Amphicheirality of links and Alexander invariants

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April 30, 2011

Abstract

We obtain an equation among invariants obtained from the Alexander module of an amphicheiral link. For special cases, it deduces necessary conditions on the Alexander polynomial. By using the present results and some known results, we show that the Alexander polynomial of an algebraically split component-preservingly $(\pm)$-amphicheiral link with even components is zero, and we determine prime amphicheiral links with at least 2 components and up to 9 crossings.

1 Introduction

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link in S^3 with $r \geq 1$. For an oriented knot K, we denote the orientation-reversed knot by $-K$. If φ is an orientation-reversing (orientation-preserving, respectively) homeomorphism of S^3 so that $\varphi(K_i) = \varepsilon_{\sigma(i)} K_{\sigma(i)}$ for all $i = 1, \ldots, r$ where $\varepsilon_i = +$ or $-$, and σ is a permutation of $\{1, 2, \ldots, r\}$, then L is called an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*amphicheiral link* (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-*invertible link*, respectively). A term "amphicheiral link" is used as a general term for an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link. A link is called an *interchangeable link* if it is an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link such that σ is not the identity. An $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link is called an *invertible link* simply if there exists $1 \leq i \leq r$ such that $\varepsilon_i = -$. If σ is the identity, then an amphicheiral link is called a *component-preservingly amphicheiral link*, and σ may be omitted from the notation. If every $\varepsilon_i = \varepsilon$ for all $i = 1, \ldots, r$ (including the case that σ is not the identity), then an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-amphicheiral link (an $(\varepsilon_1, \ldots, \varepsilon_r; \sigma)$-invertible link, respectively) is called an (ε)-amphicheiral link (an (ε)-invertible link, respectively). We use the notations $+ = +1 = 1$ and $- = -1$. A link L with at least 2-component is called an *algebraically split link* if the linking number of every 2-component sublink of L is zero. We note that a component-preservingly (ε)-amphicheiral link is an algebraically split link.

Necessary conditions for the Alexander polynomials of amphicheiral knots are studied by R. Hartley [3], R. Hartley and the second author [4], and the second author

2010 *Mathematics Subject Classification*: 57M25, 57M27.
Keywords: amphicheiral link; quadratic form; signature invariants; Alexander polynomial.

[11]. In [11], non-invertibility of 8_{17} is firstly proved by those conditions. On the other hand, T. Sakai [21] proved that any one-variable Laurent polynomial $f(t)$ over $\mathbb{Z}$ such that $f(t) = f(t^{-1})$ and $f(1) = 1$ is realized by the Alexander polynomial of a strongly invertible knot in S^3. B. Jiang, X. Lin, Shicheng Wang and Y. Wu [6] showed that (1) a twisted Whitehead doubled knot is amphicheiral if and only if it is the unknot or the figure eight knot, and (2) a prime link with at least 2 components and up to 9 crossings is component-preservingly (+)-amphicheiral if and only if it is the Borromean rings (cf. Theorem 1.4 (3)). They used S. Kojima and M. Yamasaki's η-function [15]. Shida Wang [25] determined prime component-preservingly (+)-amphicheiral links with at least 2 components and up to 11 crossings by the same method as [6]. There are four such links. For geometric studies of symmetries of arborescent knots, see F. Bonahon and L. C. Siebenmann [2]. The first author [8] studied necessary conditions for the Alexander polynomials of algebraically split component-preservingly amphicheiral links by computing the Reidemeister torsions of surgered manifolds along the link. The second author [14] defined and studied invariants obtained from the quadratic form of a link, and the Seifert quadratic form of a link associated with a Seifert surface of the link. In the present paper, we deduce necessary conditions for the invariants of amphicheiral links.

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link in S^3 with $r \geq 1$, and $E = \overline{S^3 \setminus N(L)}$ the exterior of L where $N(L)$ is a regular neighborhood of L. Let $\gamma : \pi_1(E) \to \mathbb{Z}$ be the surjective homomorphism sending every oriented meridian of L to 1, and $p : \tilde{E} \to E$ the covering associated with the kernel of γ, which is called the *infinite cyclic covering* of L. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the one variable Laurent polynomial ring over $\mathbb{Z}$. If we correspond the meridians of L to t, then the homology groups $H_*(\tilde{E}; \mathbb{Z})$, $H_*(\partial\tilde{E}; \mathbb{Z})$ and $H_*(\tilde{E}, \partial\tilde{E}; \mathbb{Z})$ are finitely generated Λ-modules. We set $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the homology groups $H_*(\tilde{E}; \mathbb{Q})$, $H_*(\partial\tilde{E}; \mathbb{Q})$ and $H_*(\tilde{E}, \partial\tilde{E}; \mathbb{Q})$ are finitely generated $\Lambda_{\mathbb{Q}}$-modules. Let $TH_1(\tilde{E}; \mathbb{Q})$ be the $\Lambda_{\mathbb{Q}}$-torsion submodule of $H_1(\tilde{E}; \mathbb{Q})$, and $BH_1(\tilde{E}; \mathbb{Q}) = H_1(\tilde{E}; \mathbb{Q})/TH_1(\tilde{E}; \mathbb{Q})$. Let $\rho_\varepsilon : TH_1(\tilde{E}; \mathbb{Q}) \to TH_1(\tilde{E}; \mathbb{Q})$ be the $\Lambda_{\mathbb{Q}}$-homomorphism multiplied by $\rho_\varepsilon = t - \varepsilon$ where $\varepsilon = 1$ or -1. We define by

$$\begin{aligned}
\kappa_\varepsilon(L) &= \dim_{\mathbb{Q}}(\mathrm{Ker}\,(\rho_\varepsilon; TH_1(\tilde{E}; \mathbb{Q}))), \\
\beta(L) &= \mathrm{rank}_{\Lambda_{\mathbb{Q}}}(BH_1(\tilde{E}; \mathbb{Q})), \\
\ell(L) &= \sum_{1 \leq i < j \leq r} \mathrm{lk}\,(K_i, K_j) \text{ for } r \geq 2, \text{ and } 0 \text{ for } r = 1
\end{aligned}$$

where the righthand side of $\kappa_\varepsilon(L)$ implies the $\mathbb{Q}$-dimension of the kernel of ρ_ε, the righthand side of $\beta(L)$ implies the $\Lambda_{\mathbb{Q}}$-rank of $BH_1(\tilde{E}; \mathbb{Q})$, and $\mathrm{lk}\,(K_i, K_j)$ is the linking number of K_i and K_j. We call $\kappa_\varepsilon(L)$ ($\beta(L)$ and $\ell(L)$, respectively) the *κ_ε-dimension* (the *β-rank* and the *total linking number*, respectively) of L.

Our main theorem is the following:

Theorem 1.1 *Let $L = K_1 \cup \ldots \cup K_r$ be an r-component amphicheiral link. Then we have*

$$\kappa_{-1}(L) \equiv r - 1 - \beta(L) + \ell(L) \pmod 2.$$

In particular, if L is an (ε)-amphicheiral link where $\varepsilon = +$ or $-$, then we have $\ell(L) = 0$, $\kappa_{-1}(L) \equiv 0 \pmod 2$ and

$$r - 1 \equiv \beta(L) \pmod 2.$$

We show the theorem in Section 3.

Let $\Delta_L(t_1, \ldots, t_r)$ be the r-variable Alexander polynomial of L which is an element of the r-variable Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ over $\mathbb{Z}$ where t_i $(i = 1, \ldots, r)$ is a variable corresponding to a meridian of K_i. We have the following corollary:

Corollary 1.2 *Let $L = K_1 \cup \ldots \cup K_r$ be an r-component amphicheiral link such that $r + \ell(L)$ is even. Then we have*

$$\Delta_L(-1, \ldots, -1) = 0.$$

In particular, if L is an (ε)-amphicheiral link where $\varepsilon = +$ or $-$, and r is even, then we have

$$\Delta_L(t, \ldots, t) = 0.$$

The following is a partial affirmative answer for [8, Conjecture 1.1] (see also Conjecture 5.1 in Section 5).

Theorem 1.3 *For an r-component component-preservingly (ε)-amphicheiral link L with r even, we have*

$$\Delta_L(t_1, \ldots, t_r) = 0.$$

The referee pointed out us that L. Traldi [23, Section 6] has already shown that if L is an r-component component-preservingly $(-)$-amphicheiral link, or a 2-component component-preservingly $(+)$-amphicheiral link, then the i-th Alexander ideal $E_i(L)$ of L vanishes for every $i < r$. Traldi's result is stronger than our result in Theorem 1.3 for an r-component component-preservingly $(-)$-amphicheiral link with $r > 2$. Our result is shown by a different method and contains a result that $\Delta_L(t_1, \ldots, t_r) = 0$ for any r-component component-preservingly $(+)$-amphicheiral link L for every even $r > 0$. It would be interesting to observe that connected sums of copies of the Borromean rings give an example of an r-component component-preservingly $(+)$-amphicheiral link L with $\Delta_L(t_1, \ldots, t_r) \neq 0$ for every odd $r > 1$.

We determined prime amphicheiral links with at least 2 components and up to 9 crossings. Let $\mathcal{A}_n$ ($\mathcal{C}_n$, respectively) be the set of prime amphicheiral links (component-preservingly amphicheiral links, respectively) with at least 2 components and up to n crossings, and $\mathcal{A}_n^\varepsilon$ the subset of $\mathcal{A}_n$ consisting of (ε)-amphicheiral links ($\mathcal{C}_n^\varepsilon$ the subset of $\mathcal{C}_n$ consisting of component-preservingly (ε)-amphicheiral links, respectively) where $\varepsilon = +$ or $-$. It is clear that $\mathcal{A}_n \supset \mathcal{C}_n$, $\mathcal{A}_n^\pm \supset \mathcal{C}_n^\pm$, $\mathcal{A}_n \supset \mathcal{A}_n^\pm$ and $\mathcal{C}_n \supset \mathcal{C}_n^\pm$. For a link with the crossing number up to 9, we use the notation of D. Rolfsen's book [20].

Theorem 1.4 *Under the setting above, we have the following:*

(1) $\mathcal{C}_9 = \{2_1^2, 6_2^2, 6_2^3, 8_8^2, 8_6^3, 8_3^4\}$.

(2) $\mathcal{A}_9 \setminus \mathcal{C}_9 = \{8_4^3, 9_{61}^2\}$.

(3) $\mathcal{C}_9^+ = \{6_2^3\}$, $\mathcal{C}_9^- = \emptyset$.

(4) $\mathcal{A}_9^+ \setminus \mathcal{C}_9^+ = \{8_4^3, 8_6^3, 8_3^4\}$, $\mathcal{A}_9^- \setminus \mathcal{C}_9^- = \{6_2^3, 8_4^3, 8_6^3, 8_3^4\}$.

We remark that Theorem 1.4 (3) corresponds to a theorem in [6]. We could detect non-amphicheirality by Corollary 1.2, some other conditions on the Alexander polynomials, and geometric conditions (cf. Lemma 4.1, Lemma 4.2, Lemma 4.3, and Lemma 4.4).

In Section 2, we define the quadratic form of a link and invariants obtained from them, and prepare properties of them. In Section 3, we prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. In Section 4, we show Theorem 1.4. In Section 5, we give some remarks related to the first author's previous results.

2 Quadratic form and its related invariants

For a symmetric matrix A of size k over $\mathbb{R}$, we define

$$\begin{aligned} n(A) &= k - \mathrm{rank}\,(A) = \sharp(0 \text{ eigenvalues of } A), \\ s(A) &= \sharp(\text{positive eigenvalues of } A) - \sharp(\text{negative eigenvalues of } A) \end{aligned}$$

where $\mathrm{rank}\,(A)$ means the rank of A over $\mathbb{R}$, and $\sharp(\cdot)$ implies the number of elements. We call $n(A)$ ($s(A)$, respectively) the *nullity* (the *signature*, respectively) of A. Suppose that $b : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ is a bilinear form presented by A. Since both $n(A)$ and $s(A)$ do not change by base changes, they are also invariants of b, and we denote them by $n(b)$ and $s(b)$, respectively. We can define the similar ones for a hermitian matrix over $\mathbb{C}$. The following lemma is clear.

Lemma 2.1 *Let A be a hermitian matrix of size k over $\mathbb{C}$. Then we have*

$$k \equiv s(A) + n(A) \pmod 2.$$

For an oriented link L, we take a Seifert surface F of L. Let S be a Seifert matrix associated with F, and $A = S + {}^tS$ where ${}^tS$ is the transposed matrix of S. Then $s(A)$ is the *Murasugi signature* of L which is an invariant of L. So we denote $s(A)$ by $s(L)$. Let $\ell(L)$ be the *total linking number* of L defined in Section 1. K. Murasugi [18] showed the following lemma. The second author [12] obtained the same results by another method.

Lemma 2.2 *Let L be an oriented link. Then we have the following:*

(1) [18, Theorem 1] *$s(L)+\ell(L)$ does not depend on the orientation of L.*

(2) [18, Theorem 2] *If L is an amphicheiral link, then $s(L)+\ell(L)=0$.*

By Lemma 2.2 (1), the statement of Lemma 2.2 (2) does not depend on the orientation of L.

For a Λ-module H, let TH denote the Λ-torsion submodule of H, $BH = H/TH$, and $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the $\Lambda_{\mathbb{Q}}$-torsion submodule of $H_{\mathbb{Q}}$ is $(TH)_{\mathbb{Q}}$, and we have $(BH)_{\mathbb{Q}} = H_{\mathbb{Q}}/(TH)_{\mathbb{Q}}$. We denote $(TH)_{\mathbb{Q}}$ and $(BH)_{\mathbb{Q}}$ by $TH_{\mathbb{Q}}$ and $BH_{\mathbb{Q}}$, respectively. We have a natural $\Lambda_{\mathbb{Q}}$-isomorphism $\mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(H_{\mathbb{Q}},\mathbb{Q})$, and a natural short exact sequence:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(BH,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(H,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(TH,\mathbb{Q}) \to 0.$$

If H is a finitely generated Λ-module, then TH and BH are finitely generated Λ-modules, so that $TH_{\mathbb{Q}}$ splits into finitely many cyclic $\Lambda_{\mathbb{Q}}$-torsion modules, and hence is (non-canonically) $\Lambda_{\mathbb{Q}}$-isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(TH,\mathbb{Q})$, and $BH_{\mathbb{Q}}$ is $\Lambda_{\mathbb{Q}}$-free of finite rank. Let E be a compact connected oriented n-dimensional manifold, E' a submanifold of E, $p:\tilde{E} \to E$ an infinite cyclic covering, and $\tilde{E}' = p^{-1}(E')$. Then we have $H_*(\tilde{E},\tilde{E}';\mathbb{Q}) = H_*(\tilde{E},\tilde{E}';\mathbb{Z})_{\mathbb{Q}}$ which is a finitely generated $\Lambda_{\mathbb{Q}}$-module. We use the following notations:

$$\begin{aligned} T^*(\tilde{E},\tilde{E}';\mathbb{Q}) &= \mathrm{Hom}_{\mathbb{Z}}(TH_*(\tilde{E},\tilde{E}';\mathbb{Z}),\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(TH_*(\tilde{E},\tilde{E}';\mathbb{Q}),\mathbb{Q}),\\ B^*(\tilde{E},\tilde{E}';\mathbb{Q}) &= \mathrm{Hom}_{\mathbb{Z}}(BH_*(\tilde{E},\tilde{E}';\mathbb{Z}),\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(BH_*(\tilde{E},\tilde{E}';\mathbb{Q}),\mathbb{Q}). \end{aligned}$$

Since the $\mathbb{Q}$-cohomology $H^*(\tilde{E},\tilde{E}';\mathbb{Q})$ is identified with $\mathrm{Hom}_{\mathbb{Z}}(H_*(\tilde{E},\tilde{E}';\mathbb{Z}),\mathbb{Q})$, we obtain the following short exact sequence:

$$0 \to B^*(\tilde{E},\tilde{E}';\mathbb{Q}) \to H^*(\tilde{E},\tilde{E}';\mathbb{Q}) \to T^*(\tilde{E},\tilde{E}';\mathbb{Q}) \to 0.$$

Let $L = K_1 \cup \ldots \cup K_r$ be an oriented r-component link in S^3. From now on, we take the exterior E of L as $E = \overline{S^3 \setminus N(L)}$ where $N(L)$ is a regular neighborhood of L. If we take a connected Seifert surface F of L, then a connected lift F_E of $F \cap E$ in $\tilde{E}$ represents the fundamental class in $H_2(\tilde{E},\partial\tilde{E};\mathbb{Z})$, and we denote $\mu = [F_E] \in H_2(\tilde{E},\partial\tilde{E};\mathbb{Z})$. We have the *quadratic form* b_L of L which is the pairing:

$$b_L : T^1(\tilde{E},\partial\tilde{E};\mathbb{Q}) \times T^1(\tilde{E},\partial\tilde{E};\mathbb{Q}) \to \mathbb{Q}$$

defined by $b_L(x,y) = \langle x \cup (t-t^{-1})y, \mu\rangle$ for all $x,y \in T^1(\tilde{E},\partial\tilde{E};\mathbb{Q})$ (cf. [10, 14]). This definition is an extension of J. Milnor [17]. Then b_L have the symmetry $b_L(x,y) = b_L(y,x)$ and the t-symmetry $b_L(tx,ty) = b_L(x,y)$ for all $x,y \in T^1(\tilde{E},\partial\tilde{E};\mathbb{Q})$. Let

$$\begin{aligned} T_\delta(L;\mathbb{Q}) &= \mathrm{Im}\,(\delta : T^0(\partial\tilde{E};\mathbb{Q}) \to T^1(\tilde{E},\partial\tilde{E};\mathbb{Q})),\\ \hat{T}^1(\tilde{E},\partial\tilde{E};\mathbb{Q}) &= T^1(\tilde{E},\partial\tilde{E};\mathbb{Q})/T_\delta(L;\mathbb{Q}) \end{aligned}$$

where the map δ is a restriction of the coboundary homomorphism. We set

$$\rho_\varepsilon : TH_1(\tilde{E};\mathbb{Q}) \to TH_1(\tilde{E};\mathbb{Q})$$

and

$$\rho_\varepsilon : \hat{T}^1(\tilde{E},\partial\tilde{E};\mathbb{Q}) \to \hat{T}^1(\tilde{E},\partial\tilde{E};\mathbb{Q})$$

as multiplied homomorphisms by $\rho_\varepsilon = t - \varepsilon$ where $\varepsilon = 1$ or -1. Then we define by

$$\begin{aligned}
\kappa_\varepsilon(L) &= \dim_{\mathbb{Q}}(\mathrm{Ker}\,(\rho_\varepsilon; TH_1(\tilde{E};\mathbb{Q}))),\\
\hat{\kappa}_\varepsilon(L) &= \dim_{\mathbb{Q}}(\mathrm{Ker}\,(\rho_\varepsilon; \hat{T}^1(\tilde{E},\partial\tilde{E};\mathbb{Q}))),\\
\beta(L) &= \mathrm{rank}_{\Lambda_{\mathbb{Q}}}(BH_1(\tilde{E};\mathbb{Q}))
\end{aligned}$$

where the righthand sides of $\kappa_\varepsilon(L)$ and $\hat{\kappa}_\varepsilon(L)$ imply the $\mathbb{Q}$-dimensions of the kernels of ρ_ε, respectively, and the righthand side of $\beta(L)$ implies the $\Lambda_{\mathbb{Q}}$-rank of $BH_1(\tilde{E};\mathbb{Q})$. We call $\kappa_\varepsilon(L)$ ($\hat{\kappa}_\varepsilon(L)$ and $\beta(L)$, respectively) the *κ_ε-dimension* (the *$\hat{\kappa}_\varepsilon$-dimension* and the *β-rank*, respectively) of L.

Let $V = (v_{ij})$ be the *canonical linking matrix* of L of size r defined by

$$v_{ij} = \begin{cases} \mathrm{lk}\,(K_i, K_j) & (i \neq j),\\ -\displaystyle\sum_{k\neq i} \mathrm{lk}\,(K_i, K_k) & (i = j). \end{cases}$$

We define by $\mathrm{r}_V(L) = \mathrm{rank}\,(V)$ and $v(L) = s(V)$. We call $\mathrm{r}_V(L)$ ($v(L)$, respectively) the *canonical rank* (the *canonical signature*, respectively) of L.

We define the local signature invariants of the quadratic form b_L by extending the coefficient field $\mathbb{Q}$ into the real numbers $\mathbb{R}$:

$$b_L^{\mathbb{R}} : T^1(\tilde{E},\partial\tilde{E};\mathbb{R}) \times T^1(\tilde{E},\partial\tilde{E};\mathbb{R}) \to \mathbb{R}$$

where $T^1(\tilde{E},\partial\tilde{E};\mathbb{R}) = T^1(\tilde{E},\partial\tilde{E};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ is a $\Lambda_{\mathbb{R}}$-module with $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We set $\sigma(L) = s(b_L^{\mathbb{R}})$, $n(L) = n(b_L^{\mathbb{R}})$ and $\rho_a = t^2 - 2at + 1$ for every $a \in (-1,1)$. Recall that $\rho_\varepsilon = t - \varepsilon$ where $\varepsilon = 1$ or -1. For a $\Lambda_{\mathbb{R}}$-module H and $a \in [-1,1]$, let H_{ρ_a} denote the ρ_a-primary component of H. Let

$$b_L^{\rho_a} : T^1(\tilde{E},\partial\tilde{E};\mathbb{R})_{\rho_a} \times T^1(\tilde{E},\partial\tilde{E};\mathbb{R})_{\rho_a} \to \mathbb{R}$$

be the induced (restricted) quadratic form from $b_L^{\mathbb{R}}$. We define the *local signature at* a and the *local nullity at* a of L by $\sigma_a(L) = s(b_L^{\rho_a})$ and $n_a(L) = n(b_L^{\rho_a})$ ($a \in [-1,1]$), respectively. We note that $\sigma_a(L)$ is zero except a finite number of $a \in [-1,1]$, $\sigma_a(L)$ is even for all $a \in (-1,1)$, and $n_a(L)$ is zero except $a \in \{1,-1\}$. Further we have

$$\sigma(L) = \sum_{x\in[-1,1]} \sigma_x(L) \quad \text{and} \quad n(L) = n_1(L) + n_{-1}(L).$$

For $x \in (-1,1)$, we set $\omega_x = x + \sqrt{1-x^2}\sqrt{-1}$. Let S be a Seifert matrix of L associated with F. We define by

$$\begin{aligned}
s_{[a,1]}(L) &= \lim_{x\to a-0} s((1-\omega_x)S + (1-\overline{\omega}_x)^tS) \text{ for each } a \in (-1,1],\\
s_{(a,1]}(L) &= \lim_{x\to a+0} s((1-\omega_x)S + (1-\overline{\omega}_x)^tS) \text{ for each } a \in [-1,1),\\
s_a(L) &= s_{[a,1]}(L) - s_{(a,1]}(L) \text{ for each } a \in [-1,1),\\
s_1(L) &= s_{[1,1]}(L),\\
s_{-1}(L) &= s_{[-1,1]}(L) - s_{(-1,1]}(L).
\end{aligned}$$

We call $s_a(L)$ ($a \in [-1,1]$) the *local Seifert signature at* a of L.

The second author [14] showed the following:

Lemma 2.3 *Let L be an oriented r-component link in S^3. Then we have the following:*

(1) $\mathrm{r}_V(L) \equiv v(L) \pmod 2$.

(2) $\sigma(L) \equiv \sigma_1(L) + \sigma_{-1}(L) \pmod 2$.

(3) [14, Lemma 3.2.2] $\hat{\kappa}_1(L) + \mathrm{r}_V(L) = \kappa_1(L)$.

(4) [14, Lemma 3.2.3] $\beta(L) + \kappa_1(L) = r - 1$.

(5) [14, Theorem 5.3] $\sigma(L) = s(L) - v(L)$.

(6) [14, Corollary 5.4] $\sigma_a(L) = s_a(L)$ *for all* $a \in [-1,1)$.

(7) [14, Lemma 5.7.1] $\sigma_1(L) \equiv \hat{\kappa}_1(L) \pmod 2$.

(8) [14, Lemma 5.7.2] $\sigma_{-1}(L) \equiv \kappa_{-1}(L) \pmod 2$.

We note that (1) is obtained from Lemma 2.1, and (2) is obtained from the definition of the local signatures. In the present paper, we do not use (6) explicitly.

To prove Theorem 1.3, we need more general settings. Let $L = K_1 \cup \ldots \cup K_r$ be an oriented r-component link in S^3, E the exterior of L, T_i the boundary of a regular neighborhood of K_i ($i = 1,\ldots,r$) (i.e. $\partial E = T_1 \cup \ldots \cup T_r$), m_i and l_i the meridian and the longitude of K_i on T_i respectively, F a Seifert surface of L, M the result of Seifert framing surgery along L which is obtained by attaching meridians of solid tori to E along ∂F, and m'_i and l'_i the meridian and the core of the i-th attaching torus respectively. Note that M is uniquely determined from the "oriented" link L. In particular, if L is algebraically split, then M is obtained as the result of $(0,\ldots,0)$-surgery along L which is independent from the orientation of L. Let $\ell_{ij} = \mathrm{lk}\,(K_i, K_j)$ be the linking number of K_i and K_j for $1 \leq i \neq j \leq r$. In $H_1(E)$ (we write elements of homologies multiplicatively), we have

$$[l_i] = \prod_{j\neq i} [m_j]^{\ell_{ij}} \tag{2.1}$$

Hence we have

$$H_1(E) = \langle [m_1], \ldots, [m_r] \rangle \cong \mathbb{Z}^r.$$

Since $\partial F \cap T_i$ represents an element of the form $[m_i]^{c_i}[l_i]$ in $H_1(T_i)$ for some $c_i \in \mathbb{Z}$, and $[\partial F] = 1$ in $H_1(E)$, we have

$$[\partial F] = \prod_{i=1}^{r} [m_i]^{c_i}[l_i] = \prod_{i=1}^{r} [m_i]^{c_i + \sum_{k \neq i} \ell_{ik}} = 1,$$

and hence

$$c_i = -\sum_{k \neq i} \ell_{ik}$$

by (2.1). In $H_1(M)$, we have

$$\begin{aligned} [m_i'] &= [m_i]^{c_i}[l_i] = [m_i]^{-\sum_{k \neq i} \ell_{ik}} \prod_{j \neq i} [m_j]^{\ell_{ij}} = 1, \\ [l_i'] &= [m_i] \end{aligned} \qquad (2.2)$$

The first equation in (2.2) implies that the canonical linking matrix V is a presentation matrix of $H_1(M)$, and

$$H_1(M; \mathbb{Q}) \cong \mathbb{Q}^{r - \mathrm{r}_V(L)} \qquad (2.3)$$

Let $\gamma : H_1(E) \to \mathbb{Z} = \langle t \rangle$ (The operation of $\langle t \rangle$ is multiplicative) be an epimorphism with $\gamma([m_i]) = t^{u_i}$ where $u_i \in \mathbb{Z}$ $(i = 1, \ldots, r)$, $p_\gamma : \tilde{E}_\gamma \to E$ the infinite cyclic covering associated with the kernel of γ, and $\Delta_L^\gamma(t)$ the Alexander polynomial of $H_1(\tilde{E}_\gamma)$. We set a column vector $\mathbf{u} = {}^t(u_1, \ldots, u_r)$ which is the transpose vector of a row vector $(u_1, \ldots, u_r)$. Then p_γ extends to the infinite cyclic covering $\tilde{M}_\gamma \to M$ if and only if $\mathbf{u}$ is in the null space of V (i.e. $V\mathbf{u} = \mathbf{0}$). We note that ${}^t(1, \ldots, 1)$ is in the null space of V (i.e. the dimension of the null space of V is at least one). In particular, if L is algebraically split, then V is the zero matrix, and p_γ extends to $\tilde{M}_\gamma \to M$ for every γ. We also denote $\tilde{M}_\gamma \to M$, the induced homomorphism $H_1(M) \to \mathbb{Z}$, and the Alexander polynomial of $H_1(\tilde{M}_\gamma)$ by the same symbols p_γ, γ, and $\Delta_M^\gamma(t)$, respectively. Let

$$\begin{aligned} \kappa_\varepsilon^\gamma(L) &= \dim_{\mathbb{Q}}(\mathrm{Ker}\,(\rho_\varepsilon; TH_1(\tilde{E}_\gamma; \mathbb{Q}))), \\ \kappa_\varepsilon^\gamma(M) &= \dim_{\mathbb{Q}}(\mathrm{Ker}\,(\rho_\varepsilon; TH_1(\tilde{M}_\gamma; \mathbb{Q}))) \end{aligned}$$

where $\varepsilon = 1$ or -1, and $\rho_\varepsilon = t - \varepsilon$,

$$\begin{aligned} T_\delta^\gamma(L; \mathbb{Q}) &= \mathrm{Im}\,(\delta^\gamma : T^0(\partial \tilde{E}_\gamma; \mathbb{Q}) \to T^1(\tilde{E}_\gamma, \partial \tilde{E}_\gamma; \mathbb{Q})), \\ \delta^\gamma(L) &= \dim_{\mathbb{Q}}(T_\delta^\gamma(L; \mathbb{Q})), \\ \delta_\varepsilon^\gamma(L) &= \dim_{\mathbb{Q}}(T_\delta^\gamma(L; \mathbb{Q})_{\rho_\varepsilon}) \end{aligned}$$

where the map δ^γ is a restriction of the coboundary map,

$$\begin{aligned} b_L^\gamma &: T^1(\tilde{E}_\gamma; \mathbb{Q}) \times T^1(\tilde{E}_\gamma; \mathbb{Q}) \to \mathbb{Q}, \\ b_M^\gamma &: T^1(\tilde{M}_\gamma; \mathbb{Q}) \times T^1(\tilde{M}_\gamma; \mathbb{Q}) \to \mathbb{Q}, \end{aligned}$$

the *quadratic forms* on $\tilde{E}_\gamma$ and $\tilde{M}_\gamma$ respectively defined by $b_L^\gamma(x,y) = \langle x \cup (t-t^{-1})y, \mu^\gamma\rangle$ for all $x, y \in T^1(\tilde{E}_\gamma;\mathbb{Q})$ and $b_M^\gamma(x,y) = \langle x \cup (t-t^{-1})y, \mu_M^\gamma\rangle$ for all $x, y \in T^1(\tilde{M}_\gamma;\mathbb{Q})$ where μ^γ and μ_M^γ are the fundamental classes in $H_2(\tilde{E}_\gamma;\mathbb{Q})$ and $H_2(\tilde{M}_\gamma;\mathbb{Q})$ respectively, $(b_L^\gamma)^{\mathbb{R}} = b_L^\gamma \otimes \mathbb{R}$, $(b_M^\gamma)^{\mathbb{R}} = b_M^\gamma \otimes \mathbb{R}$, $(b_L^\gamma)^{\mathbb{R}}_{\rho_\varepsilon}$ and $(b_M^\gamma)^{\mathbb{R}}_{\rho_\varepsilon}$ the restrictions of $(b_L^\gamma)^{\mathbb{R}}$ and $(b_M^\gamma)^{\mathbb{R}}$ on $TH_1(\tilde{E}_\gamma;\mathbb{R})_{\rho_\varepsilon}$ and $TH_1(\tilde{M}_\gamma;\mathbb{R})_{\rho_\varepsilon}$ respectively, $\sigma_\varepsilon^\gamma(L) = s((b_L^\gamma)^{\mathbb{R}}_{\rho_\varepsilon})$, $\sigma_\varepsilon^\gamma(M) = s((b_M^\gamma)^{\mathbb{R}}_{\rho_\varepsilon})$, $n_\varepsilon^\gamma(L) = n((b_L^\gamma)^{\mathbb{R}}_{\rho_\varepsilon})$ and $n_\varepsilon^\gamma(M) = n((b_M^\gamma)^{\mathbb{R}}_{\rho_\varepsilon})$.

Lemma 2.4 *We suppose the situation above.*

(1) We have

$$\Delta_L^\gamma(t) \doteq (t-1)\Delta_L(t^{u_1}, \dots, t^{u_r}).$$

If $\gamma([l_i']) = t^{u_i} \neq 1$ *(i.e.* $u_i \neq 0$*) for every* i $(i = 1, \dots, r)$*, then we have*

$$\Delta_M^\gamma(t) \doteq \Delta_L^\gamma(t)(t-1)\prod_{i=1}^{r}(t^{u_i}-1)^{-1}.$$

(2) $\kappa_1^\gamma(L) = r - \mathrm{r}_V(L) - 1 - \beta(L)$*. In particular, if* L *is algebraically split, then we have* $\kappa_1^\gamma(L) = r - 1 - \beta(L)$*.*

(3) $n_1^\gamma(L) - \kappa_1^\gamma(L) = n_1^\gamma(M) = \kappa_1^\gamma(M)$*.*

(4) $\dim_{\mathbb{Q}}(TH_1(\tilde{M}_\gamma;\mathbb{Q})) \equiv \dim_{\mathbb{Q}}(TH_1(\tilde{M}_\gamma;\mathbb{Q})_{\rho_1}) \equiv \sigma_1^\gamma(M) + \kappa_1^\gamma(M) \equiv 0 \pmod 2$*.*

Proof (1) By the second equation in (2.2), and the surgery formula of Reidemeister torsions [24], we have the result.

(2) We have the following Wang exact sequence:

$$H_1(\tilde{M}_\gamma;\mathbb{Q}) \xrightarrow{t-1} H_1(\tilde{M}_\gamma;\mathbb{Q}) \to H_1(M;\mathbb{Q}) \to \mathbb{Q} \to 0.$$

By (2.3) and $BH_1(\tilde{M}_\gamma;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}^{\beta(L)}$, we have

$$TH_1(\tilde{M}_\gamma;\mathbb{Q})/(t-1)TH_1(\tilde{M}_\gamma;\mathbb{Q}) \cong \mathbb{Q}^{r-\mathrm{r}_V(L)-1-\beta(L)},$$

and hence we have $\kappa_1^\gamma(L) = r - \mathrm{r}_V(L) - 1 - \beta(L)$. In particular, if L is algebraically split, then we can take V as the zero matrix ($\mathrm{r}_V(L) = 0$), and we have $\kappa_1^\gamma(L) = r - 1 - \beta(L)$.

(3) By the argument in [14, p101], we have exact sequences

$$TH_1(\partial\tilde{E}_\gamma;\mathbb{Q}) \xrightarrow{i_*^\gamma} TH_1(\tilde{E}_\gamma;\mathbb{Q}) \to TH_1(\tilde{M}_\gamma;\mathbb{Q}) \to 0$$

and

$$TH_1(\partial\tilde{E}_\gamma;\mathbb{Q})_{\rho_\varepsilon} \xrightarrow{(i_*^\gamma)_{\rho_\varepsilon}} TH_1(\tilde{E}_\gamma;\mathbb{Q})_{\rho_\varepsilon} \to TH_1(\tilde{M}_\gamma;\mathbb{Q})_{\rho_\varepsilon} \to 0$$

where $\varepsilon = 1$ or -1, and $(i_*^\gamma)_{\rho_\varepsilon}$ is a restriction of i_*^γ. The exact sequences and an isomorphism

$$TH_1(\partial\tilde{E}_\gamma;\mathbb{Q}) \cong \bigoplus_{i=1}^{r} \Lambda_{\mathbb{Q}}/(t^{u_i}-1)$$

induce

$$n_\varepsilon^\gamma(L) - \delta_\varepsilon^\gamma(L) = n_\varepsilon^\gamma(M) = \kappa_\varepsilon^\gamma(M) \quad \text{and} \quad \delta_\varepsilon^\gamma(L) = \kappa_\varepsilon^\gamma(L).$$

(4) Since the cup product on $T^1(\tilde{M}_\gamma;\mathbb{Q})$ is non-singular skew-symmetric, we have

$$\dim_{\mathbb{Q}}(TH_1(\tilde{M}_\gamma;\mathbb{Q})) \equiv \dim_{\mathbb{Q}}(TH_1(\tilde{M}_\gamma;\mathbb{Q})_{\rho_1}) \equiv 0 \pmod 2.$$

By (3), Lemma 2.1, and that $\dim_{\mathbb{Q}}(TH_1(\tilde{M}_\gamma;\mathbb{Q})_{\rho_1})$ is even, we have the result. ∎

3 Proofs of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1 Suppose that L is an r-component amphicheiral link. By Lemma 2.2 (2), Lemma 2.3 (2) and (5), we have

$$\sigma_1(L) + \sigma_{-1}(L) \equiv -\ell(L) - v(L) \pmod 2.$$

By this, Lemma 2.3 (1) and (3), we have

$$\sigma_1(L) + \sigma_{-1}(L) \equiv -\ell(L) + \hat{\kappa}_1(L) - \kappa_1(L) \pmod 2.$$

By this, Lemma 2.3 (4) and (7), we have

$$\sigma_{-1}(L) \equiv -\ell(L) + \beta(L) - r + 1 \pmod 2.$$

By this and Lemma 2.3 (8), we have

$$\begin{aligned} \kappa_{-1}(L) &\equiv -(r-1-\beta(L)+\ell(L)) \pmod 2 \\ &\equiv r-1-\beta(L)+\ell(L) \pmod 2. \end{aligned}$$

Suppose that L is an oriented link. We set the linking number of K_i and K_j $(1 \le i \ne j \le r)$ as $\ell_{ij} = \mathrm{lk}\,(K_i, K_j)$, and $\mathcal{L}(L) = \{\ell_{ij} \mid 1 \le i < j \le r\}$. Let L^* be the mirror image of L with the induced orientation, and $-L^*$ the oriented link obtained from L^* by introducing the opposite orientations on every components. Then we have $\mathcal{L}(L^*) = \mathcal{L}(-L^*) = \{-\ell_{ij} \mid 1 \le i < j \le r\}$, and

$$\ell(L^*) = \ell(-L^*) = -\ell(L) \tag{3.1}$$

Suppose that L is an (ε)-amphicheiral link where $\varepsilon = +$ or $-$. Then the sets $\mathcal{L}(L)$ and $\mathcal{L}(\varepsilon L^*)$ should be identical. By (3.1), we have $\ell(L) = 0$. Since $\sigma_{-1}(L) \equiv 0 \pmod 2$, we have $\kappa_{-1}(L) \equiv 0 \pmod 2$ by Lemma 2.3 (8). Therefore we have

$$r - 1 \equiv \beta(L) \pmod 2. \; ∎$$

Proof of Corollary 1.2 Suppose that L is an r-component amphicheiral link such that $r+\ell(L)$ is even. Then by Theorem 1.1, we have

$$\kappa_{-1}(L) \equiv -1-\beta(L) \pmod 2.$$

Suppose that $\beta(L)=0$. Then we have $\kappa_{-1}(L)>0$. It implies that $\Delta_L(t,\ldots,t)$ is divisible by $t+1$, and

$$\Delta_L(-1,\ldots,-1)=0.$$

Suppose that $\beta(L)>0$. It implies that

$$\Delta_L(t,\ldots,t)=0.$$

If L is an (ε)-amphicheiral link where $\varepsilon=+$ or $-$, and r is even, then we have $\beta(L)>0$ by Theorem 1.1. ∎

Proof of Theorem 1.3 Suppose $\Delta_L(t_1,\ldots,t_r)\neq 0$. Then there are coprime integers $u_1,\ldots,u_r$ with $\Delta_L(t^{u_1},\ldots,t^{u_r})\neq 0$. Note that $\{u_1,\ldots,u_r\}$ does not include 0 by the Torres condition [22]. Let E be the exterior of L, and M the result of $(0,\ldots,0)$-surgery along L. We take the epimorphism $\gamma: H_1(E)\to\mathbb{Z}=\langle t\rangle$ defined by $\gamma([m_i])=t^{u_i}$ $(i=1,\ldots,r)$ where $[m_i]$ is the representing element of the i-th meridian m_i in $H_1(E)$. Since $H_1(E)\cong H_1(M)\cong\mathbb{Z}^r$ naturally, γ induces $\gamma: H_1(M)\to\mathbb{Z}$. Here we used the same notation. By Lemma 2.4 (1), we have

$$\begin{aligned}\Delta_L^{\gamma}(t) &= (t-1)\Delta_L(t^{u_1},\ldots,t^{u_r})\neq 0,\\ \Delta_M^{\gamma}(t) &= \Delta_L^{\gamma}(t)(t-1)\prod_{i=1}^{r}(t^{u_i}-1)^{-1}\neq 0.\end{aligned}$$

Hence $H_1(\tilde{M}_\gamma;\mathbb{Q})$ is $\Lambda_{\mathbb{Q}}$-torsion, and $\beta(L)=0$. By Lemma 2.4 (2), we have

$$\kappa_1^{\gamma}(M)=r-1\equiv 1 \pmod 2$$

which is odd. By Lemma 2.4 (4), $\sigma_1^{\gamma}(M)$ is odd.

Since (M,γ) is orientation-preservingly homeomorphic to $(-M,-\varepsilon\gamma)$, we have

$$\sigma_1^{\gamma}(M)=\sigma_1^{-\varepsilon\gamma}(-M)=-\sigma_1^{\gamma}(M)=0.$$

It is a contradiction. Therefore we have $\Delta_L(t_1,\ldots,t_r)=0$. ∎

Remark 3.1 (1) The condition "$r+\ell(L)$ is even" in Corollary 1.2 is necessary. If L is the Hopf link ($r=2$ and $\ell(L)=\pm1$), then we have

$$\Delta_L(-1,-1)=\pm1\neq 0.$$

If L is the Borromean rings ($r=3$ and $\ell(L)=0$), then we have

$$\Delta_L(-1,-1,-1)=\pm8\neq 0.$$

(2) If $r = 2$, then the condition "$r+\ell(L)$ is even" in Corollary 1.2 implies that the linking number of L is even. Since R. Hartley [3] showed that a 2-component link with non-zero even linking number is not component-preservingly amphicheiral (see also Lemma 4.3), the condition "$r + \ell(L)$ is even" is effective only if L is an algebraically split link in this case. However the statement works for general amphicheiral links, so we include the case of non-zero even linking numbers (see also Remark 4.7 (1)).

(3) Though a component-preservingly (ε)-amphicheiral link is an algebraically split link, a general (ε)-amphicheiral link for $r \geq 3$ is not always an algebraically split link. For example, if $r = 3$, then the connected sum of the positive Hopf link and the negative Hopf link is a $(+)$-amphicheiral link which is not an algebraically split link. The reader can find such examples in Theorem 1.4 (4).

(4) Y. Matsumoto and G. Venema [16] applied the invariants in Theorem 1.1 and Lemma 2.3 effectively to study 4-dimensional topology.

4 Amphicheiral links with up to 9 crossings (Proof of Theorem 1.4)

In this section, we determine prime amphicheiral links with at least 2 components and up to 9 crossings. For a link with the crossing number up to 9, we use the notation of D. Rolfsen's book [20]. In Rolfsen's table [20], an r-component link such that $r \geq 2$ and the crossing number c is denoted by c^r_k where k is the ordering of the link in the table.

We raise results from R. Hartley [3] and the previous results due to the first author [8]. Since most of their statements are on component-preservingly amphicheiral links, we modify them for the case of general amphicheiral links. We do not give proofs of Lemma 4.1 and Lemma 4.4, and could not modify Lemma 4.4 into a statement on general amphicheiral links.

For two elements A and B in the r-variable Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, we denote by $A \doteq B$ if they are equal up to multiplications of units.

Lemma 4.1 ([8, Lemma 2.5]) *Let $L = K_1 \cup \dots \cup K_r$ be an r-component $(\varepsilon_1, \dots, \varepsilon_r; \sigma)$-amphicheiral link where $\varepsilon_i = +$ or $-$ $(i = 1, \dots, r)$, and σ is a permutation of $\{1, 2, \dots, r\}$. Then we have*

$$\Delta_L(t_1, \dots, t_r) \doteq \Delta_L\left(t_{\sigma(1)}^{\varepsilon_{\sigma(1)}}, \dots, t_{\sigma(r)}^{\varepsilon_{\sigma(r)}}\right).$$

Lemma 4.2 ([8, Lemma 3.1]) *Let $L = K_1 \cup \dots \cup K_r$ be an oriented r-component link.*

(1) If L is an $(\varepsilon_1, \dots, \varepsilon_r; \sigma)$-amphicheiral link, then a sublink $L' = K_{i_1} \cup \dots \cup K_{i_s}$ $(1 \leq i_1 < \dots < i_s \leq r)$ is an $(\varepsilon_{i_1}, \dots, \varepsilon_{i_s}; \rho)$-amphicheiral link where ρ is a permutation of $\{i_1, i_2, \dots, i_s\}$ closed under the action of σ.

(2) If $\ell_{1,2} \cdot \ell_{2,3} \cdot \ell_{3,1} \neq 0$ where $\ell_{p,q}$ is the linking number of K_p and K_q, then L is not amphicheiral.(Thank you for R. Nikkuni and K. Taniyama for this correct statement.)

Proof (1) We can see it without difficulty.

(2) Let L^* and K_i^* $(i = 1, 2, 3)$ be the mirror images of L and K_i^* with the induced orientations, respectively. We set the product of the linking numbers $N = \ell_{1,2} \cdot \ell_{2,3} \cdot \ell_{3,1} \neq 0$. Since the linking number of K_p^* and K_q^* is $-\ell_{p,q}$, the product N changes into $-N$ on L^*. Even if the orientations of some components of L^* are changed, the product of the linking numbers as above does not change. Therefore L cannot be amphicheiral. ∎

Lemma 4.3 *Let $L = K_1 \cup K_2$ be a 2-component link with non-zero even linking number e. Then we have the following:*

(1) (Hartley [3], [8, Lemma 3.2]) *L is not component-preservingly amphicheiral.*

(2) If $e \equiv 2 \pmod 4$, then L is not $(\pm, \mp; (1\ 2))$-amphicheiral where $(1\ 2)$ is the non-trivial permutation of $\{1, 2\}$.

Proof We show only (2). Suppose that L is a $(+, -; (1\ 2))$-amphicheiral link with the linking number e where $e > 0$ and $e \equiv 2 \pmod 4$.

By the duality and the Torres condition [22] on 2-variable Alexander polynomials, we can normalize $\Delta_L(t_1, t_2)$ as

$$\Delta_L(t_1, t_2) = (t_1 t_2)^{-\frac{e-2}{2}} \cdot \frac{(t_1 t_2)^e - 1}{t_1 t_2 - 1} \Delta_{K_1}(t_1) \Delta_{K_2}(t_2) + (t_1 - 1)(t_2 - 1) f(t_1, t_2) \quad (4.1)$$

where $\Delta_{K_i}(t_i)$ $(i = 1, 2)$ is normalized as $\Delta_{K_i}(t_i^{-1}) = \Delta_{K_i}(t_i)$ and $\Delta_{K_i}(1) = 1$, and $f(t_1, t_2) = f(t_1^{-1}, t_2^{-1})$. Then we have

$$\Delta_L(t_1, t_2) = t_1 t_2 \Delta_L(t_1^{-1}, t_2^{-1}) \quad (4.2)$$

By the assumption, the knot types of K_1 and K_2 are identical up to orientations and mirror images, and hence we have $\Delta_{K_1}(t) = \Delta_{K_2}(t)$ where we set $t = t_1 = t_2$.

By Lemma 4.1, we may assume

$$\Delta_L(t_1, t_2) = \eta t_1^{b_1} t_2^{b_2} \Delta_L(t_2^{-1}, t_1) \quad (4.3)$$

where $\eta = +$ or $-$, and $b_1, b_2 \in \mathbb{Z}$. By substituting $t_2 = 1$ to (4.3), we have

$$\Delta_{K_1}(t_1) = \eta t_1^{b_1} \Delta_{K_2}(t_1) = \eta t_1^{b_1} \Delta_{K_1}(t_1) \neq 0,$$

and $\eta = +$ and $b_1 = 0$ by (4.1). By substituting $t_1 = 1$ to (4.3), we have

$$\Delta_{K_2}(t_2) = \eta t_2^{b_2 - 1} \Delta_{K_1}(t_2^{-1}) = \eta t_2^{b_2 - 1} \Delta_{K_2}(t_2) \neq 0,$$

and $\eta = +$ and $b_2 = 1$ by (4.1). Hence we have

$$\Delta_L(t_1, t_2) = t_2 \Delta_L(t_2^{-1}, t_1) \tag{4.4}$$

By substituting $t_2 = -1$ to (4.4) and (4.1), we have

$$\Delta_L(t_1, -1) = -\Delta_L(-1, t_1)$$

and

$$-\Delta_{K_1}(-1)(-t_1)^{-\frac{e-2}{2}} \cdot \frac{t_1^e - 1}{t_1 + 1} \Delta_{K_1}(t_1) = (t_1 - 1)\{f(t_1, -1) + f(-1, t_1)\} \tag{4.5}$$

Substitute $t_1 = -1$ to (4.5). Then the lefthand side is divisible by 2, but is not divisible by 4, and the righthand side is divisible by 4. This is a contradiction. ∎

Lemma 4.4 ([8, Corollary 1.4]) *If $L = K_1 \cup K_2$ is an algebraically split component-preservingly amphicheiral link, then $\Delta_L(t_1, t_2)$ is divisible by $(t_1 - 1)^2 (t_2 - 1)^2$.*

We determine prime amphicheiral links with at least 2 components and up to 9 crossings as in Figure 1. For this class, most of them are detected not to be amphicheiral only by Lemma 4.1, Lemma 4.2, and Lemma 4.3. Firstly, we raise such examples.

Example 4.5 (1) Let L be a $(2p, 2q)$-torus link where p and q are positive integers with $\gcd(p, q) = 1$. Then L is a 2-component link, and its Alexander polynomial is

$$\Delta_L(t_1, t_2) \doteq \frac{(t_1 t_2)^{pq} - 1}{t_1 t_2 - 1} \cdot \frac{\{(t_1 t_1)^{pq} - 1\}(t_1 t_2 - 1)}{\{(t_1 t_1)^p - 1\}\{(t_1 t_2)^q - 1\}}$$

where the orientation of L is the torus braid orientation. It is easy to see that the linking number of L is pq, and the degree of $\Delta_L(t_1, t_2)$ about t_i $(i = 1, 2)$ is $(p-1)(q-1)+pq-1$. We can see by Lemma 4.1 that L is not amphicheiral except the case $p = q = 1$ (i.e. the Hopf link $(= 2_1^2)$ is amphicheiral as in Figure 1). Hence 4_1^2, 6_1^2 and 8_1^2 are not amphicheiral. Moreover by Lemma 4.2 (1), 8_1^3, 8_2^3, 8_7^3, 8_8^3, 8_9^3, 8_{10}^3, 9_5^3, 9_6^3, 9_8^3, 9_{19}^3 and 9_{20}^3 are not amphicheiral. The case of 8_{10}^3 is subtle. The 2-component sublinks of 8_{10}^3 are one 2-component trivial link and two $(2, 4)$-torus links. Since the 2-component sublinks of the mirror image of 8_{10}^3 are one 2-component trivial link and two $(2, -4)$-torus links, 8_{10}^3 is not amphicheiral. On the other hand, 8_4^3 is amphicheiral as in Figure 1. We set $L = 8_4^3 = K_1 \cup K_2 \cup K_3$, its 2-component sublinks $L_1 = K_1 \cup K_2$, $L_2 = K_1 \cup K_3$ and $L_3 = K_2 \cup K_3$, and their mirror images L^*, L_1^*, L_2^* and L_3^*, respectively where L_1 is the 2-component trivial link, L_2 is the $(2, 4)$-torus link and L_3 is the $(2, -4)$-torus link. Then there is an orientation-reversing homeomorphism φ of S^3 such that $\varphi(K_1) = K_2^*$, $\varphi(K_2) = K_1^*$, $\varphi(K_3) = K_3^*$, $\varphi(L_1) = L_1^*$, $\varphi(L_2) = L_3^*$ and $\varphi(L_3) = L_2^*$, respectively. Hence 8_4^3 is $(+)$-amphicheiral and $(-)$-amphicheiral by a suitable orientation, but it is not component-preservingly amphicheiral (cf. Figure 2).

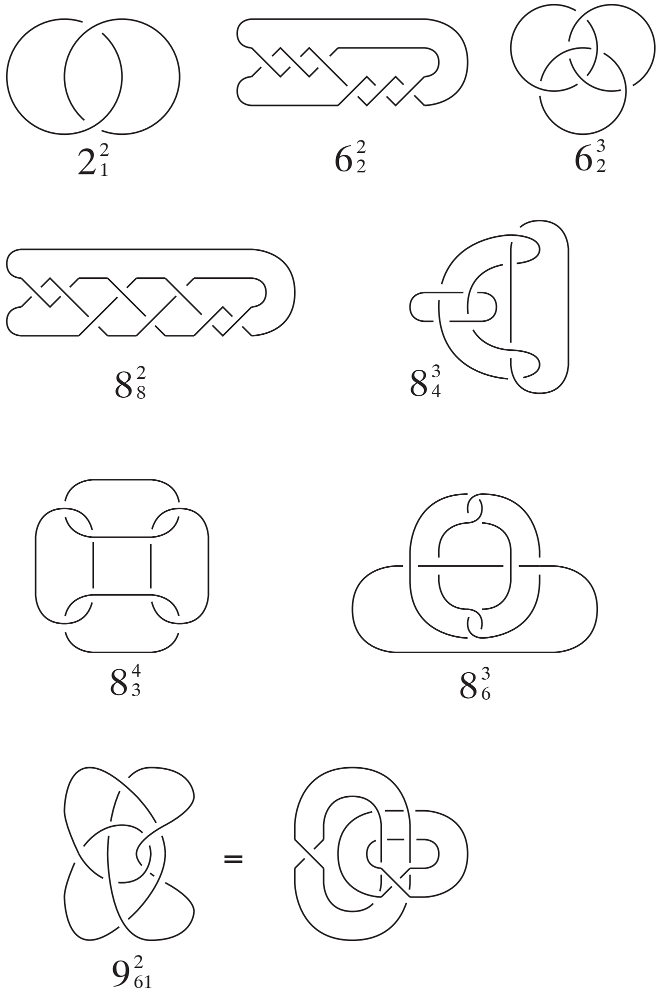


Figure 1: Prime amphicheiral links with up to 9 crossings

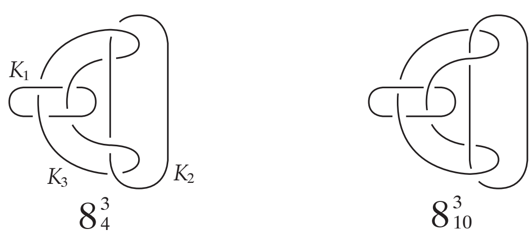


Figure 2: 8^3_4 and 8^3_{10}

(2) Let L be a 2-bridge link. It is well-known that a 2-bridge link is interchangeable, and $(-,-)$-invertible. Suppose $L = S(p,q)$ where $S(p,q)$ is Schubert's notation as in [13, Section 2]. Then L is amphicheiral if and only if $q^2 \equiv -1 \pmod{p}$. By [7], we can also detect amphicheirality of L from Conway's notation, and L cannot be (ε)-amphicheiral. By the fact, $6^2_2 = S(10,3) = C(3,3)$ and $8^2_8 = S(34,13) = C(2,1,1,1,1,2)$, where $C(\ldots)$ is Conway's notation as in [13, Section 2], are amphicheiral (cf. Figure 1), and 5^2_1, 6^2_3, 7^2_1, 7^2_2, 7^2_3, 8^2_2, 8^2_3, 8^2_4, 8^2_5, 8^2_6, 8^2_7, 9^2_1, 9^2_2, 9^2_3, 9^2_4, 9^2_5, 9^2_7, 9^2_8, 9^2_9, 9^2_{10} and 9^2_{11} are not amphicheiral. We note that a $(2,2p)$-torus link is a 2-bridge link $S(2p,1) = C(2p)$. We also see that it is not amphicheiral if $p \geq 2$. In the following table, 'A' means that the link is amphicheiral, and 'N' means that the link is not amphicheiral.

link	Conway	Schubert	Amphicheirality
2^2_1	$C(2)$	$S(2,1)$	A
4^2_1	$C(4)$	$S(4,1)$	N
5^2_1	$C(2,1,2)$	$S(8,3)$	N
6^2_1	$C(6)$	$S(6,1)$	N
6^2_2	$C(3,3)$	$S(10,3)$	A
6^2_3	$C(2,2,2)$	$S(12,5)$	N
7^2_1	$C(2,1,4)$	$S(14,5)$	N
7^2_2	$C(2,1,1,3)$	$S(18,7)$	N
7^2_3	$C(2,3,2)$	$S(16,7)$	N
8^2_1	$C(8)$	$S(8,1)$	N
8^2_2	$C(3,5)$	$S(16,5)$	N
8^2_3	$C(2,2,4)$	$S(22,9)$	N
8^2_4	$C(3,2,3)$	$S(24,7)$	N
8^2_5	$C(2,2,1,3)$	$S(26,11)$	N
8^2_6	$C(2,4,2)$	$S(20,9)$	N
8^2_7	$C(2,1,2,1,2)$	$S(30,11)$	N
8^2_8	$C(2,1,1,1,1,2)$	$S(34,13)$	A
9^2_1	$C(2,1,6)$	$S(20,7)$	N
9^2_2	$C(2,1,1,5)$	$S(28,11)$	N
9^2_3	$C(2,3,4)$	$S(30,13)$	N
9^2_4	$C(4,1,4)$	$S(24,5)$	N
9^2_5	$C(3,1,1,4)$	$S(32,9)$	N
9^2_7	$C(2,1,1,2,3)$	$S(44,17)$	N
9^2_8	$C(2,3,1,3)$	$S(34,14)$	N
9^2_9	$C(3,1,1,1,3)$	$S(40,11)$	N
9^2_{10}	$C(2,5,2)$	$S(24,11)$	N
9^2_{11}	$C(2,1,2,2,2)$	$S(46,17)$	N

(3) By Lemma 4.2 (1), 7^2_4, 7^2_5, 7^2_6, 7^2_7, 7^2_8, 8^2_{11}, 8^2_{12}, 8^2_{14}, 9^2_{13}, 9^2_{14}, 9^2_{15}, 9^2_{16}, 9^2_{17}, 9^2_{18}, 9^2_{19}, 9^2_{20}, 9^2_{21}, 9^2_{22}, 9^2_{27}, 9^2_{28}, 9^2_{29}, 9^2_{30}, 9^2_{31}, 9^2_{39}, 9^2_{40}, 9^2_{43}, 9^2_{44}, 9^2_{45}, 9^2_{46}, 9^2_{47}, 9^2_{48}, 9^2_{49}, 9^2_{50}, 9^2_{51}, 9^2_{52}, 9^2_{55}, 9^2_{56}, 9^2_{59}, 9^2_{60}, 9^3_1, 9^3_2, 9^3_3, 9^3_4, 9^3_9, 9^3_{11}, 9^3_{12}, 9^3_{13}, 9^3_{14}, 9^3_{15}, 9^3_{16}, 9^3_{17}, 9^3_{18} and 9^3_{21} are not

amphicheiral. We give a precise explanation particularly on the cases of 9^3_9 and 9^3_{21}. Let $L = K_1 \cup K_2 \cup K_3$ be 9^3_9 or 9^3_{21} such that its 2-component sublinks are $L_1 = K_1 \cup K_2$, $L_2 = K_1 \cup K_3$ and $L_3 = K_2 \cup K_3$, respectively where L_1 is the 2-component trivial link, and both L_2 and L_3 are the positive Whitehead link ($= 5^2_1$) (cf. Figure 3). Since 5^2_1 is not amphicheiral, L is not amphicheiral.

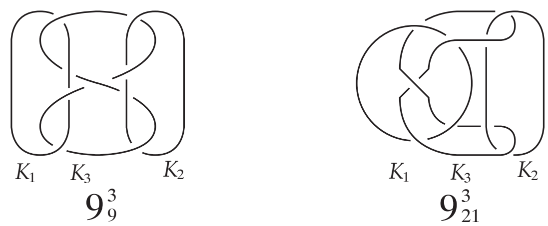


Figure 3: 9^3_9 and 9^3_{21}

(4) By Lemma 4.2 (2), 6^3_1, 6^3_3, 7^3_1, 8^3_3 and 9^3_7 are not amphicheiral.

(5) By Lemma 4.3, 8^2_9, 8^2_{16}, 9^2_6, 9^2_{23}, 9^2_{26}, 9^2_{38}, 9^2_{57} and 9^2_{58} are not amphicheiral. Since the linking number of 9^2_{53} is 4, it is not component-preservingly amphicheiral, but it may be $(\pm, \mp; (1\ 2))$-amphicheiral. The Alexander polynomial of 9^2_{53} is

$$\Delta_{9^2_{53}}(t_1, t_2) \doteq (t_1^2 t_2 + 1)(t_1 t_2^2 + 1),$$

and

$$\Delta_{9^2_{53}}(t_2, t_1^{-1}) \doteq (t_1^2 + t_2)(t_1 + t_2^2).$$

By Lemma 4.1, it is not amphicheiral (cf. Figure 4).

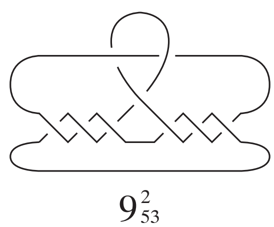


Figure 4: 9^2_{53}

(6) By Lemma 4.1, 9^2_{12}, 9^2_{24}, 9^2_{34}, 9^2_{35}, 9^2_{42} and 9^2_{54} are not amphicheiral. Since the Alexander polynomial of 9^2_{12} is

$$\Delta_{9^2_{12}}(t_1, t_2) \doteq (t_1 t_2 - t_1 - 2t_2 + 1)(t_1 t_2 - 2t_1 - t_2 + 1),$$

we have

$$\Delta_{9^2_{12}}(t_1, t_2^{-1}) \doteq \Delta_{9^2_{12}}(t_2, t_1^{-1}) \doteq (t_1 t_2 - t_1 - t_2 + 2)(2t_1 t_2 - t_1 - t_2 + 1).$$

Since the linking number of 9^2_{12} is 1 (non-zero), it is $(\pm,\mp)$-amphicheiral or $(\pm,\mp;(1\ 2))$-amphicheiral if it is amphicheiral. However by Lemma 4.1, it is not amphicheiral. The rest cases can be shown in the similar way.

The Borromean rings $(= 6^3_2)$ is well-known to be amphicheiral, and 8^4_3, 8^4_6 and 9^2_{61} are amphicheiral (cf. Figure 1). Then the rest links are 8^2_{10}, 8^2_{13}, 8^2_{15}, 8^3_5, 8^4_1, 8^4_2, 9^2_{25}, 9^2_{32}, 9^2_{33}, 9^2_{36}, 9^2_{37}, 9^2_{41}, 9^3_{10} and 9^4_1. We apply Lemma 4.4 and Corollary 1.2 to them.

Example 4.6 (1) By Lemma 4.1 and Lemma 4.4, 8^2_{10}, 8^2_{13}, 8^2_{15}, 9^2_{25}, 9^2_{32}, 9^2_{33}, 9^2_{36}, 9^2_{37} and 9^2_{41} are not amphicheiral.

(2) By Corollary 1.2, 8^3_5, 8^4_1, 8^4_2, 9^3_{10} and 9^4_1 are not amphicheiral. Since $\ell(8^3_5) = \pm 1$, 8^3_5 satisfies the condition in Corollary 1.2. Since the Alexander polynomial of 8^3_5 is

$$\Delta_{8^3_5}(t_1,t_2,t_3) \doteq (t_1-1)(t_2-1)(t_3-1)(t_2t_3+1),$$

we have

$$\Delta_{8^3_5}(-1,-1,-1) = \pm 16 \neq 0,$$

and hence 8^3_5 is not amphicheiral. Similarly we can see that 8^4_1, 8^4_2 and 8^4_3 satisfy the condition in Corollary 1.2. The Alexander polynomials of them are

$$\begin{aligned}
\Delta_{8^4_1}(t_1,t_2,t_3,t_4) &\doteq t_1t_2t_3+t_2t_3t_4+t_3t_4t_1+t_4t_1t_2 \\
&\quad -t_1t_2-t_3t_4-t_1t_4-t_2t_3-2t_1t_3-2t_2t_4 \\
&\quad +t_1+t_2+t_3+t_4, \\
\Delta_{8^4_2}(t_1,t_2,t_3,t_4) &\doteq t_1t_2t_3+t_4t_1t_2-t_1t_2-t_2t_3-t_3t_4-t_4t_1+t_3+t_4, \\
\Delta_{8^4_3}(t_1,t_2,t_3,t_4) &\doteq (t_1-t_3)(t_2-t_4),
\end{aligned}$$

respectively. Since

$$\begin{aligned}
\Delta_{8^4_1}(-1,-1,-1,-1) &= \pm 16 \neq 0, \\
\Delta_{8^4_2}(-1,-1,-1,-1) &= \pm 8 \neq 0, \\
\Delta_{8^4_3}(-1,-1,-1,-1) &= 0,
\end{aligned}$$

respectively, 8^4_1 and 8^4_2 are not amphicheiral (cf. Figure 5). The rest cases can be shown in the similar way.

Since the linking number of 9^2_{61} is 4, we have $9^2_{61} \in \mathcal{A}_9 \setminus \mathcal{C}_9$ by Lemma 4.3 (1). Since 8^3_4, 8^3_6 and 8^4_3 are component-preservingly $(-)$-invertible, they are $(+)$-amphicheiral and $(-)$-amphicheiral. The Borromean rings 6^3_2 is component-preservingly $(+)$-amphicheiral, and $(-,-,-;(1\ 2))$-amphicheiral by a suitable orientation. We show that 6^3_2 cannot be component-preservingly $(-)$-amphicheiral. Suppose that $6^3_2 = K_1 \cup K_2 \cup K_3$ is component-preservingly $(-)$-amphicheiral. Then there exists an orientation-reversing autohomeomorphism h of the exterior of 6^3_2 which preserves the orientation of the i-th meridian m_i $(h(m_i) = m_i)$, and reverses the orientation of the i-th longitude l_i

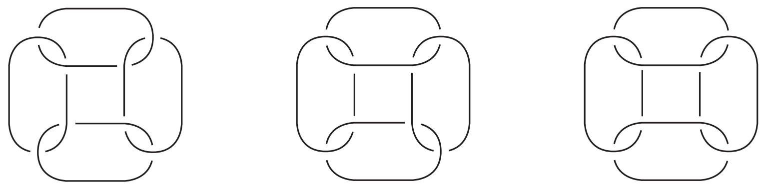


Figure 5: 8^4_1, 8^4_2 and 8^4_3

$(h(l_i) = -l_i)$ for every i $(i = 1, 2, 3)$, and h induces an orientation-reversing autohomeomorphism $\tilde{h}$ of the $(0, 0, 0)$-surgered manifold T^3 which is the 3-torus $S^1 \times S^1 \times S^1$ and we have $H_1(T^3) \cong H^1(T^3) = \mathrm{Hom}\,(H_1(T^3), \mathbb{Z}) \cong \mathbb{Z}^3$. We can take two bases of $H^1(T^3)$ induced by the dual of the meridians $\langle c_1, c_2, c_3\rangle$, and the dual of the longitudes $\langle d_1, d_2, d_3\rangle$. The cup products of $H^1(T^3)$ induces a trilinear map

$$\cup : H^1(T^3) \times H^1(T^3) \times H^1(T^3) \to H^3(T^3) \cong \mathbb{Z}.$$

We may suppose $\cup(d_1, d_2, d_3) = 1$. Let $(\tilde{h})^*$ be an autohomomorphism of $H^1(T^3)$ induced by $\tilde{h}$. Since $(\tilde{h})^*(c_i) = c_i$ for every i $(i = 1, 2, 3)$, the trilinear form $\cup$ is stable by $(\tilde{h})^*$. On the other hand, we have

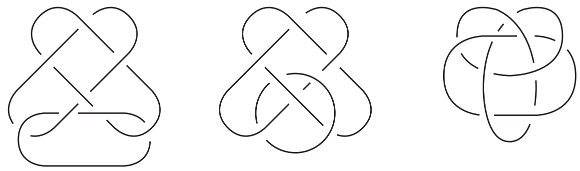
$$\cup((\tilde{h})^*(d_1), (\tilde{h})^*(d_2), (\tilde{h})^*(d_3)) = \cup(-d_1, -d_2, -d_3) = - \cup (d_1, d_2, d_3) = -1.$$

It implies that the trilinear form $\cup$ is not stable by $(\tilde{h})^*$. This is a contradiction. 6^3_2 is not component-preservingly $(-)$-amphicheiral. This argument also works for the case that a 3-component algebraically split link L has the non-zero trilinear cup product $\cup$ as above. The condition is equivalent to that a Milnor's $\overline{\mu}$-invariant $\overline{\mu}_L(123)$ is non-zero. That is, a 3-component algebraically split link L with $\overline{\mu}_L(123) \neq 0$ is not component-preservingly $(-)$-amphicheiral. Therefore we could determine the sets $\mathcal{A}_9$, $\mathcal{C}_9$, $\mathcal{A}^{\pm}_9$ and $\mathcal{C}^{\pm}_9$ completely.

Remark 4.7 (1) The condition "$e \equiv 2 \pmod 4$" in Lemma 4.3 (2) is necessary. 9^2_{61} is an amphicheiral with the linking number 4. Note that it is not component-preservingly amphicheiral by Lemma 4.3 (1). For any positive odd integer m, let L be a 2-component link obtained by taking $(2, 1)$-cable and $(2, -1)$-cable of each component of a 2-bridge link $C(m, m)$ which is amphicheiral with the linking number m. Then L is an amphicheiral link with the linking number $4m$. By repeating these processes (taking $(2, 1)$-cable and $(2, -1)$-cable of each component), for any positive integer n, we can construct a 2-component amphicheiral link with the linking number $4^n m$. R. Nikkuni and K. Taniyama [19] found a sequence of amphicheiral links with the linking number $4k$ for any integer k.

(2) For a link L, let $V_L(t)$ be the Jones polynomial of L. If L is amphicheiral, then $V_L(t^{-1})$ is equal to $V_L(t)$ up to multiplications of positive trivial units (cf. [9, Lemma 3.1

(4)], for example). It implies that an amphicheiral link has the Jones polynomial with symmetric coefficients. In general, this obstruction detects very well non-amphicheiral links. However it is not a complete one. For example, 8^3_2, 8^3_7 and 9^2_{49} in Figure 6 have the Jones polynomials with symmetric coefficients, and they are not amphicheiral by Example 4.5 (1) and (3). The reader can find more such examples for prime links with 10 or 11 crossings in [9].



8^3_2 8^3_7 9^2_{49}

Figure 6: 8^3_2, 8^3_7 and 9^2_{49}

(3) To determine the *link-symmetric group* (cf. [5, 7, 26]) for any link is an important problem. It has information of amphicheirality, invertibility and interchangeability. In the present paper, we deal with only a part of amphicheirality and interchangeability.

5 Further remarks

In [8], the first author raised a conjecture:

Conjecture 5.1 ([8, Conjecture 1.1]) *For an r-component algebraically split component-preservingly amphicheiral link L with r even, we have $\Delta_L(t_1, \ldots, t_r) = 0$.*

This is one of motivations of our present study. Our results in the present paper support it. In particular, Theorem 1.3 is a very strong partial affirmative answer for the conjecture. Some interesting examples are found in the table of prime links with crossing numbers 10 and 11 (see D. Bar-Natan and S. Morrison's website [1] and the first author's paper [9]). Though the examples support the conjecture, the condition "component-preservingly" is needed

One of the supporting results in [8] for the conjecture is Lemma 4.4, and another is the following:

Theorem 5.2 ([8, Theorem 1.5]) *If $L = K_1 \cup \ldots \cup K_r$ is an r-component component-preservingly (ε)-amphicheiral link with r even, then the Alexander polynomial of L satisfies*

$$\Delta_L(t^{\eta_1}, \ldots, t^{\eta_r}) = 0$$

where $\eta_i \in \{1, -1\}$ $(i = 1, \ldots, r)$.

It was shown by spanning Seifert surfaces following the argument due to R. Hartley [3]. We note that Corollary 1.2 and Theorem 1.3 include properly Theorem 5.2.

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