diagram:

$$
\begin{array}{ccc}
\tau H^2(\tilde{E}, \partial \tilde{E}; Z) & \xrightarrow{\cap \mu} & tH_1(\tilde{E}; Z) \\
\tilde{i}^* \downarrow & & \tilde{i}_* \\
tH^2(V, \partial V; Z) & \cong & tH_1(V; Z).
\end{array}
$$

In this diagram, we have

$$\tilde{i}_* = 0 : tH_1(V; Z) \to tH_1(\tilde{E}; Z)$$

by Corollary 3.4 and hence

$$\cap \mu = 0 : \tau H^2(\tilde{E}, \partial \tilde{E}; Z) \to tH_1(\tilde{E}; Z).$$

By Lemma 2.1, we have

$$\cap \mu = 0 : \text{hom}(DH_1(\tilde{E}, \partial \tilde{E}; Z)^\theta, Q/Z) \cong DH_1(\tilde{E}; Z)^\theta,$$

which implies $DH_1(\tilde{E}, \partial \tilde{E}; Z)^\theta = DH_1(\tilde{E}; Z)^\theta = 0$. $\Box$

Here is another corollary to Theorem 3.1.

**Corollary 3.5.** For any canonical Seifert hypersurface $V$ of a pseudo-ribbon surface-link $F$, the natural homomorphism $j_* : tH_1(V; Z) \to tH_1(V, \partial V; Z)$ is trivial.

**Proof.** From the boundary isomorphism $\partial : H_2(S^4, V \times I; Z) \cong H_1(V \times I; Z)$ and the excision isomorphism $H_2(E_V, V^+ \cup V^-; Z) \cong H_2(S^4, V \times I; Z)$, the composite of the natural homomorphisms

$$tH_2(E_V, \partial E_V; Z) \xrightarrow{\partial} tH_1(\partial E_V; Z) \xrightarrow{i_*} tH_1(V \times I; Z)$$
is an isomorphism. Since $\partial E_V = \partial (V \times I)$, the Poincaré duality implies that the composite of the natural homomorphisms
\[
\hom(t H_1(E_V; Z), Q/Z) \xrightarrow{i^\#} \hom(t H_1(\partial E_V; Z), Q/Z) \xrightarrow{\partial'^\#} \hom(t H_2(V \times I, \partial(V \times I); Z), Q/Z)
\]
is an isomorphism and hence the composite of the natural homomorphisms
\[
t H_2(V \times I, \partial(V \times I); Z) \xrightarrow{\partial} t H_1(\partial E_V; Z) \xrightarrow{i} t H_1(E_V; Z)
\]
is an isomorphism by applying $\hom(\ , Q/Z)$ to the homomorphisms above. Further, composing a suspension isomorphism
\[
\sigma : t H_1(V, \partial V; Z) \cong H_2((V, \partial V) \times (I, \partial I); Z) = t H_2(V \times I, \partial(V \times I); Z)
\]
to this composite isomorphism, we obtain an isomorphism
\[
\theta = i_* \partial \sigma : t H_1(V, \partial V; Z) \cong t H_1(E_V; Z).
\]
For the natural homomorphism $j_* : t H_1(V; Z) \longrightarrow t H_1(V, \partial V; Z)$, the composite $j_* \theta : t H_1(V; Z) \longrightarrow t H_1(E_V; Z)$ is equal to the map $i^+_* - i^-_*$ which is the zero map. Thus, $j_* = 0$. \qed

We note that Corollary 3.5 does not mean that $t H_1(V; Z) = 0$. It is unknown whether every pseudo-ribbon surface-link $F$ admits a Seifert hypersurface $V$ with $t H_1(V; Z) = 0$.

4. The triple point cancelling number of a surface-link

The triple point number of a surface-link $F$ in $R^4$, denoted by $T(F)$ is the minimum on the triple point number of the singular surface $\rho(F')$ for all generic surface-links $F'$ ambient isotopic to $F$. In this section, we shall discuss a similar but distinct concept on a surface-link $F$. Let $F'$ be a generic surface-link ambient isotopic to $F$. By an ambient deformation of $F'$ without changing $\rho(F')$, we can consider that the set $F' \cap \rho^{-1}(B_x)$ for a 3-ball neighborhood $B_x$ of every triple point $x \in \rho(F')$ in $R^3$ is the union $D_1[t_1] \cup D_2[t_2] \cup D_3[t_3]$ where $D_i$ is a proper disk in $B_x$ and $t_1 < t_2 < t_3$. Then we make an orientation-preserving cut on $D_i \cup D_{i+1} \subset B_x$ for $i = 1$ or 2 to obtain from $F'$ a new generic surface-link $F'_1$ in $R^4$ (see J. S. Carter-M. Saito [2,Figure N]). When we compare $\rho(F'_1)$ with $\rho(F')$, $\rho(F'_1)$ has the triple points decreased by one point and the branch points increased by two points. We call the operation $F' \Rightarrow F'_1$ a triple point cancelling operation on $\rho(F')$. The triple point cancelling number of the singular surface $\rho(F')$ is the minimum of the number of triple point cancelling operations on $\rho(F')$ needed to obtain a pseudo-ribbon surface-link $F'_* \subset R^4$.
Definition 4.1. The triple point cancelling number of a surface-link $F$ in $R^4$, denoted by $\mathcal{T}(F)$ is the minimum on the triple point cancelling number of the singular surface $\rho(F')$ for all generic surface-links $F'$ ambient isotopic to $F$.

If we compare $\mathbb{T}(F)$ to the crossing number of a classical knot, then we could compare $\mathcal{T}(F)$ to the unknotting number of a classical knot. The following lemma is useful to understand a triple point cancelling operation:

Lemma 4.2. Let $F'_*$ be a surface-link obtained by doing $m$ triple point cancelling operations on $\rho(F')$ for a generic surface-link $F'$ ambient isotopic to a surface-link $F$ in $R^4$. Then $F'_*$ is ambient isotopic to a surface-link $F_*$ obtained from $F$ by an embedded surgery along $m$ mutually disjoint 1-handles on $F$. Conversely, if $F_*$ is a surface-link obtained from $F$ by an embedded surgery along $m$ mutually disjoint 1-handles on $F$, then $F_*$ and $F$ are respectively ambient isotopic to generic surface-links $F'_*$ and $F'$ such that $F'_*$ is a surface-link obtained by doing $m$ triple point cancelling operations on $\rho(F')$.

Proof. The proof of the first half part is obvious from the definition of a triple point cancelling operation. To prove the second half part, we use three 2-spheres $S_i \subset R^3[t_i]$ ($i = 1, 2, 3$) with $t_1 < t_2 < t_3$ such that the singularity image $\rho S(S_1 \cup S_2 \cup S_2)$ is homeomorphic to a suspension of a three point set and hence has just two triple points. Let $B^3$ be a 3-ball in $R^3$ such that $D_i = S_i \cap B^3[t_i]$ is a disk with $\rho(\text{cl}(S_i - D_i))$ ($i = 1, 2, 3$) mutually disjoint disks in $R^3$. We find a 1-handle $h \subset B^3[t_1, t_2]$ on $D_1 \cup D_2 \cup D_3$ connecting $D_1$ and $D_2$ such that $\rho(h)$ induces a triple point cancelling operation on $\rho(S_1 \cup S_2 \cup S_3)$. Let $h_i$ ($i = 1, 2, \ldots, m$) be mutually disjoint 1-handles on $F$ to produce a surface-link $F_*$. Then $F$ is ambient isotopic to a generic surface-link $F'$ such that $F' \cap B^3[t_1, t_3] = F' \cap \rho^{-1}(B^3_i)$ and $(B^3_i[t_1, t_3]; F' \cap B^3_i[t_1, t_3], h_i)$ is $[t_1, t_3]$-level-preservingly homeomorphic to $(B^3_i[t_1, t_3]; D_1 \cup D_2 \cup D_3, h)$ for some $m$ mutually disjoint 3-balls $B^3_i (i = 1, 2, \ldots, m)$ in $R^3$. The surface-link $F_*$ is ambient isotopic to a surface-link $F'_*$ obtained by doing $m$ triple point cancelling operations on $\rho(F')$. $\square$

We use the following result later:

Corollary 4.3. Let $F$ be an $S^2$-knot obtained from any non-trivial 2-bridge knot by the 2-twist spinning. Then $\mathcal{T}(F) = 1$.

Proof. Since $DH_1(\tilde{E}; Z) = H_1(\tilde{E}; Z) \cong \Lambda/(p, t + 1)$ for an integer $p \geq 3$ (see M. Teragaito [20]) and $BH_2(\tilde{E}, \partial \tilde{E}; Z) = 0$, the torsion linking $\ell_F$ is not zero by [7]. Hence we have $\mathcal{T}(F) \geq 1$ by Theorem 2.2. On the other hand, the $S^2$-knot $F$ has a Seifert hypersurface $V$ homeomorphic to a punctured Lens space. Then there is a 1-handle $h$ on $F$ such that $h \subset V$ with $\text{cl}(V - h)$ is a solid torus, so that the surface $F_1$ obtained from $F$ by the embedded surgery along $h$ is a trivial $T^2$-surface which is a pseudo-ribbon surface-knot. Hence $\mathcal{T}(F) \leq 1$ and $\mathcal{T}(F) = 1$. $\square$
The following remark concerns the difference $\mathcal{T}(F) - \mathcal{T}(F)$:

**Remark 4.4.** For a surface-link $F$, $\mathcal{T}(F) = 0$ if and only if $\mathcal{T}(F) = 0$ if and only if $F$ is pseudo-ribbon by the definitions. S. Satoh observed that the difference $\mathcal{T}(F) - \mathcal{T}(F)$ is positive for every non-pseudo-ribbon surface-link $F$. In fact, we have $\mathcal{T}(F) \geq \mathcal{T}(F) > 0$ by the definitions. If $\mathcal{T}(F) = \mathcal{T}(F) > 0$, then we have a generic surface-link $F'$ with $\mathcal{T}(F') = \mathcal{T}(F') = 1$ by taking $\mathcal{T}(F) - 1$ times of triple point cancelling operations. Then we see from a result of S. Satoh [16] that we find a simple double line connecting to the triple point and a branch point in $\rho(F')$, so that we can eliminate the triple point by moving this branch point along this double line, meaning that $F'$ is a pseudo-ribbon surface-link, contradicting to $\mathcal{T}(F') = 1$. Hence $\mathcal{T}(F) - \mathcal{T}(F) > 0$. S. Satoh and A. Shima [17] showed that $\mathcal{T}(S(3_1)) = 4$ for the $S^2$-knot $S(3_1)$ obtained from the trefoil knot $3_1$ by the 2-twist spinning. By Corollary 4.3, we have $\mathcal{T}(S(3_1)) = 1$, so that $\mathcal{T}(S(3_1)) - \mathcal{T}(S(3_1)) = 3$. There are open questions asking whether $\mathcal{T}(F) - \mathcal{T}(F) \geq 3$ for every non-pseudo-ribbon surface-knot $F$ and whether there is a surface-knot $F$ such that $\mathcal{T}(F) - \mathcal{T}(F)$ is greater than any previously given positive integer.

The inequality $\mathcal{T}(F_1 \# F_2) \leq \mathcal{T}(F_1) + \mathcal{T}(F_2)$ holds for any surface-knots $F_1$ and $F_2$, and the equality does not appear to hold in general (see T. Kanenobu [6]). **It is also an open question whether there is such an example.**

From now on, we shall establish an estimate of the triple point cancelling number of a general surface-link. Let $E = \text{cl}(S^4 - N)$ be the knot-exterior of a surface-knot $F$ in $S^4$ where $N = F \times D^2$ is a tubular neighborhood of $F$ in $S^4$ with the specified trivialization. Let $V_0$ be the handlebody such that $\partial V_0 = F$. Let $M_\phi$ be the closed 4-manifold obtained from the exterior $E$ and $V_0 \times S^1$ by attaching the boundaries by a homeomorphism $\phi : \partial E = F \times S^1 \to \partial V_0 \times S^1$ which preserves the $S^1$-factor. Then $M_\phi$ is a closed connected oriented 4-manifold with $H_1(M_\phi; \mathbb{Z}) \cong \mathbb{Z}$, which we call a $Z^{H_1}$-manifold. We use the concept of exactness of $Z^{H_1}$-manifold in [8, 9] in our argument.

**Lemma 4.5.** Let $F$ be a pseudo-ribbon surface-knot in $S^4$. Then there exists an attachment $\phi$ such that the $Z^{H_1}$-manifold $M_\phi$ is a spin exact $Z^{H_1}$-manifold.

**Proof.** Let $V$ be a canonical Seifert hypersurface for $F$ in $S^4$. Let $C$ be the image of the boundary homomorphism $\partial : H_2(V, F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$. By Corollary 3.5, we have a subgroup $\tilde{C}$ of $H_1(F; \mathbb{Z})$ such that $\tilde{C} \supset C$ and the natural monomorphism $H_1(F; \mathbb{Z})/C \to H_1(V; \mathbb{Z})$ induces a monomorphism

$$H_1(F; \mathbb{Z}))/\tilde{C} \to bH_1(V; \mathbb{Z}).$$

Then $\tilde{C}$ is a self-orthogonal complement of $H_1(F; \mathbb{Z})$ with respect to the intersection form $\text{Int} : H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}$. Let $x_i, y_i$ be a $\mathbb{Z}$-basis for $H_1(F; \mathbb{Z})$ such
that $x_i \ (i = 1, 2, \ldots, g)$ is a $\mathbb{Z}$-basis for $\mathbb{C}$ and $\text{Int}(x_i, x_j) = \text{Int}(y_i, y_j) = 0$ and $\text{Int}(x_i, y_j) = \delta_{ij}$ for all $i, j$. Let $K_i^x$ and $K_i^y$ be simple loops on $F$ such that $K_i^x \cap K_j^x = K_i^y \cap K_j^y = K_i^x \cap K_j^y = \emptyset$ for all $i, j$ with $i \neq j$ and $K_i^x \cap K_j^y$ is one point for all $i$. Let $V_0$ be a handlebody with $\partial V_0 = F$ such that $K_i^x \ (i = 1, 2, \ldots, g)$ bound mutually disjoint meridian disks $D_i \ (i = 1, 2, \ldots, g)$ in $V_0$. Let $\bar{V} = V \cup V_0$ be a closed oriented 3-manifold pasting $F$ with these data which produces a $\mathbb{Z}^{H_1}$-manifold $M_\phi$. From the homology exact sequence of the pair $(\bar{V}, V)$, we obtain a natural isomorphism $bH_1(V; \mathbb{Z}) \cong H_1(\bar{V}; \mathbb{Z})$. When we regard $V$ as $V^+$ in $E_V$, we see from Theorem 3.1 that the simple loop $K_i^x \subset V^+$ bounds a compact oriented surface $F_i$ in $E_V$. Let $\bar{F}_i = F_i \cup D_i \ (i = 1, 2, \ldots, g)$ be closed oriented surfaces in $M_\phi$. Let $T_i = K_i^y \times S^1 \subset F \times S^1 = \partial E \subset M_\phi \ (i = 1, 2, \ldots, g)$. Then the closed oriented surfaces $\bar{F}_i$ and $T_i \ (i = 1, 2, \ldots, g)$ form a $\mathbb{Z}$-basis for $H_2(M_\phi; \mathbb{Z})$ with $\text{Int}(\bar{F}_i, \bar{F}_j) = \text{Int}(T_i, T_j) = 0$ and $\text{Int}(\bar{F}_i, T_j) = \delta_{ij}$ for all $i, j$ with respect to the intersection form $\text{Int} : H_2(M_\phi; \mathbb{Z}) \times H_2(M_\phi; \mathbb{Z}) \longrightarrow \mathbb{Z}$. [To see that $\text{Int}(\bar{F}_i, \bar{F}_j) = 0$, we note that $K_i^x \ (i = 1, 2, \ldots, g)$ represent torsion elements in $H_1(V; \mathbb{Z})$, which implies that for each $i$ there is a non-zero integer $m_i$ such that $m_i \bar{F}_i$ is homologous to a cycle $C_i + C_i'$ in $M_\phi$ where $C_i$ is a cycle in $\bar{V}$ and $C_i'$ is a cycle in $\text{int} E_V \subset S^4$. Then we have]

$$m_im_j\text{Int}(\bar{F}_i, \bar{F}_j) = \text{Int}(m_i\bar{F}_i, m_j\bar{F}_j) = 0$$

showing $\text{Int}(\bar{F}_i, \bar{F}_j) = 0.$] In particular, $M_\phi$ is spin. Using a collar of $\bar{V}$ in $M_\phi$, we take mutually disjoint closed oriented surfaces $\bar{F}_i' \ (i = 1, 2, \ldots, g)$ in $M_\phi$ such that $\bar{F}_i' \cap \bar{V} = \emptyset$ and $\bar{F}_i'$ is homologous to $\bar{F}_i$ in $M_\phi$. Since the normal disk bundle $N'_i$ of $\bar{F}_i'$ in $M_\phi$ is trivial, the leaf $V^*$ of $M_\phi$ obtained by taking connected sums of $\bar{V}$ and $\partial N'_i \ (i = 1, 2, \ldots, g)$ in $M_\phi$ satisfies the condition that $tH_1(V^*; \mathbb{Z}) = 0$ and the image of the natural homomorphism $H_2(V^*; \mathbb{Z}) \longrightarrow H_2(M_\phi; \mathbb{Z})$ is a self-orthogonal complement with respect to the intersection form $\text{Int} : H_2(M_\phi; \mathbb{Z}) \times H_2(M_\phi; \mathbb{Z}) \longrightarrow \mathbb{Z}$. Let $M'_\phi$ be the 4-manifold obtained from $M_\phi$ by splitting along $V^*$. Then by [8; (4.7.2)] $tH_1(V^*; \mathbb{Z}) = 0$ implies $tH_1(M'_\phi; \mathbb{Z}) = tH_2(M_\phi, V^*; \mathbb{Z}) = 0$. We see from [9] that $V^*$ is an exact leaf of $M_\phi$ and thus $M_\phi$ is exact. \ 

For a finitely generated $\Lambda$-module $H$, let $e(H)$ denote the minimal number of $\Lambda$-generators of $H$. By convention, $e(0) = 0$. We show the following theorem, which improves and generalizes Kamada’s estimate in [5]:

**Theorem 4.6.** Let $F$ be a surface-link with $r$ components and total genus $g$. Then for every $\Lambda$-submodule $H' \subset H = H_1(\bar{E}, \partial \bar{E}; \mathbb{Z})$ such that $D = H/H'$ is a $(t - 1)$-divisible finite $\Lambda$-module, there is a $\Lambda$-submodule $D'$ of $D$ such that

$$e(D') \leq T(F) \quad \text{and} \quad e(E^2(D/D')) \leq g + e(H') - r + 1 + T(F).$$

**Proof.** Let $m = e(H')$. Let $F'$ be a surface-link obtained from $F$ by an embedded surgery along $m$ mutually disjoint 1-handles representing $\Lambda$-generators for
Let $E'$ be the compact exterior of $F'$ in $S^4$. Then we see that $H_1(\tilde{E}'; Z) \cong H/H' = D$, so that $F'$ is a surface-knot of genus $g' = g + m - (r - 1)$. Since the 1-handle surgery can be done apart from the triple points of $\rho(F)$, we obtain a pseudo-ribbon surface-knot $F''$ of genus

$$g'' = g' + \mathcal{T}(F) = g + m - (r - 1) + \mathcal{T}(F)$$

from $F'$ by $\mathcal{T}(F)$ times of triple point cancelling operations. Let $E''$ be the knot-exterior of $F''$. Then we have $H_1(\tilde{E}''; Z) \cong D/D'$ for a $\Lambda$-submodule $D'$ of $D$ with $e(D') \leq \mathcal{T}(F)$. By Theorem 2.2, we have a $t$-anti isomorphism

$$D/D' = H_1(\tilde{E}''; Z) = DH_1(\tilde{E}''; Z) \cong E^1(BH_2(\tilde{E}'', \partial \tilde{E}''; Z)).$$

Let $M_\phi$ be an exact $\mathbb{Z}^{H_1}$-manifold obtained from $F''$ by Lemma 4.5. By an argument in [8], we have a $\Lambda$-isomorphism

$$E^1(BH_2(\tilde{M}_\phi; Z)) \cong E^1(BH_2(\tilde{E}'', \partial \tilde{E}''; Z)).$$

Since $M_\phi$ is exact, we see from [9] that there is a splitting

$$BH_2(\tilde{M}_\phi; Z) \cong X \oplus \Lambda g''$$

for a torsion-free $\Lambda$-module $X$ with $E^0E^0(X) = \Lambda g''$, so that

$$E^1(BH_2(\tilde{E}'', \partial \tilde{E}''; Z)) \cong E^1(X)$$

and we have a $t$-anti isomorphism

$$E^2(D/D') = E^2(H_1(\tilde{E}''; Z)) \cong E^2E^1(X).$$

Using a natural $\Lambda$-epimorphism $\Lambda g'' \cong E^0E^0(X) \to E^2E^1(X)$ (see [7]), we have

$$e(E^2(D/D')) = e(E^2E^1(X)) \leq g''.$$

A surface-link $F$ is concordant to a surface-link $F'$ if there is a proper locally-flat embedding $f : F \times [0, 1] \to S^4 \times [0, 1]$ such that $f(F \times 0) = F \times 0$ and $f(F \times 1) = F' \times 1$. Since the triple point cancelling number of every trivial surface-link is zero, the following corollary also implies that every positive integer is the triple point cancelling number of a surface-link with any previously given genera of the components.

**Corollary 4.7.** For every surface-link $F$ and every integer $m \geq \mathcal{T}(F)$, there is a surface-link $F_*$ such that $F_*$ is concordant to $F$ and $\mathcal{T}(F_*) = m$. 

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Proof. Let $S_n$ be the $n$-fold connected sum of any $S^2$-knot in Corollary 4.3. Then we have $T(S_n) \leq n$. Let $F_n$ be any connected sum of $F$ and $S_n$ and $F_0 = F$. Then $F_n$ is concordant to $F$, since every $S^2$-knot is concordant to the trivial $S^2$-knot (see M. A. Kervaire [12]). Let $E_n$ be the link-exterior of $F_n$ in $S^4$. Let $H_n = H_1(\partial E_n, \partial \bar{E}_n; \mathbb{Z})$. Then we have

$$H_n = H_0 \oplus [\Lambda/(p, t+1)]^n.$$ 

In Theorem 4.6, we take $H = H_n$ and $H' = H_0$. Then by Theorem 4.6, there is a finite $\Lambda$-submodule $D'$ of $D = H/H' = [\Lambda/(p, t+1)]^n$ such that $e(D') \leq T(F_n)$ and $e(E^2(D/D')) \leq g + e(H_0) - r + 1 + T(F_n)$. Since $t$ acts on $D$ as the $(-1)$-multiple map, we have a $\Lambda$-isomorphism

$$E^2(D/D') = \text{hom}(D/D', Q/\mathbb{Z}) \cong D/D'.$$

Thus, we have

$$n = e(D) \leq e(D') + e(D/D') \leq 2T(F_n) + g + e(H_0) - r + 1,$$

where $g$ and $r$ denote the total genus and the component number of $F$, respectively.

Using that $g$, $e(H_0)$ and $r$ are independent of $n$, we see that

$$\lim_{n \to +\infty} T(F_n) = +\infty.$$

Using that $T(F_{n+1}) \leq T(F_n) + 1$ for all $n$, we find an integer $n$ such that $T(F_n) = m$ for every integer $m \geq T(F)$. □

References