Splitting a 4-manifold with infinite cyclic fundamental group, revised

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ABSTRACT

This article is a revised version of the author's earlier paper on a TOP-splitting of a closed connected oriented 4-manifold with infinite cyclic fundamental group. We show that a closed connected oriented 4-manifold with infinite cyclic fundamental group is TOP-split if it is virtually TOP-split. As a consequence, we see that a closed connected oriented 4-manifold with infinite cyclic fundamental group is TOP-split if the intersection form is indefinite. This also implies that every closed connected oriented smooth spin 4-manifold with infinite cyclic fundamental group is TOP-split.

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1. Introduction

A closed connected oriented 4-manifold M is called a Z^{π_1}*-manifold* if the fundamental group $\pi_1(M)$ is isomorphic to Z, and a Z^{H_1}*-manifold* if the first homology group $H_1(M;Z)$ is isomorphic to Z. A Z^{π_1}-manifold M is *TOP-split* if M is homeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for a simply connected closed 4-manifold M_1, and *virtually* TOP-split if a finite covering of M is TOP-split. Here, we do not assume that a closed 4-manifold is a smooth or piecewise-linear manifold, but we can use smooth and piecewise-linear techniques for it because a punctured manifold of it is smoothable (see Freedman-Quinn [2]). The purpose of this paper is to make a revised version of the author's earlier paper [7] on TOP-splitting of a Z^{π_1}-manifold, which was needed because a non-TOP-split Z^{π_1}-manifold was given by Hambleton-Teichner in [4] (see also [8, 10, 11] for some discussions and partial results). We shall show the following theorem:

Theorem 1.1. Every virtually TOP-split Z^{π_1}-manifold is TOP-split.

The non-TOP-split Z^{π_1}-manifold given by Hambleton-Teichner is obtained from their non-trivial Λ-Hermitian form by using a construction technique of a topological 4-manifold with a given Λ-Hermitian form by Freedman-Quinn [2], where $\Lambda = Z[Z] = Z[t, t^{-1}]$ denotes the integral Laurent polynomial ring in t. As it is observed by Friedl, Hambleton, Melvin and Teichner in [3] and seen also from Theorem 1.1, the Hambleton-Teichner example is virtually non-TOP-split. In the proof of [4], the non-TOP-splitting comes from the property that the intersection form of the Hambleton-Teichner example is definite. For a Z^{π_1}-manifold with indefinite intersection form, we show the following theorem.

Theorem 1.2. Every Z^{π_1}-manifold with indefinite intersection form is TOP-split.

A key to this theorem is to show the following lemma with which Theorem 1.1 implies Theorem 1.2.

Lemma 1.3. Every Z^{π_1}-manifold with indefinite intersection form is virtually TOP-split.

By the proof of Hillman-Kawauchi [5] using Theorem 1.2 in place of [7], we have:

Corollary 1.4 (Hillman-Kawauchi). Every orientable surface-knot F in S^4 is topologically unknotted if the fundamental group $\pi_1(S^4 \backslash F)$ is isomorphic to Z.

For an S^2-knot K in a simply connected 4-manifold M_1, we have the following unknotting result, where K is *of Dehn's type* in M_1 if there is a map f from the 3-disk D^3 to M_1 such that the image $f(\partial D^3) = K$ and the singular set $\Sigma(f) \subset \mathrm{int}D^3$.

Corollary 1.5. Let M_1 be a closed simply connected 4-manifold with indefinite intersection form. An S^2-knot K in M_1 is topologically unknotted if we have one of the following two conditions:

(1) The fundamental group $\pi_1(M_1 \backslash K)$ is isomorphic to Z.

(2) The S^2-knot K is of Dehn's type in M_1.

Proof. We can obtain a Z^{π_1}-manifold M with indefinite intersection form from M_1 by surgery replacing a normal disk-bundle $K \times D^2$ of K in M_1 with $D^3 \times \partial D^2$. By Theorem 1.2, M is TOP-split. Since a simple loop ℓ in M representing a generator of $\pi_1(M) \cong Z$ is unique up to isotopies of M, we see that K is topologically unknotted in M_1. If K is of Dehn's type, then we have also $\pi_1(M_1 \backslash K) \cong Z$ by the proof of [5, Corollary 4.2], so that K is topologically unknotted. □

By Donaldson's famous result [1], there is no smooth spin 4-manifold with definite intersection form. Hence we have the following corollary.

Corollary 1.6. Every smooth spin Z^{π_1}-manifold is TOP-split.

Friedl, Hambleton, Melvin and Teichner in [3] showed that the non-TOP-split Z^{π_1}-manifold given by Hambleton-Teichner is non-smoothable and further virtually non-smoothable. It appears unknown whether every smooth non-spin Z^{π_1}-manifold with definite intersection form is TOP-split (see [10, p.209] as well as [3]). By [2], it is known that every non-singular Λ-Hermitian form on a free Λ-module of finite rank is realized as the Λ-intersection form $\mathrm{Int}_\Lambda : H_2(\tilde{M}; Z) \times H_2(\tilde{M}; Z) \to \Lambda$ of the infinite cyclic covering $\tilde{M}$ of a Z^{π_1}-manifold M. Thus, we obtain from Theorems 1.1 and 1.2 the following purely algebraic result:

Corollary 1.7. A non-singular Λ-Hermitian form

$$I_\Lambda : \Lambda^n \times \Lambda^n \to \Lambda$$

admits a Λ-basis $x_1, x_2, \ldots, x_n$ of Λ^n such that $I_\Lambda(x_i, x_j)$ is an integer for all i, j if we have one of the following two conditions:

(1) For a positive integer m, we regard Λ^n as a free $\Lambda^{(m)}$-module of rank mn over the subring $\Lambda^{(m)} = Z[t^m, t^{-m}]$ of Λ, so that I_Λ induces a non-singular $\Lambda^{(m)}$-Hermitian form

$I_{\Lambda^{(m)}}$. Then for some positive integer m, there is a $\Lambda^{(m)}$-basis x_{ik} $(i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)$ for Λ^n such that $I_{\Lambda^{(m)}}(x_{ik}, x_{i'k'})$ is an integer for all i, i', k, k'.

(2) The nonsingular symmetric bilinear form $I : Z^n \times Z^n \to Z$ obtained from I_Λ by taking $t = 1$ is indefinite.

The idea of proof of Theorem 1.1 is to find an exact leaf of a Z^{π_1}-manifold, whose notion was developed in [10, 11] for the infinite cyclic covering $\tilde{M}$ of a Z^{H_1}-manifold M. The idea of proof of Lemma 1.3 which is a key to Theorem 1.2 is to find a connected summand $S^2 \times S^2$ in the infinite cyclic covering $\tilde{M}$ of a Z^{π_1}-manifold M.

We note that this paper is done on a basis of the author's earlier paper [7]. In fact, in this paper we shall use the results [10, Corollary 3.4] and [11, Theorem 2.3, Lemma 3.6] on exactness of a Z^{π_1}-manifold, but behind them we used the result that every Z^{π_1}-manifold M is homology cobordant to the connected sum $S^1 \times S^3 \# M_1$ which has been shown in [7].

2. Proof of Theorem 1.1.

The following lemma is needed to reduce the proof of Theorem 1.1 to an argument on the double covering of Z^{π_1}-manifold.

Lemma 2.1. Let M be a virtually topological split Z^{π_1}-manifold. Then there is a positive integer m such that the 2^m-fold cyclic covering $M^{(2^m)}$ of M is TOP-split.

Proof. Let $\tilde{M}$ be the infinite cyclic covering of M. Then we have a 3-sphere S^3 in $\tilde{M}$ with $[S^3] \in H_3(\tilde{M}; Z) \cong Z$ a generator such that there is a constant $n_0 > 0$ with $S^3 \cap t^n(S^3) = \emptyset$ for all $n > n_0$. Take $2^m > n_0$. □

By induction on m in Lemma 2.1, Theorem 1.1 follows from the following lemma:

Lemma 2.2. A Z^{π_1}-manifold M is TOP-split if $M^{(2)}$ is TOP-split.

A leaf V of a Z^{H_1}-manifold M is *exact* if the natural semi-exact sequence

$$0 \to \mathrm{tor} H_2(\tilde{M}, \tilde{V}; Z) \to \mathrm{tor} H_1(\tilde{V}; Z) \to \mathrm{tor} H_1(\tilde{M}; Z)$$

on the Z-torsion parts induced from the homology exact sequence of the pair $(\tilde{M}, \tilde{V})$ is exact, where the pair $(\tilde{M}, \tilde{V})$ denotes the lift of the pair (M, V) under the infinite cyclic connected covering $\tilde{M} \to M$ (see [10, 11]). A Z^{H_1}-manifold M is *exact* if M admits an exact leaf. Because an exact Z^{π_1}-manifold M is TOP-split (see [10, Corollary 3.4]), Lemma 2.2 is obtained by a combination of the following Lemmas 2.3 and 2.4:

Lemma 2.3. For a Z^{π_1}-manifold M, if $M^{(2)}$ is TOP-split, then M has a connected leaf V with $H_1(V;Z)$ a free abelian group.

Lemma 2.4. If a Z^{H_1}-manifold M has a connected leaf V with $H_1(V;Z)$ a free abelian group, then M is exact.

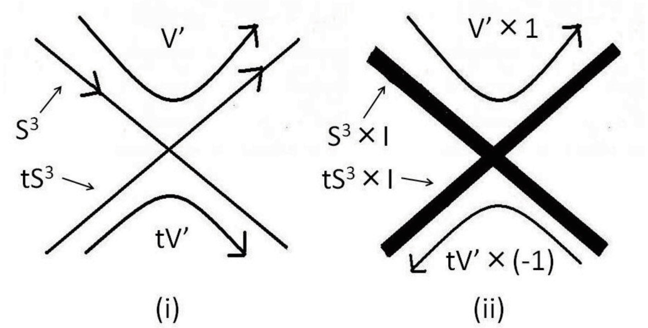

Figure 1: Creating t-interchangable 3-manifolds

Proof of Lemma 2.3. Let S^3 be an oriented 3-sphere leaf of $M^{(2)}$ such that S^3 and $t(S^3)$ meet transversely for the covering involution t of $M^{(2)}$. Then the intersection $F = S^3 \cap t(S^3)$ is a closed orientable (possibly disconnected) 2-manifold F. Let E_+, E_- be the oriented 3-manifolds obtained from S^3 by splitting along F, and tE_+, tE_- be the oriented 3-manifolds obtained from the oriented 3-sphere $t(S^3)$ given by the orientation of S^3 and t by splitting along $tF = F$. We consider that F is oriented so that $\partial E_+ = \partial tE_+ = F$ and $\partial E_- = \partial tE_- = -F$. The 3-manifolds $V' = E_+ \cup tE_-$ and $tV' = tE_+ \cup E_-$ are closed oriented (possibly disconnected) 3-manifolds which are interchangable by the t-action (see Fig. 1(i)[1]). By the same local modification, the closed oriented 3-manifold V' can be also obtained from the immersed image $S^3_\#$ of S^3 in M under the double covering projection $M^{(2)} \to M$. We note that the homomorphism $H_1(M^{(2)};Z) \to H_1(M;Z)$ induced from the covering projection sends a generator $x^{(2)} \in H_1(M^{(2)};Z) \cong Z$ to the double of a generator $x \in H_1(M;Z) \cong Z$.

[1]A precise local picture is obtained from this picture by taking the product of this picture and an open disk.

Since $\mathrm{Int}_{M^{(2)}}(S^3, x^{(2)}) = +1$ and $[S^3_\#] = [V'] \in H_3(M; Z)$, we have

$$\mathrm{Int}_M(S^3_\#, 2x) = \mathrm{Int}_M(V', 2x) = +2.$$

Hence $\mathrm{Int}_M(V', x) = +1$. Let V be a component of V' with $\mathrm{Int}_M(V, x) > 0$. Representing x by a simple loop ℓ in M, we can construct from ℓ a simple loop ℓ' in M meeting V transversely in a single point because V is connected. This means that V and ℓ' represent generators of $H_3(M; Z) \cong Z$ and $H_1(M; Z) \cong Z$, respectively, and the natural homomorphism $H_1(V; Z) \to H_1(M; Z)$ is the zero map. Thus, V is a connected leaf of M. We show that $H_1(V; Z)$ is a free abelian group. To see this, for $I = [-1, 1]$ we consider normal I-bundles $S^3 \times I$ and $t(S^3) \times I$ of S^3 and $t(S^3)$, respectively, whose union W is a compact connected oriented 4-manifold (see Fig. 1(ii)[2]). We observe the following sublemma:

Sublemma 2.3.1. For the 4-manifold $W = S^3 \times I \cup t(S^3 \times I)$, the homology groups $H_d(W; Z)$ $(d = 1, 2)$ are free abelian groups and the intersection form $\mathrm{Int} : H_2(W; Z) \times H_2(W; Z) \to Z$ is the zero form.

Assuming this sublemma, we obtain by Poincaré duality that $H_2(W, \partial W; Z)$ is a free abelian group and the natural homomorphism $H_2(W; Z) \to H_2(W, \partial W; Z)$ is the zero map, because this homomorphism induces the intersection form

$$\mathrm{Int} : H_2(W; Z) \times H_2(W; Z) \to Z$$

from the non-singular intersection form

$$\mathrm{Int} : H_2(W; Z) \times H_2(W, \partial W; Z) \to Z.$$

Then the exact sequence

$$0 \to H_2(W, \partial W; Z) \to H_1(\partial W; Z) \to H_1(W; Z)$$

implies that $H_1(\partial W; Z)$ is a free abelian group. Since V is a component of the boundary ∂W, we see that $H_1(V; Z)$ is a free abelian group. □

Proof of Sublemma 2.3.1. By Mayer-Vietoris sequence, we have

$$H_2(W; Z) \cong H_1(F; Z) \cong Z^{2g} \quad \text{and} \quad H_1(W; Z) \cong \tilde{H}_0(F; Z) \cong Z^{c-1},$$

[2]A precise local picture is obtained from this picture by taking the product of this picture and an open disk.

where g and c denote the total genus of F and the number of the connected components of F, respectively. Thus, the homology groups $H_d(W;Z)$ $(d = 1,2)$ are free abelian groups. Since the boundary operators $\partial_* : H_2(S^3, F; Z) \to H_1(F;Z)$ and $\partial'_* : H_2(t(S^3), F; Z) \to H_1(F;Z)$ are isomorphisms, we see from the excision isomorphisms

$$\begin{aligned} H_2(S^3, F; Z) &\cong H_2(E^+, \partial E^+; Z) \oplus H_2(E^-, \partial E^-; Z), \\ H_2(t(S^3), F; Z) &\cong H_2(tE^+, \partial tE^+; Z) \oplus H_2(tE^-, \partial tE^-; Z) \end{aligned}$$

that $H_1(F;Z)$ has a basis $[\partial C_i^+]$, $[\partial C_i^-]$ $(i = 1, 2, \ldots, g)$ where C_i^+ is a 2-chain in E^+ and C_i^- is a 2-chain in E^-, and further we can write $\partial C_i^+ = \partial D_i^{++} + \partial D_i^{+-}$ and $\partial C_i^- = \partial D_i^{-+} + \partial D_i^{--}$ where D_i^{++}, D_i^{-+} are 2-chains in tE^+ and D_i^{+-}, D_i^{--} are 2-chains in tE^-. The homology classes $z_i = [C_i^+ - (D_i^{++} + D_i^{+-})], z'_i = [C_i^- - (D_i^{-+} + D_i^{--})]$ $(i = 1, 2, \ldots, g)$ form a basis for $H_2(W;Z)$. Using the thickness $t(S^3) \times I$ of $t(S^3)$ in W, we see that $\mathrm{Int}(z_i, z'_j) = 0$ for all i, j. Since $\mathrm{Int}(\partial C_i^+, \partial C_j^+) = \mathrm{Int}(\partial C_i^-, \partial C_j^-) = 0$ in F for all i, j, we also see that $\mathrm{Int}(z_i, z_j) = \mathrm{Int}(z'_i, z'_j) = 0$ for all i, j. □

Proof of Lemma 2.4. If $H_2(M, V; Z)$ and $H_1(V;Z)$ are free abelian groups, then we have $\mathrm{tor} H_2(\tilde{M}, \tilde{V}; Z) = \mathrm{tor} H_1(\tilde{V}; Z) = 0$ and V is an exact leaf. Assume that $H_2(M, V; Z)$ is not free abelian. We shall construct a connected leaf V^* of M such that $H_1(V^*; Z)$ and $H_2(M, V^*; Z)$ are free abelian groups, which is an exact leaf of M. To see this, we consider a free abelian subgroup G of $H_2(M;Z)$ such that the quotient group $H_2(M;Z)/G$ is a free abelian group and the image H of the natural homomorphism $H_2(V;Z) \to H_2(M;Z)$ is a finite index subgroup of G. Then there are a basis $x_i (i = 1, 2, \ldots, v)$ for $H_2(V;Z)$ and a basis $y_i (i = 1, 2, \ldots, u)$ for G with $u \leqq v$ such that the natural homomorphism $H_2(V;Z) \to H_2(M;Z)$ sends the first $m (\leqq u)$ elements $x_i (i = 1, 2, \ldots, m)$ to the elements $k_i y_i (i = 1, 2, \ldots, m)$ for some integers $k_i > 1 (i = 1, 2, \ldots, m)$ and the elements x_i $(i = m+1, m+2, \ldots, u)$ to y_i $(i = m+1, m+2, \ldots, u)$ and the elements x_i $(i = u+1, m+2, \ldots, v)$ to 0. Then we have

$$G/H \cong Z_{k_1} \oplus Z_{k_2} \oplus \cdots \oplus Z_{k_m}.$$

By an argument of [9], every x_i is represented by a closed connected oriented surface S_i^x in V. Regarding x_i as an element in $H_2(M;Z)$, we have $\mathrm{Int}(x_i, x_j) = \mathrm{Int}(y_i, y_j) = \mathrm{Int}(x_i, y_j) = 0$ in M for all i, j. Thus, we can represent y_i $(i = 1, 2, \ldots, m)$ by mutually disjoint closed connected oriented surfaces S_i^y $(i = 1, 2, \ldots, m)$ in M. Let $\ell_1 = V \cap S_1^y$ be a closed oriented 1-manifold. Since $k_1 S_1^y$ is homologous to S_1^x in M and hence $\mathrm{Int}(k_1 S_1^y, S_i^x) = 0$ in M for all i, the intersection number of $k_1 \ell_1$ and S_i^x in V must be 0 for all i. Using that $H_1(V;Z)$ is free abelian, we obtain by Poincaré duality that $k_1 \ell_1$ and hence ℓ_1 are null-homologous in V. Thus, the 1-manifold ℓ_1 bounds an oriented surface Δ_1 in V. Let S_1^* be a closed (possibly disconnected) oriented surface in $M\backslash V$ obtained from S_1^y by cutting along ℓ_1 and then adding parallel copies of Δ_1 in a collar neighborhood of V in M. Since V is connected, the complement $M\backslash V$ is also connected. We can construct a closed connected oriented surface S_1^{**} by piping the components of S_1^* in the complement $M\backslash V$. Then we have $y_1 = [S_1^{**}] \in H_2(M;Z)$. Since S_1^{**} admits a trivial normal D^2-bundle $S_1^{**} \times D^2$ in M, we can take a connected sum of V and $S_1^{**} \times \partial D^2$ to obtain a connected leaf V' of M. Then $H_1(V';Z)$ is a free abelian group. Let H' be the image of the natural homomorphism $H_2(V';Z) \to H_2(M;Z)$. By construction, we have $H' \subset G$ and

$$G/H' \cong Z_{k_2} \oplus Z_{k_3} \oplus \cdots \oplus Z_{k_m}.$$

By continuing this process, we have a connected leaf V^* with $H_1(V^*;Z)$ a free abelian group such that the image of the natural homomorphism $H_2(V^*;Z) \to H_2(M;Z)$ coincides with G. The exact sequence

$$0 \to H_2(M;Z)/G \to H_2(M,V^*;Z) \to H_1(V^*;Z)$$

induced from the homology exact sequence of (M,V) implies that $H_2(M,V^*;Z)$ is a free abelian group. □

3. Proof of Lemma 1.3.

First, we show the following lemma.

Lemma 3.1. If the intersection form on Z^{H_1}-manifold M is indefinite, then there is a pair of elements $x, y \in H_2(\tilde{M};Z)$ such that the ordinary intersection numbers of x, y in $\tilde{M}$ have

$$\mathrm{Int}(x,x) = \mathrm{Int}(y,y) = 0 \quad \text{and} \quad \mathrm{Int}(x,y) = 1.$$

Proof of Lemma 3.1. For $\Lambda = Z[t,t^{-1}]$, we consider the subring

$$\Lambda^+ = \{\frac{f(t)}{g(t)} \in Q(\Lambda) |\, f(t), g(t) \in \Lambda \text{ with } g(1) = \pm 1\}$$

of the quotient field $Q(\Lambda)$. For a Λ-module H, we denote the Λ^+-module $H \otimes_\Lambda \Lambda^+$ by H^+. The Λ-intersection number $\mathrm{Int}_\Lambda(x,y)$ for $x, y \in H_2(\tilde{M};Z)$ is defined by

$$\mathrm{Int}_\Lambda(x,y) = \sum_{i=-\infty}^{+\infty} \mathrm{Int}(x, t^{-i}y)t^i \in \Lambda,$$

which is extended to the Λ^+-intersection number $\mathrm{Int}_{\Lambda^+}(x^+, y^+)$ for any elements x^+, y^+ of the second Λ^+-homology $H_2(\tilde{M};Z)^+$. By [11, Lemma 3.6], we see that the Λ^+-homology $H_2(\tilde{M};Z)^+$ of every Z^{H_1}-manifold M is a Λ^+-free module with a Λ^+-basis $x_i^+ (i = 1, 2, \ldots, n)$ such that the Λ^+-intersection numbers $\mathrm{Int}_{\Lambda^+}(x_i^+, x_j^+)$ are integers for all i, j. Since the intersection form $\mathrm{Int} : H_2(M;Z) \times H_2(M;Z) \to Z$ is indefinite (i.e., there is a non-zero element $x_0 \in H_2(M;Z)$ with $\mathrm{Int}(x_0, x_0) = 0$), we may assume that $\mathrm{Int}_{\Lambda^+}(x_1^+, x_1^+) = 0$, $\mathrm{Int}_{\Lambda^+}(x_1^+, x_2^+) = 1$, $\mathrm{Int}_{\Lambda^+}(x_i^+, x_j^+) = 0$ $(i = 1, 2, j = 3, 4, \ldots, n)$. We take elements $g_i(t) \in \Lambda$ with $g_i(1) = 1$ $(i = 1, 2)$ such that the elements $x' = g_1(t)x_1^+, y' = g_2(t)x_2^+$ are in $H_2(\tilde{M})$. Then we have

$$\begin{aligned} \mathrm{Int}_\Lambda(x', x') &= g_1(t^{-1})g_1(t)\mathrm{Int}_{\Lambda^+}(x_1^+, x_1^+) = 0, \\ \mathrm{Int}_\Lambda(x', y') &= g_1(t^{-1})g_2(t)\mathrm{Int}_{\Lambda^+}(x_1^+, x_2^+) = g_1(t^{-1})g_2(t). \end{aligned}$$

Thus, if we let $x = \sum_{i=-N}^{N} t^i x'$ for a large positive integer N, then we have

$$\mathrm{Int}(x, x) = 0, \quad \text{and} \quad \mathrm{Int(x, y')} = \mathrm{g_1(1)g_2(1)} = 1.$$

If $\mathrm{Int}(y', y')$ is an even integer, say $2m$, then we take $y = y' - mx$ to obtain a desired pair x, y. Otherwise, we can assume that $\mathrm{Int}(y', y') = 1$ by replacing y' with $y' - mx$ when $\mathrm{Int}(y', y') = 2m+1$. The covering translated elements $z = t^n(x)$ and $w = t^n(y')$ for a large integer n are represented by 2-cycles disjoint from 2-cycles represented by x, y'. Since $\mathrm{Int}(z, z) = 0$ and $\mathrm{Int}(z, w) = \mathrm{Int}(w, w) = 1$, the pair of x and $y = y' + z - w$ gives a desired pair. □

By the integral duality on the infinite cyclic covering $\tilde{M}$ of a Z^{H_1}-manifold M (see [6]), we have $H_3(\tilde{M};Z) \cong Z$, whose generator is called the fundamental class of the infinite cyclic covering $\tilde{M} \to M$ (see [9]). By considering a closed oriented 3-manifold representing the fundamental class, we can complete the proof of Lemma 1.3.

Completion of the proof of Lemma 1.3. We shall find a 3-sphere S^3 in $\tilde{M}$ representing a generator of $H_3(\tilde{M};Z)$. Then this 3-sphere S^3 is embedded in the p-fold cyclic covering $M^{(p)}$ of M for a large p so that S^3 represents a generator of $H_3(M^{(p)};Z) \cong Z$ because the covering projection $\tilde{M} \to M^{(p)}$ induces an isomorphism $H_3(\tilde{M};Z) \cong H_3(M^{(p)};Z)$ by the Wang exact sequence. A generator of $H_1(M^{(p)};Z) \cong Z$ is represented by an embedded circle S^1 in $M^{(p)}$ intersecting S^3 transversely in a single point. The regular neighborhood of the bouquet $S^1 \vee S^3$ in $M^{(p)}$ gives a connected summand $S^1 \times S^3$ of $M^{(p)}$, so that $M^{(p)}$ is TOP-split. Since $\tilde{M}$ is a simply connected 4-manifold, the pair of x, y in Lemma 3.1 is represented by a pair of 2-spheres in M with geometric intersection number one by [2]. Thus, $\tilde{M}$ has a connected summand $S^2 \times S^2$. Then we see that the q-fold cyclic covering $M^{(q)}$ of M for a large q has a connected summand of $S^2 \times S^2$. Let $M^{(q)} = M' \# S^2 \times S^2$ for a Z^{π_1}-manifold M'. It turns out that the Z^{π_1}-manifold $M' \# k S^2 \times S^2$ for any positive integer k has the infinite cyclic covering (non-equivariantly) homeomorphic to $\tilde{M}$. The connected sum $M' \# k S^2 \times S^2$ for a large k is exact by [11, Theorem 2.4] and hence TOP-split by [10, Corollary 3.4]. We note that another proof of the TOP-splitting of the connected sum $M' \# k S^2 \times S^2$ for a large k is known by Matumoto [12]. Then we can obtain a 3-sphere S^3 in $\tilde{M}$ representing a generator of $H_3(\tilde{M}; Z)$. □

There is also another proof of Lemma 1.3. For the proof, let M is a Z^{π_1}-manifold with indefinite intersection form. Then we have the Witt index

$$w(M) = \frac{\beta_2(M; Z) - |\mathrm{sign}(M)|}{2} \geqq 1.$$

Let M' be the d-fold cyclic covering of M. By the covering properties of Euler characteristic and signature, we have

$$\beta_2(M'; Z) = d\,\beta_2(M; Z) \quad \text{and} \quad \mathrm{sign}(M') = d\,\mathrm{sign}(M),$$

so that $w(M') \geqq d$. Thus, we have the Witt index $w(M') \geqq 3$ for $d \geqq 3$. Then Hambleton-Teichner in [4] observed that there is a Λ-basis $x'_i (i = 1, 2, \ldots, n')$ for the second homology $H_2(\tilde{M}'; Z)$ of the infinite cyclic covering $\tilde{M}'$ of M' such that the Λ-intersection $\mathrm{Int}_\Lambda(x'_i, x'_j)$ is an integer for all i, j. By [11], M' is exact and hence by [10, Corollary 3.4] M' is TOP-split, implying that M is virtually TOP-split.

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