# Almost identical link imitations and the skein polynomial

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*Dedicated to Professor Kunio Murasugi on his 60th birthday*

The imitation theory gives a method constructing from a given good (3,1)-manifold pair $(M, L)$ a new good (3,1)-manifold pair $(M, L^*)$ with a map $q : (M, L^*) \to (M, L)$ close to a diffeomorphism in several senses (cf.[**K0**],[**K1**]). In [**K2**], for any given good (3,1)-manifold pair $(M, L)$, an infinite family of almost identical imitations $(M, L^*)$ of $(M, L)$ with exteriors $E(L^*, M)$ hyperbolic is constructed. In [**K3**] it is shown that we can take as this family an infinite family of (3,1)-manifold pairs $(M, L^*)$ which have the hyperbolic covering property. In this paper, this result is applied to some graph in the 3-sphere $S^3$ to construct from any link $L$ in $S^3$ an infinite family $\Im$ of almost identical imitations $(S^3, L^*)$ of $(S^3, L)$ with several properties, one of which is related to the skein (= two-variable Jones, HOMFLY, FLYPMOTH) polynomials (cf.[**F/Y/H/L/M/O**]) of the links $L^*, L$. A link $L^*$ will be simply called an *almost identical link imitation* of a link $L$ if $(S^3, L^*)$ is an almost identical imitation of $(S^3, L)$.

For an link $L$ and any positive number $C$, we shall show that this family $\Im$ of almost identical link imitations $L^*$ of $L$ can be taken so as to have all of the following properties:

(1) Each $L^* \in \Im$ has the hyperbolic covering property, and there is a number $C^+ > C$ such that the hyperbolic volume $\mathrm{Vol}E(L^*, S^3) < C^+$ and $\sup_{L^* \in \Im} \mathrm{Vol}E(L^*, S^3) = C^+$,

(2) Each $L^* \in \Im$ is obtained as a band sum of $L$ and a trivial knot $O$ (In other words, $L^*$ is a fusion of the split union $L + O$) and has the unlinking number $u(L^*) \leq \max\{u(L), 1\}$,

(3) The skein polynomial is constant on all $L^* \in \Im$, and the skein polynomial of each $L^* \in \Im$ is 'close' to the skein polynomial of $L$.

A certain generalization of (2) will be shown in Theorem 3.1 which is our main theorem. The precise meaning of the term 'close' in (3) will be clear in Theorem 3.1 using *coefficient polynomials* , essentially the coefficients of the skein polynomial, regarded as a polynomial in $m$ in the convention of Lickorish/Millett

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in [L/M]. Infinitely many knots with the same skein polynomial constructed by Kanenobu [K] have mutually non-isomorphic Alexander modules. By a property of the imitation in [K1], any imitation map $q : (S^3, L^*) \to (S^3, L)$ induces an isomorphism between the Alexander modules of $L^*$ and $L$. Thus, for each link $L$, we have infinitely many links with the same skein polynomial and the same Alexander module isomorphic to the Alexander module of $L$.

Throughout this paper, some terminologies of [K0],[K1],[K2],[K3] will be used without mentions. However, the following terms are reconfirmed here. Namely, a compact connected oriented 3-manifold $M$ is said to be *hyperbolic* if $\mathrm{int}_0 M = M - \partial_0 M$ ( when $\partial M = \partial_0 M$ ) or its double $D(\mathrm{int}_0 M)$ ( when $\partial M \neq \partial_0 M$ ) has a complete Riemannian structure of constant curvature -1, where $\partial_0 M$ denotes the union of all tori in the boundary $\partial M$ of $M$. Then the volume Vol($\mathrm{int}_0 M$) or Vol($D(\mathrm{int}_0 M)$)/2 (known to be finite) is a topological invariant of $M$, called the *hyperbolic volume* of $M$ and denoted by Vol$M$. By a *good* (3,1)-manifold pair $(M, L)$ ( or a *good* 1-manifold $L$ in $M$ ), we mean that $M$ is a compact connected oriented 3-manifold and $L$ is a compact proper smooth 1-submanifold of $M$ such that any 2-sphere component of $\partial M$ meets $L$ with at least three points. It is said to have the *hyperbolic covering property* if for every pair of component unions $L_0, L_1$ ( possibly, $\emptyset$ ) of $L$ with $L - L_0 = L_1$, any finite regular covering space of the exterior $E(L_0, M)$ branched along $L_1$ is hyperbolic after spherical completion.

## §1. Construction

Let $L$ be a link in $S^3$ and $b$ an oriented band spanning $L$ with orientation coherent with the orientation of $L$ (cf. Fig.1(1)).

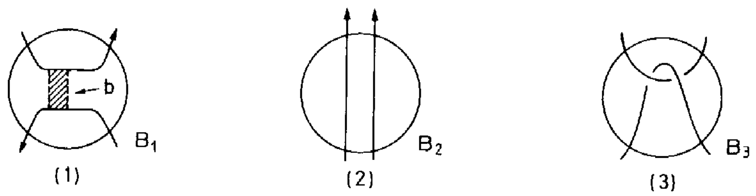


**Figure 1**

Let $L'$ be a link obtained from $L$ by surgery along $b$. For our application to the first half of the statement (2) in the introduction, we take $b$ so that $L'$ is a split union of $L$ and a trivial knot $O$. We choose mutually disjoint 3-balls $B_i, i = 1, 2, 3$, in $S^3$ so that $L \cap B_i$ is a 2-string tangle in $B_i$, as it is illustrated in Fig.1$(i)$, $i$=1,2,3, where $L \cap B_2$ should belong to one component of $L$ and we do not specify the string orientation of $L \cap B_3$. Let $L''$ be a link obtained from $L$ by a crossing change at a crossing point in Fig.1(3). We call this crossing point a *clasp crossing point*. For our application to the latter half of the statement (2) in the introduction, we take $L''$ so that the unlinking number $u(L'') = \max\{u(L) - 1, 0\}$. Next, we replace the 2-string tangle $L \cap B_i$ by the graph $G_i$ in Fig.2($i$), $i$= 1,2,3.

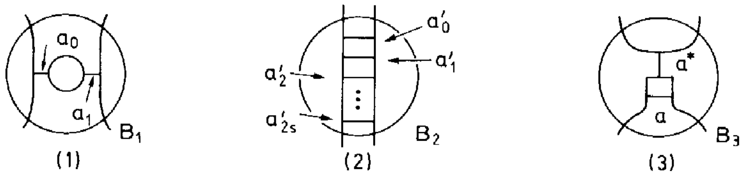


**Figure 2**

By $\Gamma$ we denote a spatial graph occurring from $L$ by this replacement. Let $(S^3_o, \Gamma_o)$ be a good (3,1)-manifold pair obtained from $(S^3, \Gamma)$ by removing a small open ball neighborhood of each vertex of $\Gamma$ of degree 3. Given an almost identical imitation

$$q_o : (S^3_o, \Gamma^*_o) \to (S^3_o, \Gamma_o),$$

we have an almost identical graph imitation

$$q : (S^3, \Gamma^*) \to (S^3, \Gamma)$$

by taking a spherical completion (cf.[**K2**]). We consider an H-graph $H_i$ in a 3-ball neighborhood $V_i$ around the arc $a_i, i = 0, 1$ and an H-graph $H'_j$ in a 3-ball neighborhood $V'_j$ around the arc $a'_j, j = 0, 1, ..., 2s$, and H-graphs $H, H^*$ in 3-ball neighborhoods $V, V^*$ around the arcs $a, a^*$ which are illustrated in Fig.3(1), (2) and (3), respectively.

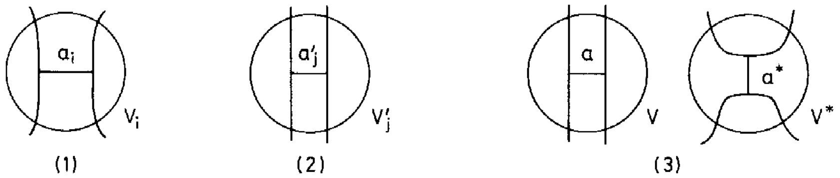


**Figure 3**

We replace these H-graphs by certain 2-tangles with $m_i, m'_j, m$ and $m^*$ full twists, respectively, as they are illustrated in Fig.4(1),(2),(3) (In the figure, the case of positive twists is illustrated and a negative twist is the mirror image of a positive twist).

We impose the following condition on $m_i, m'_j, m$ and $m^*$:

(#) $$m_0 + m_1 = m'_0 + m'_1 + ... + m'_{2s} = m + m^* = 0.$$

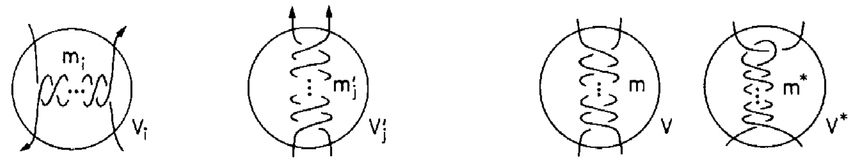


**Figure 4**

Then $\Gamma$ changes into a link with the same oriented link type as $L$. Identifying this link with $L$, we see that any almost identical graph imitation $q : (S^3, \Gamma^*) \to (S^3, \Gamma)$ induces an almost identical imitation $q^* : (S^3, L^*) \to (S^3, L)$. Let $b^*$ be a band spanning $L^*$ obtained by thickening the arc $a_0$ in Fig.3(1) so that $q^*$ maps $b^*$ diffeomorphically onto $b$. The following lemma is obvious from our construction:

**Lemma 1.1.** *The map $(S^3, L^{*\prime}) \to (S^3, L')$ induced from $q^*$ by the surgeries along $b^*, b$ is homotopic to a diffeomorphism, and the map $(S^3, L^{*\prime\prime}) \to (S^3, L'')$ induced from $q^*$ by the crossing changes at the clasp crossing points corresponding by $q^*$ is homotopic to a diffeomorphism.*

Let $M$ be a 3-manifold obtained from $S^3$ by removing $\mathrm{int}V_i$, $\mathrm{int}V'_j$ for all $i, j$, $\mathrm{int}V$ and $\mathrm{int}V^*$. Let $U, U'$ be 3-balls obtained from $S^3$ by splitting along a 2-sphere such that

(1) $L \cap U$ is a trivial $r(\geq 3)$-string tangle in $U$ , and
(2) $V_i, V'_j, V$ and $V^*$ for all $i, j$ are contained in $\mathrm{int}U$.

**Lemma 1.2.** *For any positive number $C$, there is an almost identical graph imitation $q : (S^3, \Gamma^*) \to (S^3, \Gamma)$ extending an almost identical graph imitation $q_U : (U, (\Gamma \cap U)^*) \to (U, \Gamma \cap U)$ and an almost identical imitation $q_{U'} : (U', (L \cap U')^*) \to (U', L \cap U')$ such that $(U', (L \cap U')^*)$ has the hyperbolic covering property and the double covering spaces $M_2, (U \cap M)_2$ of $M, U \cap M$ branched along $\Gamma^* \cap M, \Gamma^* \cap U \cap M$, respectively, are hyperbolic with* $\mathrm{Vol}M_2 > 2C$.

*Proof.* Let $\Gamma^+$ be a split union in $S^3$ of an $n$-component trivial link $O$ in $U$ and $\Gamma$. By **[K3]**, we have an almost identical graph imitation $q_U^+ : (U, (\Gamma^+ \cap U)^*) \to (U, \Gamma^+ \cap U)$ such that the good (3,1)-manifold pair $(U, (\Gamma^+ \cap U)^*) \cap S_0^3$ has the hyperbolic covering property. By **[K3]** and **[K2,** Corollary 3.3], we have an almost identical imitation $q_{U'} : (U', (L \cap U')^*) \to (U', L \cap U')$ such that $(U', (L \cap U')^*)$ has the hyperbolic covering property and the extension $(S^3, (L \cap U')^* \cup (L \cap U)) \to (S^3, L)$ of $q_{U'}$ by the identity on $(U, L \cap U)$ is homotopic to a diffeomorphism. Using $q_U^+$ and $q_{U'}$, we have an almost identical graph imitation $q^+ : (S^3, \Gamma^{+*}) \to (S^3, \Gamma^+)$. Let $O^*$ be the preimage of $O$ by $q^+$ and $E_{U \cap M} = E(O^*, M \cap U)$ and $E_M = E(O^*, M)$. The double covering space $(E_{U \cap M})_2$ of $E_{U \cap M}$ branched along

$\Gamma^{++}\cap U\cap M-O^*$ and lifting the tori around $O^*$ in $\partial E_{U\cap M}$ trivially is hyperbolic by the hyperbolic covering property of $(U,\Gamma^{++}\cap U)\cap S_0^3$. By Myers gluing lemma (cf.[**K2,** Lemma 5.3]), the double covering space $(E_M)_2$ of $E_M$ branched along $\Gamma^{++}\cap M-O^*$ which extends the covering $(E_{U\cap M})_2\to E_{U\cap M}$ is also hyperbolic, for $(U',\Gamma^{++}\cap U')=(U',(L\cap U')^*)$ has the hyperbolic covering property. By J$\phi$rgensen's theorem [**T1**],[**T2**], we have $\mathrm{Vol}(E_M)_2>2C$ by taking $n$ so large. By Thurston's hyperbolic Dehn surgery [**T1**],[**T2**], if $q:(S^3,\Gamma^*)\to(S^3,\Gamma)$ is obtained from $q^+$ by Dehn surgery along the components of $O^*$ and $O$ with the same coefficient $1/m$ for a large positive integer $m$, then $q$ is an almost identical graph imitation and we have that $M_2$ and $(U\cap M)_2$ are hyperbolic with $\mathrm{Vol}M_2>2C$. This completes the proof.

From now on, we consider an almost identical imitation

$$q^*:(S^3,L^*)\to(S^3,L)$$

induced from an almost identical graph imitation

$$q:(S^3,\Gamma^*)\to(S^3,\Gamma)$$

stated in Lemma 1.2.

**Lemma 1.3.** *For any positive number $C$, there is a positive constant $c$ such that for all $m_i,m'_j,m$ and $m^*$ with (#) and $|m_i|,|m'_j|,|m|,|m^*|>c$, $(S^3,L^*)$ has the hyperbolic covering property and*

$$C<\mathrm{Vol}E(L^*,S^3)<\sup_{\{m_0,m_1\}}\mathrm{Vol}E(L^*,S^3)<+\infty$$

*if we fix $m'_j,m$ and $m^*$ for all $j$.*

*Proof.* The branched covering spaces $S^3(L^*)_2,U(L^*\cap U)_2$ of $S^3,U$ branched along $L^*,L^*\cap U$ are obtained from $M_2,(U\cap M)_2$ by attaching solid tori whose attaching meridians vary by the values $m_i,m'_j,m$ and $m^*$. Hence by Thurston's hyperbolic Dehn surgery, there is a number $c>0$ such that for all $m_i,m'_j,m,m^*$ with $|m_i|,|m'_j|,|m|,|m^*|>c,S^3(L^*)_2$ and $U(L^*\cap U)_2$ are hyperbolic with $\mathrm{Vol}S^3(L^*)_2>2C$. Since the surface $\partial U(L^*\cap U)_2$ is incompressible in $S^3(L^*)_2$ which is hyperbolic, we see from [**K3,** Lemma 1.7] that $(S^3,L^*)$ has the hyperbolic covering property. Let $L_2^*$ be the lift of $L^*$ to $S^3(L^*)_2$. Then

$$\mathrm{Vol}E(L^*,S^3)=\mathrm{Vol}E(L_2^*,S^3(L^*)_2)/2>\mathrm{Vol}S^3(L^*)_2/2>C.$$

Let $L_0^*$ be the link $L^*$ with $m_i=m'_j=m=m^*=0$ for all $i,j$. Let $L^\#$ be the link obtained from $L_0^*$ by adding the components $k_i,k'_j,k,k^*$ for all $i,j$ indicated in Fig. 5.

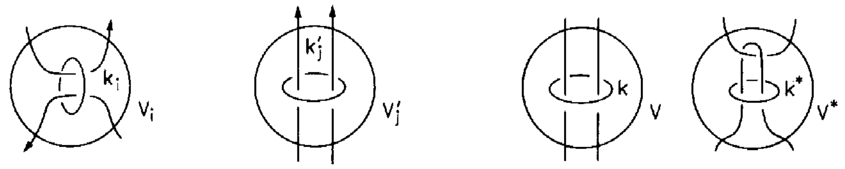


**Figure 5**

Note that $L^*$ is obtained from $L^\#$ by the $1/m_i, 1/m'_j, 1/m$, and $1/m^*$-surgeries along the components $k_i, k'_j, k, k^*$, respectively, for all $i, j$. Using that $S^3(L^*)_2$ is hyperbolic, we see that $L^\#$ is a prime link. Since $E(L^*, S^3)$ is hyperbolic for all $m_i, m'_j, m$ and $m^*$ with $|m_i|, |m'_j|, |m|$ and $|m^*|$ greater than $c$, we can conclude from the torus decomposition of $E(L^\#, S^3)$ that $L^\#$ is a hyperbolic link. Hence by Thurston's hyperbolic Dehn surgery,

$$\mathrm{Vol}E(L^*, S^3) < \sup_{\{m_0, m_1\}} \mathrm{Vol}E(L^*, S^3) < +\infty$$

if we take and fix $m'_j$ and $m, m^*$ for all $j$ with $|m'_j|, |m|, |m^*| > c$. This complete the proof.

## §2. The coefficient polynomials

The skein polynomial $P_L(\ell, m)$ of a link $L$ is calculable in principle by the *initial condition*

(1) $$P_O(\ell, m) = 1,$$

where $O$ is a trivial knot, and the *skein relation*

(2) $$\ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) + m P_{L_0}(\ell, m) = 0,$$

where $L_+, L_-$ and $L_0$ are links, identical except the part in a 3-ball $B$ in which the 2-strand braids with positive half twist, negative half twist and 0 twist are occurring, respectively. We call an operation producing from one of the links $L_+, L_-, L_0$ the other two links a *skein move within* $B$. For the component number $r$ of $L$, we let

$$P_\#(L; \ell, m) = (\ell m)^{r-1} P_L(\ell, m).$$

Then the initial condition and the skein relation are written as

($1_\#$) $$P_\#(O; \ell, m) = 1,$$

($2_\#$) $$-\ell^2 P_\#(L_+; \ell, m) - P_\#(L_-; \ell, m) = (\ell^2 m^2)^\delta P_\#(L_0; \ell, m),$$

where we let $\delta = (r_+ - r_0 + 1)/2$ (=1 or 0) for the component numbers $r_+, r_0$ of $L_+, L_0$, respectively.

Then we see easily that $P_\#(L;\ell,m)$ is written as

$$\Sigma_{n=0}^{+\infty} p_{2n}(L;\ell)m^{2n},$$

where $p_{2n}(L;\ell)$ is a Laurent polynomial in $\ell^2$, being 0 except a finite number of $n$. We denote $-\ell^2$ and $-m^2$ by $x$ and $y$, respectively, and then $P_\#(L;\ell,m)$ and $p_{2n}(L;\ell)$ by $C_\#(L;x,y)$ and $c_n(L;x)(-x)^n$, respectively. Clearly, $c_n(L;x)$ is a Laurent polynomial in $x$ and we have

$$C_\#(L;x,y) = \Sigma_{n=0}^{+\infty} c_n(L;x)(xy)^n.$$

Taking $c_n(L;x) = 0$ for $n < 0$, we obtain the following, giving an alternative description of the initial condition and the skein relation of the skein polynomial:

**Theorem 2.1.** *There is one and only one link invariant family of Laurent polynomials in $x$ of a link $L$ which is denoted by $c_n(L;x), n \in Z$, and has the following identities:*

(1) *For a trivial knot $O$,*

$$c_n(O;x) = \begin{cases} 0 & (n \neq 0) \\ 1 & (n = 0), \end{cases}$$

(2) $xc_n(L_+;x) - c_n(L_-;x) = c_{n-\delta}(L_0;x)$ *for all $n$, with $\delta = (r_+ - r_0 + 1)/2$ ( = 0 or 1 ) for the component numbers $r_+, r_0$ of $L_+, L_0$, respectively.*

We call the Laurent polynomial $c_n(L;x)$ the $n$th *coefficient polynomial* of the skein polynomial $P_L(\ell,m)$ (or simply, of the link $L$). Some calculations of the coefficient polynomials has been made in **[K4]**. For example, for any link $L$ with the components $k_i, i = 1,2,...,r$, and the total linking number $\lambda$, we have

$$c_0(L;x) = (x-1)^{r-1}x^{-\lambda}c_0(k_1;x)c_0(k_2;x)...c_0(k_r;x),$$
$$c_0(k_i;1) = 1, \frac{d}{dx}c_0(k_i;1) = 0, i = 1,2,...,r,$$

and this characterizes the zeroth coefficient polynomial. Since $c_0(L;x)$ determines the component number $r$ of $L$, we see that for any link $L$, the family of the coefficient polynomials $c_n(L;x)$ for all $n$ determines not only $P_\#(L;\ell,m)$ but also the skein polynomial $P_L(\ell,m)$· itself. From now we shall calculate the coefficient polynomial $c_n(L^*;x)$ of an almost identical link imitation $L^*$ of a link $L$ constructed in §1 for all $m_i(i = 0,1), m'_j(0 \leq j \leq 2s), m, m^*$ full twists with condition (#). We show the following three lemmas:

**Lemma 2.2.** *$c_n(L^*;x)$ is constant on all $m_0, m_1$ with $m_0 + m_1 = 0$.*

**Lemma 2.3.** *$c_n(L^*;x) = c_n(L;x)$ for all $n$ with $n \le s$.*

**Lemma 2.4.** *$c_n(L^*;x) - c_n(L;x)$ is divided by $x^m - 1$ for all $n$.*

*Proof of Lemma 2.2.* By fixing the $m'_j$'s and $m, m^*$, we write $L^*$ as $L^*_{(m_0,m_1)}$. Let $m_0 > 0, m_1 < 0$ without loss of generality. By the skein relation corresponding to a skein move in $V_i, i = 0, 1$, we have

$$c_n(L^*_{(m_0,m_1)};x) = x^{-1}c_n(L^*_{(m_0-1,m_1)};x) + x^{-1}c_{n-\delta}(L';x)$$

and

$$c_n(L^*_{(m_0-1,m_1)};x) = xc_n(L^*_{(m_0-1,m_1+1)};x) - c_{n-\delta}(L';x),$$

for some $\delta$ =0 or 1. Hence

$$c_n(L^*_{(m_0,m_1)};x) = c_n(L^*_{(m_0-1,m_1+1)};x) = ... = c_n(L^*_{(0,0)};x),$$

for $m_0 + m_1 = 0$. This completes the proof of Lemma 2.2.

Next, when we fix $m_i, i = 0, 1$, and $m, m^*$, we denote by $L^*_{(u_0,u_1,...,u_{2s})}$ a link obtained from $L^*$ by replacing $m'_j$ full twists in Fig.4(2) with $u_j$ *half* twists. Thus, $L^*_{(2m'_0,2m'_1,...,2m'_{2s})} = L^*$. Take a similar presentation of $L$ using $m_i, m'_j, m, m^*$. Then we also use the notation $L_{(u_0,u_1,...,u_{2s})}$ if we replace $m'_j$ full twists with $u_j$ half twists and fix the other $m_i, i = 0, 1$, and $m, m^*$, so that $L_{(2m'_0,2m'_1,...,2m'_{2s})} = L$.

If a Laurent polynomial $f(x)$ in $x$ is written as a sum

$$\Sigma_{i=1}^n \epsilon_i x^{r_i} f_i(x)$$

with some $\epsilon_i = \pm 1$, integers $r_i$ and Laurent polynomials $f_i(x)$ in $x$, we say that $f(x)$ is a *unit multiple sum of* $f_i(x), i = 1, 2, ..., n$.

*Proof of Lemma 2.3.* Let

$$c_{n(u_0,u_1,...,u_{2s})} = c_n(L_{(u_0,u_1,...,u_{2s})};x)$$

and

$$c^*_{n(u_0,u_1,...,u_{2s})} = c_n(L^*_{(u_0,u_1,...,u_{2s})};x).$$

Note by our assumption in §1 that $L \cap B_2$ in Fig.1(2) belongs to one component of $L$. Then there is a skein move in the 3-ball $V'_1$ changing $c^*_{n(2m'_0,2m'_1,...,2m'_{2s})}$ into a unit multiple sum of $c^*_{n(2m'_0,u_1,2m'_2,...,2m'_{2s})}$ and $c^*_{n(2m'_0,u'_1,2m'_2,...,2m'_{2s})}$ with $|u_1| = |2m'_1| - 1$ and $|u'_1| = |2m'_1| - 2$. There is also a skein move in $V'_2$ changing $c^*_{n(2m'_0,u_1,2m'_2,...,2m'_{2s})}$ into a unit multiple sum of $c^*_{n-1(2m'_0,u_1,u_2,...,2m'_{2s})}$ and $c^*_{n(2m'_0,u_1,u'_2,...,2m'_{2s})}$ with $|u_2| = |2m'_2| - 1$ and $|u'_2| = |2m'_2| - 2$. By induction on $|2m'_2|$, there are skein moves in $V'_2$ changing $c^*_{n(2m'_0,u_1,2m'_2,...,2m'_{2s})}$

into a unit multiple sum of $c^*_{n(2m'_0,u_1,0,\ldots,2m'_{2s})}$ and $c^*_{n-1(2m'_0,u_1,u_2^{(k)},\ldots,2m'_{2s})}$ for some odd integers $u_2^{(k)}, k = 1, 2, \ldots, |m'_2|$. Hence there are skein moves in $V'_1$ and $V'_2$ changing $c^*_{n(2m'_0,2m'_1,\ldots,2m'_{2s})}$ into a unit multiple sum of $c^*_{n(2m'_0,u'_1,2m'_2,\ldots,2m'_{2s})}$ and $c^*_{n(2m'_0,u_1,0,\ldots 2m'_{2s})}$ and the $c_{n-1(2m'_0,u_1,u_2^{(k)},\ldots,2m'_{2s})}$'s. Since $|u'_1| = |2m'_1| - 2$, we have by induction on $|2m'_1|$ that there are skein moves in $V'_1, V'_2$ changing $c^*_{n(2m'_0,2m'_1,\ldots,2m'_{2s})}$ into a unit multiple sum of a finite number of Laurent polynomials of the following types:

$$c^*_{n(2m'_0,v_1,0,\ldots,2m'_{2s})},$$

$$c^*_{n(2m'_0,0,v_2,\ldots,2m'_{2s})}$$

and

$$c^*_{n-1(2m'_0,u_1,u_2,\ldots,2m'_{2s})}$$

with $u_1, u_2$ odd.

Applying this process to $2m'_{2j-1}, 2m'_{2j}, j = 2, 3, \ldots, s$, we see that there are skein moves in $V'_j, j = 1, 2, \ldots, 2s$, changing $c^*_{n(2m'_0,2m'_1,\ldots,2m'_{2s})}$ into a unit multiple sum of a finite number of Laurent polynomials of the following types:

$$c^*_{n^*(2m'_0,v_1,\ldots,v_{2s})}$$

where $n - s < n^* \leq n$ and some $v_j = 0$, and

$$c^*_{n-s(2m'_0,u_1,\ldots,u_{2s})}$$

with $u_j$ odd. By a property of the almost identical graph imitation, we see that

$$L^*_{(2m'_0,v_1,\ldots,v_{2s})} = L_{(2m'_0,v_1,\ldots,v_{2s})}.$$

Hence

$$c^*_{n^*(2m'_0,v_1,\ldots,v_{2s})} = c_{n^*(2m'_0,v_1,\ldots,v_{2s})}.$$

Let $n \leq s$. Then if

$$c^*_{0(2m'_0,u_1,\ldots,u_{2s})} = c_{0(2m'_0,u_1,\ldots,u_{2s})}$$

is proved, we can see from the skein relations corresponding to the skein moves in the $V'_j$'s that

$$c_n(L^*;x) = c^*_{n(2m'_0,2m'_1,\ldots,2m'_{2s})} = c_{n(2m'_0,2m'_1,\ldots,2m'_{2s})} = c_n(L;x).$$

To show that

$$c^*_{0(2m'_0,u_1,\ldots,u_{2s})} = c_{0(2m'_0,u_1,\ldots,u_{2s})},$$

we first note that for any odd $u_0, L_{(u_0,u_1,\ldots,u_{2s})}$ is a link with two or more (in fact, $r + 1$) components and $L^*_{(u_0,u_1,\ldots,u_{2s})}$ is an almost identical link imitation of

$L_{(u_0,u_1,\dots,u_{2s})}$. By the characterization of the zeroth coefficient polynomial([**K4**]), we have

$$c^*_{0(u_0,u_1,\dots,u_{2s})} = c_{0(u_0,u_1,\dots,u_{2s})}.$$

There are skein moves in $V'_0$ changing $c^*_{0(2m'_0,u_1,\dots,u_{2s})}$ into a unit multiple sum of $c^*_{0(0,u_1,\dots,u_{2s})}$ and a finite number of Laurent polynomials of the type $c^*_{0(u_0,u_1,\dots,u_{2s})}$ with $u_0$ odd. Since

$$L^*_{(0,u_1,\dots,u_{2s})} = L_{(0,u_1,\dots,u_{2s})},$$

we see from the skein relations corresponding to the skein moves in $V'_0$ that

$$c^*_{0(2m'_0,u_1,\dots,u_{2s})} = c_{0(2m'_0,u_1,\dots,u_{2s})}.$$

This completes the proof of Lemma 2.3.

*Proof of Lemma 2.4.* Let $L^* = L^*_{(m,m^*)}$ by fixing the other $m_i$ and $m'_j$ for all $i, j$. Let $m > 0$ and $m^* < 0$. Since $m + m^* = 0$, by skein relations corresponding to skein moves in $V^*$, we have

$$c_n(L^*_{(m,m^*)}; x) = x^m c_n(L^*_{(m,0)}; x) - (1 + x + \dots + x^{m-1}) c_n(L_H; x),$$

where $L_H$ denotes a connected sum of the link $L''$ appearing in our construction of §1 and the Hopf link. By skein relations corresponding to skein moves in $V$, we have

$$c_n(L^*_{(m,0)}; x) = x^{-m} c_n(L^*_{(0,0)}; x) + x^{-m}(1 + x + \dots + x^{m-1}) c_n(L^*_H; x),$$

where $L^*_H$ is a link which is an almost identical link imitation of $L_H$. Noting $L^*_{0,0} = L$, we have

$$c_n(L^*_{(m,m^*)}; x) = c_n(L; x) + \frac{x^m - 1}{x - 1}\{x^{-m} c_n(L^*_H; x) - c_n(L_H; x)\}.$$

Note that $c_n(L^*_H; 1)$ and $c_n(L_H; 1)$ are the $z^{2n-r+1}$-coefficients of the Conway polynomials of $L^*_H$ and $L_H$, respectively. Since the Conway polynomial is known to be invariant under any link imitation(cf.[**K4**]), we have $c_n(L^*_H; 1) = c_n(L_H; 1)$ and $x^{-m} c_n(L^*_H; x) - c_n(L_H; x)$ is divided by $x - 1$, showing that $c_n(L^*; x) - c_n(L; x)$ is divided by $x^m - 1$. The same assertion for the case that $m < 0$ and $m^* > 0$ is proved similarly. This completes the proof of Lemma 2.4.

## §3. Main theorem

**Theorem 3.1.** *For any link $L$ in $S^3$, we consider links $L', L''$ such that $L'$ is obtained from $L$ by a surgery along a band $b$ with coherent orientation and $L''$ is* *obtained from $L$ by a crossing change within a 3-ball $B$ where $L \cap B$ is a trivial 2-string tangle. Then for any positive integers $N, N'$ and positive number $C$, there is an infinite family $\Im$ of almost identical link imitations $L^*$ of $L$ with all of the following properties:*

(1) *Each $L^* \in \Im$ has the hyperbolic covering property and there is a number $C^+ > C$ such that the hyperbolic volume $\mathrm{Vol}E(L^*, S^3) < C^+$ and $\sup_{L^* \in \Im} \mathrm{Vol}E(L^*, S^3) = C^+$,*

(2) *For the imitation map $q : (S^3, L^*) \to (S^3, L)$, there are a band $b^*$ spanning $L^*$ with diffeomorphism $q|b^* : b^* \to b$ and a 3-ball $B^*$ in $S^3$ with a diffeomorphism $q|B^* : B^* \to B$ such that the maps*

$$q' : (S^3, L^{*\prime}) \to (S^3, L'),$$

*and*

$$q'' : (S^3, L^{*\prime\prime}) \to (S^3, L'')$$

*induced from $q$ by surgeries along $b^*, b$ and the crossing changes within $B^*, B$ corresponding by $q$, respectively, are homotopic to diffeomorphisms,*

(3) *For any $n$, the $n$th coefficient polynomial $c_n(L^*; x)$ is constant on all $L^* \in \Im$ and $c_n(L^*; x) = c_n(L; x)$ for all $n < N$ and the difference $c_n(L^*; x) - c_n(L; x)$ is divided by $x^{N'} - 1$ for all $n \geq N$.*

*Proof.* In Lemma 1.3, we take $\Im$ to be the family of $L^*$ for all $m_0, m_1$ with $m_0 + m_1 = 0$ and $|m_i| > c$, by choosing and fixing $m'_j, m$ and $m^*$ for all $j, j = 0, 1, ..., 2s$, so that they satisfy (#) and have $s = N - 1, |m'_j| > c$ and $|m| > \max\{c, N' - 1\}$. Let $C^+ = \sup_{L^* \in \Im} \mathrm{Vol}E(L^*, S^3)$. Then for any $L^* \in \Im$, $L^*$ is an almost identical link imitation of $L$ with hyperbolic covering property and $C < \mathrm{Vol}E(L^*, S^3) < C^+$, showing (1). We obtain (2) from Lemma 1.1. We obtain (3) from Lemmas 2.2,2.3,2.4. This completes the proof.

**Remark 3.2.** For the Jones polynomial $V_L(t)$(cf.[**J**]) of a link $L$, we let

$$V_{\#}(L; t) = (\sqrt{t} - t\sqrt{t})^{r-1} V_L(t^{-1}),$$

where $r$ is the component number of $L$. Then we have

$$V_{\#}(L; t) = \Sigma_{n=0}^{+\infty} c_n(L; t^2) t^n (t-1)^{2n}$$

(cf.[**K4**]). Thus, the statement (3) of Theorem 3.1 implies that $V_{\#}(L^*; t)$ and hence $V_{L^*}(t)$ are constant on all $L^* \in \Im$ and $V_{\#}(L^*; t) - V_{\#}(L; t)$ is divided by $(t - 1)^{2N}(t^{2N'} - 1)$.

The following corollary is obtained directly from Theorem 3.1 by taking $L, L'$ and $L''$ to be a trivial knot, a two-component trivial link and a trivial knot, respectively.

**Corollary 3.3.** *For any positive integers $N, N'$ and any positive number $C$, there is an infinite family $\Im$ of knots $O^*$ with trivial Alexander polynomial and with the same skein polynomial such that*

(1) *Each $O^* \in \Im$ has the hyperbolic covering property and there is a number $C^+ > C$ such that the hyperbolic volume $\mathrm{Vol}E(O^*, S^3) < C^+$ and $\sup_{O^* \in \Im} \mathrm{Vol}E(O^*, S^3) = C^+$,*

(2) *Each $O^* \in \Im$ is a ribbon knot of 1-fusion with unknotting number $u(O^*) = 1$,*

(3)
$$c_n(O^*; x) = \begin{cases} 0 & (0 < n < N) \\ 1 & (n = 0), \end{cases}$$
*and $c_n(O^*; x)$ is divided by $x^{N'} - 1$ for all $n \geq N$.*

Finally, we note that we can obtain similar results, taking an infinite family of hyperbolic links ( but with essential Conway spheres ) as $\Im$, when we use the results of **[K2]** alone instead of **[K3]**.

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