Splitting a 4-manifold with infinite cyclic fundamental group, revised

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ABSTRACT

This article is a revised version of the author’s earlier paper on a TOP-splitting of a closed connected oriented 4-manifold with infinite cyclic fundamental group. We show that a closed connected oriented 4-manifold with infinite cyclic fundamental group is TOP-split if it is virtually TOP-split. As a consequence, we see that a closed connected oriented 4-manifold with infinite cyclic fundamental group is TOP-split if the intersection form is indefinite. This also implies that every closed connected oriented smooth spin 4-manifold with infinite cyclic fundamental group is TOP-split.

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1. Introduction

A closed connected oriented 4-manifold $M$ is called a $Z^{\pi_1}$-manifold if the fundamental group $\pi_1(M)$ is isomorphic to $Z$, and a $Z^{H_1}$-manifold if the first homology group $H_1(M; Z)$ is isomorphic to $Z$. A $Z^{\pi_1}$-manifold $M$ is TOP-split if $M$ is homeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for a simply connected closed 4-manifold $M_1$, and virtually TOP-split if a finite covering of $M$ is TOP-split. Here, we do not assume that a closed 4-manifold is a smooth or piecewise-linear manifold, but we can use smooth and piecewise-linear techniques for it because a punctured manifold of it is smoothable (see Freedman-Quinn [2]). The purpose of this paper is to make a revised version of the author’s earlier paper [7] on TOP-splitting of a $Z^{\pi_1}$-manifold, which was needed because a non-TOP-split $Z^{\pi_1}$-manifold was given by Hambleton-Teichner in [4] (see also [8, 10, 11] for some discussions and partial results). We shall show the following theorem:

**Theorem 1.1.** Every virtually TOP-split $Z^{\pi_1}$-manifold is TOP-split.

The non-TOP-split $Z^{\pi_1}$-manifold given by Hambleton-Teichner is obtained from their non-trivial $\Lambda$-Hermitian form by using a construction technique of a topological 4-manifold with a given $\Lambda$-Hermitian form by Freedman-Quinn [2], where $\Lambda = Z[Z] = Z[t, t^{-1}]$ denotes the integral Laurent polynomial ring in $t$. As it is observed by Friedl, Hambleton, Melvin and Teichner in [3] and seen also from Theorem 1.1, the Hambleton-Teichner example is virtually non-TOP-split. In the proof of [4], the non-TOP-splitting comes from the property that the intersection form of the Hambleton-Teichner example is definite. For a $Z^{\pi_1}$-manifold with indefinite intersection form, we show the following theorem.

**Theorem 1.2.** Every $Z^{\pi_1}$-manifold with indefinite intersection form is TOP-split.

A key to this theorem is to show the following lemma with which Theorem 1.1 implies Theorem 1.2.

**Lemma 1.3.** Every $Z^{\pi_1}$-manifold with indefinite intersection form is virtually TOP-split.

By the proof of Hillman-Kawauchi [5] using Theorem 1.2 in place of [7], we have:

**Corollary 1.4 (Hillman-Kawauchi).** Every orientable surface-knot $F$ in $S^4$ is topologically unknotted if the fundamental group $\pi_1(S^4 \setminus F)$ is isomorphic to $Z$. 

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For an $S^2$-knot $K$ in a simply connected 4-manifold $M$, we have the following unknottedting result, where $K$ is of Dehn’s type in $M$ if there is a map $f$ from the 3-disk $D^3$ to $M$ such that the image $f(\partial D^3) = K$ and the singular set $\Sigma(f) \subset \text{int} D^3$.

**Corollary 1.5.** Let $M$ be a closed simply connected 4-manifold with indefinite intersection form. An $S^2$-knot $K$ in $M$ is topologically unknotted if we have one of the following two conditions:

1. The fundamental group $\pi_1(M \setminus K)$ is isomorphic to $\mathbb{Z}$.
2. The $S^2$-knot $K$ is of Dehn’s type in $M$.

**Proof.** We can obtain a $\mathbb{Z}^{\pi_1}$-manifold $M$ with indefinite intersection form from $M$ by surgery replacing a normal disk-bundle $K \times D^2$ of $K$ in $M$ with $D^3 \times \partial D^2$. By Theorem 1.2, $M$ is TOP-split. Since a simple loop $\ell$ in $M$ representing a generator of $\pi_1(M) \cong \mathbb{Z}$ is unique up to isotopies of $M$, we see that $K$ is topologically unknotted in $M$. If $K$ is of Dehn’s type, then we have also $\pi_1(M \setminus K) \cong \mathbb{Z}$ by the proof of [5, Corollary 4.2], so that $K$ is topologically unknotted. □

By Donaldson’s famous result [1], there is no smooth spin 4-manifold with definite intersection form. Hence we have the following corollary.

**Corollary 1.6.** Every smooth spin $\mathbb{Z}^{\pi_1}$-manifold is TOP-split.

Friedl, Hambleton, Melvin and Teichner in [3] showed that the non-TOP-split $\mathbb{Z}^{\pi_1}$-manifold given by Hambleton-Teichner is non-smoothable and further virtually non-smoothable. It appears unknown whether every smooth non-spin $\mathbb{Z}^{\pi_1}$-manifold with definite intersection form is TOP-split (see [10, p.209] as well as [3]). By [2], it is known that every non-singular $\Lambda$-Hermitian form on a free $\Lambda$-module of finite rank is realized as the $\Lambda$-intersection form $\text{Int}_\Lambda : H_2(\hat{M}; \mathbb{Z}) \times H_2(\hat{M}; \mathbb{Z}) \to \Lambda$ of the infinite cyclic covering $\hat{M}$ of a $\mathbb{Z}^{\pi_1}$-manifold $M$. Thus, we obtain from Theorems 1.1 and 1.2 the following purely algebraic result:

**Corollary 1.7.** A non-singular $\Lambda$-Hermitian form

$$I_\Lambda : \Lambda^n \times \Lambda^n \to \Lambda$$

admits a $\Lambda$-basis $x_1, x_2, \ldots, x_n$ of $\Lambda^n$ such that $I_\Lambda(x_i, x_j)$ is an integer for all $i, j$ if we have one of the following two conditions:

1. For a positive integer $m$, we regard $\Lambda^n$ as a free $\Lambda^{(m)}$-module of rank $mn$ over the subring $\Lambda^{(m)} = \mathbb{Z}[\ell^m, \ell^{-m}]$ of $\Lambda$, so that $I_\Lambda$ induces a non-singular $\Lambda^{(m)}$-Hermitian form
Then for some positive integer \(m\), there is a \(\Lambda^{(m)}\)-basis \(x_{ik}\) \((i = 1, 2, \ldots, n; k = 1, 2, \ldots, m)\) for \(\Lambda^n\) such that \(I_{\Lambda^{(m)}}(x_{ik}, x_{i'k'})\) is an integer for all \(i, i', k, k'\).

(2) The nonsingular symmetric bilinear form \(I : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}\) obtained from \(I_{\Lambda}\) by taking \(t = 1\) is indefinite.

The idea of proof of Theorem 1.1 is to find an exact leaf of a \(\mathbb{Z}^\pi\)-manifold, whose notion was developed in [10, 11] for the infinite cyclic covering \(\tilde{M}\) of a \(\mathbb{Z}^{H_1}\)-manifold \(M\). The idea of proof of Lemma 1.3 which is a key to Theorem 1.2 is to find a connected summand \(S^2 \times S^2\) in the infinite cyclic covering \(\tilde{M}\) of a \(\mathbb{Z}^\pi\)-manifold \(M\).

We note that this paper is done on a basis of the author's earlier paper [7]. In fact, in this paper we shall use the results [10, Corollary 3.4] and [11, Theorem 2.3, Lemma 3.6] on exactness of a \(\mathbb{Z}^\pi\)-manifold, but behind them we used the result that every \(\mathbb{Z}^\pi\)-manifold \(M\) is homology cobordant to the connected sum \(S^1 \times S^3\# M_1\) which has been shown in [7].

2. Proof of Theorem 1.1.

The following lemma is needed to reduce the proof of Theorem 1.1 to an argument on the double covering of \(\mathbb{Z}^\pi\)-manifold.

**Lemma 2.1.** Let \(M\) be a virtually topological split \(\mathbb{Z}^\pi\)-manifold. Then there is a positive integer \(m\) such that the \(2^m\)-fold cyclic covering \(M^{(2^m)}\) of \(M\) is TOP-split.

**Proof.** Let \(\tilde{M}\) be the infinite cyclic covering of \(M\). Then we have a 3-sphere \(S^3\) in \(\tilde{M}\) with \([S^3] \in H_3(\tilde{M}; Z) \cong Z\) a generator such that there is a constant \(n_0 > 0\) with \(S^3 \cap t^n(S^3) = \emptyset\) for all \(n > n_0\). Take \(2^m > n_0\). \(\square\)

By induction on \(m\) in Lemma 2.1, Theorem 1.1 follows from the following lemma:

**Lemma 2.2.** A \(\mathbb{Z}^\pi\)-manifold \(M\) is TOP-split if \(M^{(2)}\) is TOP-split.

A leaf \(V\) of a \(\mathbb{Z}^{H_1}\)-manifold \(M\) is exact if the natural semi-exact sequence

\[
0 \to \text{tor}H_2(\tilde{M}, \tilde{V}; Z) \to \text{tor}H_1(\tilde{V}; Z) \to \text{tor}H_1(\tilde{M}; Z)
\]

on the \(Z\)-torsion parts induced from the homology exact sequence of the pair \((\tilde{M}, \tilde{V})\) is exact, where the pair \((\tilde{M}, \tilde{V})\) denotes the lift of the pair \((M, V)\) under the infinite cyclic connected covering \(\tilde{M} \to M\) (see [10, 11]). A \(\mathbb{Z}^{H_1}\)-manifold \(M\) is exact if \(M\) admits an exact leaf. Because an exact \(\mathbb{Z}^\pi\)-manifold \(M\) is TOP-split (see [10,
Corollary 3.4), Lemma 2.2 is obtained by a combination of the following Lemmas 2.3 and 2.4:

**Lemma 2.3.** For a $\mathbb{Z}^n$-manifold $M$, if $M^{(2)}$ is TOP-split, then $M$ has a connected leaf $V$ with $H_1(V; \mathbb{Z})$ a free abelian group.

**Lemma 2.4.** If a $\mathbb{Z}^{H_1}$-manifold $M$ has a connected leaf $V$ with $H_1(V; \mathbb{Z})$ a free abelian group, then $M$ is exact.

![Image of creating $t$-interchangeable 3-manifolds](image-url)

**Figure 1:** Creating $t$-interchangeable 3-manifolds

**Proof of Lemma 2.3.** Let $S^3$ be an oriented 3-sphere leaf of $M^{(2)}$ such that $S^3$ and $t(S^3)$ meet transversely for the covering involution $t$ of $M^{(2)}$. Then the intersection $F = S^3 \cap t(S^3)$ is a closed orientable (possibly disconnected) 2-manifold $F$. Let $E_+, E_-$ be the oriented 3-manifolds obtained from $S^3$ by splitting along $F$, and $tE_+, tE_-$ be the oriented 3-manifolds obtained from the oriented 3-sphere $t(S^3)$ given by the orientation of $S^3$ and $t$ by splitting along $tF = F$. We consider that $F$ is oriented so that $\partial E_+ = \partial tE_+ = F$ and $\partial E_- = \partial tE_- = -F$. The 3-manifolds $V' = E_+ \cup tE_-$ and $tV' = tE_+ \cup E_-$ are closed oriented (possibly disconnected) 3-manifolds which are interchangeable by the $t$-action (see Fig. 1(i)). By the same local modification, the closed oriented 3-manifold $V'$ can be also obtained from the immersed image $S^3_\#$ of $S^3$ in $M$ under the double covering projection $M^{(2)} \to M$. We note that the homomorphism $H_1(M^{(2)}; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ induced from the covering projection sends a generator $x^{(2)} \in H_1(M^{(2)}; \mathbb{Z}) \cong \mathbb{Z}$ to the double of a generator $x \in H_1(M; \mathbb{Z}) \cong \mathbb{Z}$.

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*A precise local picture is obtained from this picture by taking the product of this picture and an open disk.*
Since $\text{Int}_{M(2)}(S^3, x^{(2)}) = +1$ and $[S^3]_{#} = [V'] \in H_2(M; Z)$, we have

$$\text{Int}_{M}(S^3, 2x) = \text{Int}_{M}(V', 2x) = +2.$$ 

Hence $\text{Int}_{M}(V', x) = +1$. Let $V$ be a component of $V'$ with $\text{Int}_{M}(V, x) > 0$. Representing $x$ by a simple loop $\ell$ in $M$, we can construct from $\ell$ a simple loop $\ell'$ in $M$ meeting $V$ transversely in a single point because $V$ is connected. This means that $V$ and $\ell'$ represent generators of $H_3(M; Z) \cong Z$ and $H_1(M; Z) \cong Z$, respectively, and the natural homomorphism $H_1(V; Z) \to H_1(M; Z)$ is the zero map. Thus, $V$ is a connected leaf of $M$. We show that $H_1(V; Z)$ is a free abelian group. To see this, for $I = [-1, 1]$ we consider normal $I$-bundles $S^3 \times I$ and $t(S^3) \times I$ of $S^3$ and $t(S^3)$, respectively, whose union $W$ is a compact connected oriented 4-manifold (see Fig. 1(ii))2. We observe the following sublemma:

**Sublemma 2.3.1.** For the 4-manifold $W = S^3 \times I \cup t(S^3 \times I)$, the homology groups $H_d(W; Z)$ ($d = 1, 2$) are free abelian groups and the intersection form $\text{Int} : H_2(W; Z) \times H_2(W; Z) \to Z$ is the zero form.

Assuming this sublemma, we obtain by Poincaré duality that $H_2(W, \partial W; Z)$ is a free abelian group and the natural homomorphism $H_2(W; Z) \to H_2(W, \partial W; Z)$ is the zero map, because this homomorphism induces the intersection form

$$\text{Int} : H_2(W; Z) \times H_2(W; Z) \to Z$$

from the non-singular intersection form

$$\text{Int} : H_2(W; Z) \times H_2(W, \partial W; Z) \to Z.$$

Then the exact sequence

$$0 \to H_2(W, \partial W; Z) \to H_1(\partial W; Z) \to H_1(W; Z)$$

implies that $H_1(\partial W; Z)$ is a free abelian group. Since $V$ is a component of the boundary $\partial W$, we see that $H_1(V; Z)$ is a free abelian group. □

**Proof of Sublemma 2.3.1.** By Mayer-Vietoris sequence, we have

$$H_2(W; Z) \cong H_1(F; Z) \cong Z^{2g} \quad \text{and} \quad H_1(W; Z) \cong \tilde{H}_0(F; Z) \cong Z^{c-1},$$

2A precise local picture is obtained from this picture by taking the product of this picture and an open disk.
where \( g \) and \( c \) denote the total genus of \( F \) and the number of the connected components of \( F \), respectively. Thus, the homology groups \( H_d(W; Z) \) (\( d = 1, 2 \)) are free abelian groups. Since the boundary operators \( \partial_* : H_2(S^3, F; Z) \rightarrow H_1(F; Z) \) and \( \partial'_* : H_2(t(S^3), F; Z) \rightarrow H_1(F; Z) \) are isomorphisms, we see from the excision isomorphisms

\[
H_2(S^3, F; Z) \cong H_2(E^+, \partial E^+; Z) \oplus H_2(E^-, \partial E^-; Z), \\
H_2(t(S^3), F; Z) \cong H_2(tE^+, \partial tE^+; Z) \oplus H_2(tE^-, \partial tE^-; Z)
\]

that \( H_1(F; Z) \) has a basis \([\partial C^+_i], [\partial C^-_i] \) (\( i = 1, 2, \ldots, g \)) where \( C^+_i \) is a 2-chain in \( E^+ \) and \( C^-_i \) is a 2-chain in \( E^- \), and further we can write \( \partial C^+_i = \partial D^+_i + \partial D^-_i \) and \( \partial C^-_i = \partial D^-_i + \partial D^+_i \) where \( D^+_i, D^-_i \) are 2-chains in \( tE^+ \) and \( D^+_i, D^-_i \) are 2-chains in \( tE^- \). The homology classes \( z_i = [C^+_i - (D^+_i + D^-_i)], z'_i = [C^-_i - (D^-_i + D^+_i)] \) \( (i = 1, 2, \ldots, g) \) form a basis for \( H_2(W; Z) \). Using the thickness \( t(S^3) \times I \) of \( t(S^3) \) in \( W \), we see that \( \text{Int}(z_i, z'_j) = 0 \) for all \( i, j \). Since \( \text{Int}(\partial C^+_i, \partial C^-_j) = \text{Int}(\partial C^-_i, \partial C^+_j) = 0 \) in \( F \) for all \( i, j \), we also see that \( \text{Int}(z_i, z_j) = \text{Int}(z'_i, z'_j) = 0 \) for all \( i, j \). □

**Proof of Lemma 2.4.** If \( H_2(M, V; Z) \) and \( H_1(V; Z) \) are free abelian groups, then we have \( \text{tor} H_2(M, \bar{V}; Z) = \text{tor} H_1(V; Z) = 0 \) and \( V \) is an exact leaf. Assume that \( H_2(M, V; Z) \) is not free abelian. We shall construct a connected leaf \( V^* \) of \( M \) such that \( H_1(V^*; Z) \) and \( H_2(M, V^*; Z) \) are free abelian groups, which is an exact leaf of \( M \). To see this, we consider a free abelian subgroup \( G \) of \( H_2(M; Z) \) such that the quotient group \( H_2(M; Z)/G \) is a free abelian group and the image \( H \) of the natural homomorphism \( H_2(V; Z) \rightarrow H_2(M; Z) \) is a finite index subgroup of \( G \). Then there are a basis \( x_i (i = 1, 2, \ldots, v) \) for \( H_2(V; Z) \) and a basis \( y_i (i = 1, 2, \ldots, u) \) for \( G \) with \( u \leq v \) such that the natural homomorphism \( H_2(V; Z) \rightarrow H_2(M; Z) \) sends the first \( m \) \((\leq u)\) elements \( x_i (i = 1, 2, \ldots, m) \) to the elements \( k_i y_i (i = 1, 2, \ldots, m) \) for some integers \( k_i > 1 \) \((i = 1, 2, \ldots, m)\) and the elements \( x_i (i = m + 1, 1, m + 1, 2, \ldots, u) \) to \( y_i \) \((i = m + 1, m + 2, \ldots, u)\) and the elements \( x_i (i = u + 1, m + 2, \ldots, v) \) to \( 0 \). Then we have

\[
G/H \cong Z_{k_1} \oplus Z_{k_2} \oplus \cdots \oplus Z_{k_m}.
\]

By an argument of [9], every \( x_i \) is represented by a closed connected oriented surface \( S_i^\ast \) in \( V \). Regarding \( x_i \) as an element in \( H_2(M; Z) \), we have \( \text{Int}(x_i, x_j) = \text{Int}(y_i, y_j) = \text{Int}(x_i, y_j) = 0 \) in \( M \) for all \( i, j \). Thus, we can represent \( y_i \) \((i = 1, 2, \ldots, m)\) by mutually disjoint closed connected oriented surfaces \( S_i^\ast \) \((i = 1, 2, \ldots, m)\) in \( M \). Let \( \ell_i = V \cap S_i^\ast \) be a closed oriented 1-manifold. Since \( k_i S_i^\ast \) is homologous to \( S_i^\ast \) in \( M \) and hence \( \text{Int}(k_i S_i^\ast, S_i^\ast) = 0 \) in \( M \) for all \( i \), the intersection number of \( k_i \ell_i \) and \( S_i^\ast \) in \( V \) must be 0 for all \( i \). Using that \( H_1(V; Z) \) is free abelian, we obtain by Poincaré duality that \( k_i \ell_i \) and hence \( \ell_i \) are null-homologous in \( V \). Thus, the 1-manifold \( \ell_i \) bounds an oriented surface \( \Delta_i \) in \( V \). Let \( S_i^\ast \) be a closed (possibly disconnected)
oriented surface in $M \setminus V$ obtained from $S^1_i$ by cutting along $\ell_1$ and then adding parallel copies of $\Delta_1$ in a collar neighborhood of $V$ in $M$. Since $V$ is connected, the complement $M \setminus V$ is also connected. We can construct a closed connected oriented surface $S^1_{i^*}$ by piping the components of $S^1_i$ in the complement $M \setminus V$. Then we have $y_i = [S^1_{i^*}] \in H_2(M; \mathbb{Z})$. Since $S^1_{i^*}$ admits a trivial normal $D^2$-bundle $S^1_{i^*} \times D^2$ in $M$, we can take a connected sum of $V$ and $S^1_{i^*} \times \partial D^2$ to obtain a connected leaf $V'$ of $M$. Then $H_1(V'; Z)$ is a free abelian group. Let $H'$ be the image of the natural homomorphism $H_2(V'; Z) \to H_2(M; Z)$. By construction, we have $H' \subset G$ and

$$G/H' \cong \mathbb{Z}_{k_2} \oplus \mathbb{Z}_{k_3} \oplus \cdots \oplus \mathbb{Z}_{k_m}.$$ 

By continuing this process, we have a connected leaf $V^*$ with $H_1(V^*; Z)$ a free abelian group such that the image of the natural homomorphism $H_2(V^*; Z) \to H_2(M; Z)$ coincides with $G$. The exact sequence

$$0 \to H_2(M; Z)/G \to H_2(M, V^*; Z) \to H_1(V^*; Z)$$

induced from the homology exact sequence of $(M, V)$ implies that $H_2(M, V^*; Z)$ is a free abelian group. □

3. Proof of Lemma 1.3.

First, we show the following lemma.

**Lemma 3.1.** If the intersection form on $Z^H_1$-manifold $M$ is indefinite, then there is a pair of elements $x, y \in H_2(\tilde{M}; \mathbb{Z})$ such that the ordinary intersection numbers of $x, y$ in $\tilde{M}$ have

$$\text{Int}(x, x) = \text{Int}(y, y) = 0 \quad \text{and} \quad \text{Int}(x, y) = 1.$$ 

**Proof of Lemma 3.1.** For $\Lambda = \mathbb{Z}[t, t^{-1}]$, we consider the subring

$$\Lambda^+ = \left\{ \frac{f(t)}{g(t)} \in Q(\Lambda) \mid f(t), g(t) \in \Lambda \text{ with } g(1) = \pm 1 \right\}$$

of the quotient field $Q(\Lambda)$. For a $\Lambda$-module $H$, we denote the $\Lambda^+$-module $H \otimes_\Lambda \Lambda^+$ by $H^+$. The $\Lambda$-intersection number $\text{Int}_\Lambda(x, y)$ for $x, y \in H_2(\tilde{M}; \mathbb{Z})$ is defined by

$$\text{Int}_\Lambda(x, y) = \sum_{i=-\infty}^{+\infty} \text{Int}(x, t^{-i}y)t^i \in \Lambda,$$

which is extended to the $\Lambda^+$-intersection number $\text{Int}_{\Lambda^+}(x^+, y^+)$ for any elements $x^+, y^+$ of the second $\Lambda^+$-homology $H_2(\tilde{M}; Z)^+$. By [11, Lemma 3.6], we see that
the $\Lambda^+$-homology $H_2(\hat{M}; Z)^+$ of every $Z^{H_1}$-manifold $M$ is a $\Lambda^+$-free module with a $\Lambda^+$-basis $x_i^+ (i = 1, 2, \ldots, n)$ such that the $\Lambda^+$-intersection numbers $\text{Int}_{\Lambda^+}(x_i^+, x_j^+)$ are integers for all $i, j$. Since the intersection form $\text{Int} : H_2(M; Z) \times H_2(M; Z) \to Z$ is indefinite (i.e., there is a non-zero element $x_0 \in H_2(M; Z)$ with $\text{Int}(x_0, x_0) = 0$), we may assume that $\text{Int}_{\Lambda^+}(x_i^+, x_i^+) = 0$, $\text{Int}_{\Lambda^+}(x_i^+, x_j^+) = 1$, $\text{Int}_{\Lambda^+}(x_i^+, x_j^+) = 0$ $(i = 1, 2, j = 3, 4, \ldots, n)$. We take elements $g_i(t) \in \Lambda$ with $g_i(1) = 1$ $(i = 1, 2)$ such that the elements $x' = g_i(t)x_i^+$, $y' = g_2(t)x_2^+$ are in $H_2(\hat{M})$. Then we have

$$\text{Int}_{\Lambda^+}(x', x') = g_i(t^{-1})g_i(t)\text{Int}_{\Lambda^+}(x_i^+, x_i^+) = 0,$$

$$\text{Int}_{\Lambda^+}(x', y') = g_i(t^{-1})g_2(t)\text{Int}_{\Lambda^+}(x_i^+, x_2^+) = g_i(t^{-1})g_2(t).$$

Thus, if we let $x = \sum_{i=1}^{N} t^ix'$ for a large positive integer $N$, then we have

$$\text{Int}(x, x) = 0, \quad \text{and} \quad \text{Int}(x, y') = g_i(1)g_2(1) = 1.$$  

If $\text{Int}(y', y')$ is an even integer, say $2m$, then we take $y = y' - mx$ to obtain a desired pair $x, y$. Otherwise, we can assume that $\text{Int}(y', y') = 1$ by replacing $y'$ with $y' - mx$ when $\text{Int}(y', y') = 2m + 1$. The covering translated elements $z = t^n(x)$ and $w = t^n(y')$ for a large integer $n$ are represented by 2-cycles disjoint from 2-cycles represented by $x, y'$. Since $\text{Int}(z, z) = 0$ and $\text{Int}(z, w) = \text{Int}(w, w) = 1$, the pair of $x$ and $y = y' + z - w$ gives a desired pair. □

By the integral duality on the infinite cyclic covering $\hat{M}$ of a $Z^{H_1}$-manifold $M$ (see [6]), we have $H_3(\hat{M}; Z) \cong Z$, whose generator is called the fundamental class of the infinite cyclic covering $\hat{M} \to M$ (see [9]). By considering a closed oriented 3-manifold representing the fundamental class, we can complete the proof of Lemma 1.3.

**Completion of the proof of Lemma 1.3.** We shall find a 3-sphere $S^3$ in $\hat{M}$ representing a generator of $H_3(\hat{M}; Z)$. Then this 3-sphere $S^3$ is embedded in the $p$-fold cyclic covering $M^{(p)}$ of $M$ for a large $p$ so that $S^3$ represents a generator of $H_3(M^{(p)}; Z) \cong Z$ because the covering projection $\hat{M} \to M^{(p)}$ induces an isomorphism $H_3(\hat{M}; Z) \cong H_3(M^{(p)}; Z)$ by the Wang exact sequence. A generator of $H_1(M^{(p)}; Z) \cong Z$ is represented by an embedded circle $S^1$ in $M^{(p)}$ intersecting $S^3$ transversely in a single point. The regular neighborhood of the bouquet $S^1 \vee S^3$ in $M^{(p)}$ gives a connected summand $S^1 \times S^3$ of $M^{(p)}$, so that $M^{(p)}$ is TOP-split. Since $\hat{M}$ is a simply connected 4-manifold, the pair of $x, y$ in Lemma 3.1 is represented by a pair of 2-spheres in $M$ with geometric intersection number one by [2]. Thus, $\hat{M}$ has a connected summand $S^2 \times S^2$. Then we see that the $q$-fold cyclic covering $M^{(q)}$ of $M$ for a large $q$ has a connected summand of $S^2 \times S^2$. Let $M^{(q)} = M' \# kS^2 \times S^2$ for a $Z^{r_1}$-manifold $M'$. It turns out that the $Z^{r_1}$-manifold $M' \# kS^2 \times S^2$ for any
positive integer $k$ has the infinite cyclic covering (non-equivariantly) homeomorphic to $\tilde{\mathcal{M}}$. The connected sum $M' \# k S^2 \times S^2$ for a large $k$ is exact by [11, Theorem 2.4] and hence TOP-split by [10, Corollary 3.4]. We note that another proof of the TOP-splitting of the connected sum $M' \# k S^2 \times S^2$ for a large $k$ is known by Matumoto [12]. Then we can obtain a 3-sphere $S^3$ in $\tilde{\mathcal{M}}$ representing a generator of $H_3(\tilde{\mathcal{M}}; \mathbb{Z})$. $\square$

There is also another proof of Lemma 1.3. For the proof, let $M$ is a $\mathbb{Z}^{n-1}$-manifold with indefinite intersection form. Then we have the Witt index

$$w(M) = \frac{\beta_2(M; \mathbb{Z}) - |\text{sign}(M)|}{2} \geq 1.$$ 

Let $M'$ be the $d$-fold cyclic covering of $M$. By the covering properties of Euler characteristic and signature, we have

$$\beta_2(M'; \mathbb{Z}) = d \beta_2(M; \mathbb{Z}) \quad \text{and} \quad \text{sign}(M') = d \text{sign}(M),$$

so that $w(M') \geq d$. Thus, we have the Witt index $w(M') \geq 3$ for $d \geq 3$. Then Hambleton-Teichner in [4] observed that there is a $\Lambda$-basis $x'_i (i = 1, 2, \ldots, n')$ for the second homology $H_2(M'; \mathbb{Z})$ of the infinite cyclic covering $\tilde{M}'$ of $M'$ such that the $\Lambda$-intersection $\text{Int}_\Lambda(x'_i, x'_j)$ is an integer for all $i, j$. By [11], $M'$ is exact and hence by [10, Corollary 3.4] $M'$ is TOP-split, implying that $M$ is virtually TOP-split.

References


