Splitting a 4-manifold with infinite cyclic fundamental group, revised in a definite case

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ABSTRACT

A sufficient condition that a closed connected definite 4-manifold with infinite cyclic fundamental group is TOP-split is given. By this condition, it is shown that every closed connected definite smooth 4-manifold with infinite cyclic fundamental group is TOP-split. By combining with an earlier result, it is confirmed that every closed connected oriented smooth 4-manifold with infinite cyclic fundamental group is TOP-split. This also implies that every smooth sphere-knot in a closed simply connected smooth 4-manifold is topologically unknotted if the fundamental group of the complement is infinite cyclic.

Mathematics Subject Classification 2010: 57M10, 57M35, 57M50, 57N13

Keywords: Smooth 4-manifold, Definite intersection form, Topological splitting, Infinite cyclic covering, Topological unknotting.
1. Introduction

A closed connected oriented 4-manifold \( M \) is called a \( \mathbb{Z}^{\pi_1} \)-manifold if the fundamental group \( \pi_1(M) \) is isomorphic to \( \mathbb{Z} \), and a \( \mathbb{Z}^{H_1} \)-manifold if the first homology group \( H_1(M; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z} \). A \( \mathbb{Z}^{\pi_1} \)-manifold \( M \) is TOP-split if \( M \) is homeomorphic to the connected sum \( S^1 \times S^3 \# M_1 \) for a simply connected closed 4-manifold \( M_1 \), and virtually TOP-split if a finite covering of \( M \) is TOP-split. A \( \mathbb{Z}^{H_1} \)-manifold \( M \) is definite if the rank of the intersection form

\[
\text{Int}^M : H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \to \mathbb{Z}
\]

is equal to the absolute value of the signature, and positive definite if, furthermore, the signature is positive. A definite \( \mathbb{Z}^{H_1} \)-manifold is positive definite, if necessary, by changing an orientation of \( M \).

In this paper, a sufficient condition for a definite \( \mathbb{Z}^{\pi_1} \)-manifold to be TOP-split is given in a study following [10] of the revision of the author’s earlier paper [6] of a TOP-split \( \mathbb{Z}^{\pi_1} \)-manifold. This revision was needed because a non-TOP-split, positive definite and non-smoothable \( \mathbb{Z}^{\pi_1} \)-manifold was given by Hambleton-Teichner in [5] and Friedl, Hambleton, Melvin and Teichner in [4] (see also [7, 8, 9] for some discussions).

To explain our main result, some observations and terminologies are needed. It is not always assumed that a closed 4-manifold is a smooth or piecewise-linear manifold, but smooth and piecewise-linear techniques can be used for it because a punctured manifold of it is smoothable (see Freedman-Quinn [3]). Let \( X \) be a \( \mathbb{Z}^{H_1} \)-manifold, and \( V \) a leaf of \( X \). Let \( S \) be a closed oriented surface (embedded) in \( X \) lifting trivially to the infinite cyclic covering \( \tilde{X} \) of \( X \). Then we can assume that the intersection \( \mathcal{L} = S \cap V \) is a closed oriented possibly disconnected 1-manifold unless it is empty. Let \( D_i \) \((i = 1, 2, \ldots, r)\) be the connected regions of \( S \) divided by \( \mathcal{L} \). Let \( \alpha_{ij} \) be an oriented arc in \( S \) joining an interior point \( p_i \) of \( D_i \) to an interior point \( p_j \) of \( D_j \). The absolute value \( |\text{Int}^S(\alpha_{ij}, \mathcal{L})| \) of the intersection number \( \text{Int}^S(\alpha_{ij}, \mathcal{L}) \) is independent of any choices of \( p_i, p_j \) and \( \alpha_{ij} \), whose maximal number for all \( i, j \) is called the winding index of the surface \( S \) in \( X \) with respect to the leaf \( V \) and denoted by \( \delta(S, V; X) \).

Given a positive definite \( \mathbb{Z}^{\pi_1} \)-manifold \( M \), it is shown in [10] that the connected sum \( X = \mathbb{C}P^2 \# M \) is TOP-split because \( \text{sign}(\mathbb{C}P^2) = -1 \) and hence \( X \) is an indefinite \( \mathbb{Z}^{\pi_1} \)-manifold. Let \( S^3 \) be a 3-sphere leaf of \( X \). For the sphere \( \mathbb{C}P^1 \) in \( \mathbb{C}P^2 \), the winding index \( \delta(\mathbb{C}P^1, S^3; X) \) is simply called a winding index on \( M \). We note that there are infinitely many winding indexes on \( M \) by deforming the leaf \( S^3 \) in \( X \) isotopically. We shall show the following theorem:
Theorem 1.1. A definite $\mathbb{Z}^{\pi_1}$-manifold $M$ is TOP-split if for any given winding index $\delta$ on $M$ there is a $\mathbb{Z}$-basis $x_i$ ($i = 1, 2, \ldots, n$) of the second homology $H_2(M^{(m)}; \mathbb{Z})$ of an $m$-fold covering $M^{(m)}$ of $M$ with $m \geq \delta$ such that the intersection number $\text{Int}^{M^{(m)}}(x_i, x_i)$ has

$$1 \leq |\text{Int}^{M^{(m)}}(x_i, x_i)| \leq 2$$

for all $i$. In particular, a $\mathbb{Z}^{\pi_1}$-manifold $M$ is TOP-split if every finite covering of $M$ has an intersection matrix which is a block sum of copies of $E_8^1$ and/or (1).

If a positive definite $\mathbb{Z}^{\pi_1}$-manifold $M$ is smooth, then the intersection form $\text{Int}^{M^{(m)}}: H_2(M^{(m)}; \mathbb{Z}) \times H_2(M^{(m)}; \mathbb{Z}) \to \mathbb{Z}$ of the $m$-fold covering $M^{(m)}$ of $M$ for any $m > 0$ is standard by Donaldson’s theorem in [1], because $M^{(m)}$ is a positive definite $\mathbb{Z}^{\pi_1}$-manifold which is seen by using the Euler characteristic identity $\chi(M^{(m)}) = m\chi(M)$, the signature identity $\text{sign}(M^{(m)}) = m\text{sign}(M)$ and the Poincaré duality on $M^{(m)}$ and the intersection form $\text{Int}^{M^{(m)}}$ is isomorphic to the intersection form

$$\text{Int}^{M_1^{(m)}}: H_2(M_1^{(m)}; \mathbb{Z}) \times H_2(M_1^{(m)}; \mathbb{Z}) \to \mathbb{Z}$$

of a simply connected smooth 4-manifold $M_1^{(m)}$ obtained from $M^{(m)}$ by surgery killing $\pi_1(M^{(m)}) = Z$. Thus, there is a $Z$-basis $x_i$ ($i = 1, 2, \ldots, n$) of $H_2(M^{(m)}; \mathbb{Z})$ such that $\text{Int}^{M^{(m)}}(x_i, x_i) = 1$ for all $i$ and hence by Theorem 1.1 $M$ is TOP-split. Since it is shown in [10] that every indefinite $\mathbb{Z}^{\pi_1}$-manifold is TOP-split, we have the following corollary.

**Corollary 1.2.** Every smooth $\mathbb{Z}^{\pi_1}$-manifold is TOP-split.

This result answers affirmatively a question thought so by the author himself in [8, p.209] and also confirms affirmatively a conjecture given by Friedl, Hambleton, Melvin and Teichner in [4]. We note that there is a smooth $\mathbb{Z}^{\pi_1}$-manifold $M$ which is not diffeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for any simply connected smooth 4-manifold $M_1$ (see Fintushel and Stern [2]).

In terms of $S^2$-knot theory, Corollary 1.2 implies the following corollary by a similar reason as the proof of [10, Corollary 1.5].

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\footnote{This matrix is a square matrix $(a_{ij})$ of size 8 whose non-zero entries are given by $a_{ii} = 2$ ($i = 1, 2, \ldots, 8$), $a_{14} = a_{41} = 1$, and $a_{jj+1} = a_{j+1j} = 1$ ($j = 2, 3, \ldots, 7$).}
**Corollary 1.3.** Let $M_1$ be a closed simply connected smooth 4-manifold. A smooth $S^2$-knot $K$ in $M_1$ is topologically unknotted if we have one of the following two conditions:

1. The fundamental group $\pi_1(M_1 \setminus K)$ is isomorphic to $\mathbb{Z}$.
2. The $S^2$-knot $K$ is of Dehn’s type in $M_1$, namely if there is a map $f$ from the 3-disk $D^3$ to $M_1$ such that the image $f(\partial D^3) = K$ and the singular set $\Sigma(f) \subset \text{int} D^3$.

**2. Proof of Theorem 1.1**

We first observe the following lemma.

**Lemma 2.1.** Let $X$ be a $\mathbb{Z}^m$-manifold, $V$ a connected leaf of $X$ and $S$ a closed connected oriented surface in $X$ lifting trivially to the infinite cyclic covering $\tilde{X}$ of $X$. Let $V'$ and $S'$ be connected lifts of $V$ and $S$, respectively, to any $m$-fold covering $X^{(m)}$ of $X$ with $m \geq \delta(S, V; X)$. Then we have the winding index

$$0 \leq \delta(S', V'; X^{(m)}) \leq 1.$$

**Proof.** Let $X_V$ be a 4-manifold obtained from $X$ by cutting $X$ along $V$ whose boundary $\partial V$ is given as the disjoint union $V^+ \cup V^-$ of two copies $V^\pm$ of $V$. Then the $m$-fold covering $X^{(m)}$ of $X$ is obtained from the $m$ copies $X_i^V$ ($i = 1, 2, \ldots, m$) of $X_V$ by pasting $V^+_i$ to $V^{-}_{i+1}$ ($i = 1, 2, \ldots, m$) with $m+1 = 1$ for the $m$ copies $V^+_i \cup V^-_i$ ($i = 1, 2, \ldots, m$) of $V^+ \cup V^-$. For any $m \geq \delta(S, V; X)$, it is seen from the definition of a winding index that $0 \leq \delta(S', V'; X^{(m)}) \leq 1$. □

Throughout the remainder of this section, the proof of Theorem 1.1 is done. We assume that the $\mathbb{Z}^n_1$-manifold $M$ is positive definite by a choice of an orientation of $M$. It suffices to prove that $M$ is virtually TOP-split since “virtually TOP-split” implies “TOP-split” by [10]. For any given winding index $\delta$ on $M$, there is an $m$-fold covering $M^{(m)}$ of $M$ with $m \geq \delta$ such that there is a $\mathbb{Z}$-basis $x_i$ ($i = 1, 2, \ldots, n$) of $H_2(M^{(m)}; \mathbb{Z})$ with $1 \leq \text{Int}^{(m)}(x, x_i) \leq 2$ for all $i$. Since $\delta = \delta(\mathbb{CP}^1, S_1^3; X)$ for $X = \mathbb{CP}^2 \# M$, we see from Lemma 2.1 that any connected lift $S_1^3$ of the leaf $S_1^3$ of $X$ to the $m$-fold cyclic covering $X^{(m)} = (m\mathbb{CP}^2) \# M^{(m)}$ of $X$ has the winding index

$$0 \leq \delta(\mathbb{CP}^1_k, S_1^3; X^{(m)}) \leq 1$$

for all the connecting lifts $\mathbb{CP}^1_k$ ($k = 1, 2, \ldots, m$) of $\mathbb{CP}^1$ to $X^{(m)}$. Let $L_k = \mathbb{CP}^1_k \cap S_1^3$ ($k = 1, 2, \ldots, m$) be oriented links (possibly empty) with the orientations determined by the orientations of $\mathbb{CP}^1_k, S_1^3$ and $X^{(m)}$. Represent $x_i$ by a closed connected oriented
we can construct a compact connected oriented surface \( F \) of \( X \) by taking its connected oriented surface obtained from \( F \) trivially to \( X \). We show that the linking number \( \text{Link}(K, L_k) = 0 \) in \( S^3 \) for all \( k \). Suppose that \( \text{Link}(K_i, L_k) \neq 0 \) for some \( k \). By the fact that every meridian of the sphere \( \mathbb{CP}^1 \) in the complex projective plane bounds a disk disjointly parallel to \( \mathbb{CP}^1 \), we can construct a compact connected oriented surface \( D_i \) with \( \partial D_i = K_i \) in the union \( U \) of \( S^3 \) for all \( i \) and the connected summand \( m \mathbb{CP}^2 \) of \( X^{(m)} \) so that \( D_i \cap m \mathbb{CP}^1 = \emptyset \).

As an important note, the surface \( D_i \) has the intersection number \( -a_i \) with respect to a Seifert framing of \( K_i \) in \( S^3 \) for some \( a_i = \sum_{k=1}^{m} c_{i,k} > 0 \). The surface \( D_i \) is regarded as a surface in the \( S \)-manifold \( M^{(m)} \) obtained from \( X^{(m)} \) by replacing a normal disk bundle of \( m \mathbb{CP}^2 \) in \( m \mathbb{CP}^1 \) with the 4-disks \( m D^4 \). Let \( S_i \) be an immersed closed connected oriented surface obtained from \( F_i \) by a surgery along \( D_i \), namely take

\[
S_i = \text{cl}(F_i \setminus K_i \times [-1, 1]) \cup D_i^{-1} \cup D_i^{+1}
\]

for a collar \( K_i \times [-1, +1] \) of \( K_i \) in \( F_i \) and isotopically deformed surfaces \( D_i \) of \( D_i \) such that the boundaries \( \partial D_i^\pm \) are deformed into \( K_i \times \pm 1 \) through \( F_i \), respectively. Then we have \( [S_i] = [F_i] = x_i \neq 0 \) in \( H_2(M^{(m)}; \mathbb{Z}) \). Let \( S'_i \) be a connected lift of \( S_i \) to the double covering \( X^{(2m)} \) of \( X^{(m)} \) which is the 2-fold covering of \( X \). Since \( U \) lifts trivially to \( X^{(2m)} \), the self-intersection number of \( S'_i \) in \( X^{(2m)} \) is computed as follows:

\[
\text{Int}^{X^{(2m)}}([S'_i], [S'_i]) = \text{Int}^{X^{(m)}}(x_i, x_i) - 2a_i \leq 2 - 2a_i \leq 0.
\]

Since the surface \( S'_i \) is in the connected summand \( M^{(2m)} \) of \( X^{(2m)} \), we have

\[
\text{Int}^{M^{(2m)}}([S'_i], [S'_i]) \leq 0 \quad (i = 1, 2, \ldots, n).
\]

Since \( M^{(2m)} \) is a positive definite \( S \)-manifold, there is a real basis \( e_j \) \( (j = 1, 2, \ldots, 2n) \) of the real extension \( H_2(M^{(2m)}; \mathbb{R}) \) of the integral homology group \( H_2(M^{(2m)}; \mathbb{Z}) \) such that the real intersection number \( \text{Int}^{M^{(2m)}}(e_j, e_{j'}) = \delta_{jj'} \) (the Kronecker’s delta) for all \( j, j' \). The covering projection \( M^{(2m)} \to M^{(m)} \) induces a homomorphism \( H_2(M^{(2m)}; \mathbb{Z}) \to H_2(M^{(m)}; \mathbb{Z}) \) sending the homology class \( [S'_i] \) to \( x_i \neq 0 \), so that \( [S'_i] \in H_2(M^{(2m)}; \mathbb{R}) \) is written as

\[
[S'_i] = \sum_{j=1}^{2n} c_{ij} e_j
\]

with \( c_{ij} \neq 0 \) for some \( j \) and hence

\[
\text{Int}^{M^{(2m)}}([S'_i], [S'_i]) = \text{Int}^{M^{(2m)}}([S'_i], [S'_i]) = \sum_{j=1}^{2n} c_{ij}^2 > 0,
\]

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which is a contradiction. Thus, we have
\[ \text{Link}(K_i, L_k) = 0 \]
for all \( k \). Then \( K_i \) bounds a compact connected oriented surface \( D_i^* \) embedded in \( U_m \) such that \( D_i^* \cap m\mathbb{CP}^1 = \emptyset \) and \( D_i^* \) has the self-intersection number 0 with respect to a Seifert framing of \( K_i \) in \( S^3 \). The surface \( D_i^* \) is regarded as a surface in the \( Z \)-manifold \( M(m) \) obtained from \( X(m) \) by replacing a normal disk bundle of \( m\mathbb{CP}^1 \) in \( m\mathbb{CP}^2 \) with the 4-disks \( mD^4 \). Then a closed connected orientable surface \( S_i^* \) embedded in \( M(m) \) is obtained from \( F_i \) by a surgery along a collar \( D_i^* \times [-1, 1] \) of \( D_i^* \) with \( (D_i^* \times [-1, 1]) \cap F_i = K_i \times [-1, +1] \). This modification can be done for all \( i \). Since \( U_m \) lifts trivially to the infinite cyclic covering \( \tilde{X}(m) \) of \( X(m) \), we see that the \( Z \)-basis \( x_i = [S_i^*] \) \((i = 1, 2, \ldots, n)\) of \( H_2(M(m); Z) \) regarded as a direct summand of \( H_2(X(m); Z) \) lifts to a set of elements \( \tilde{x}_i = [\tilde{S}_i^*] \) \((i = 1, 2, \ldots, n)\) of \( H_2(\tilde{X}(m); Z) \) for a connected lift \( \tilde{S}_i^* \) of \( S_i^* \) to \( \tilde{X}(m) \) such that the \( \Lambda \)-intersection form
\[ \text{Int}^\Lambda_{\tilde{X}(m)}: H_2(\tilde{X}(m); Z) \times H_2(\tilde{X}(m); Z) \to \Lambda \]
has
\[ \text{Int}^\Lambda_{\tilde{X}(m)}(\tilde{x}_i, \tilde{x}_j) = \text{Int}^\Lambda_{X(m)}(x_i, x_j) \in Z \]
for all \( i, j \), where \( \Lambda = Z[Z] = Z[t, t^{-1}] \). Since the surfaces \( \tilde{S}_i^* \) \((i = 1, 2, \ldots, n)\) belong to the infinite cyclic covering \( \tilde{M}(m) \) of \( M(m) \), the elements \( \tilde{x}_i \) \((i = 1, 2, \ldots, n)\) are regarded as elements of \( H_2(\tilde{M}(m); Z) \) with
\[ \text{Int}^\Lambda_{\tilde{M}(m)}(\tilde{x}_i, \tilde{x}_j) = \text{Int}^\Lambda_{\tilde{X}(m)}(\tilde{x}_i, \tilde{x}_j) \]
for all \( i, j \). Using that \( H_2(\tilde{M}(m); Z) \) is a \( \Lambda \)-free module of rank \( n \), we see from the non-singularity of the \( \Lambda \)-intersection matrix \((\text{Int}^\Lambda_{\tilde{M}(m)}(\tilde{x}_i, \tilde{x}_j))\) that the elements \( \tilde{x}_i \) \((i = 1, 2, \ldots, n)\) form a \( \Lambda \)-basis for \( H_2(\tilde{M}(m); Z) \). This implies that the \( Z \)-manifold \( M(m) \) is exact (see \([8, 9]\)), so that by \([8, \text{Corollary 3.4}]\) \( M(m) \) is TOP-split. Thus, \( M \) is virtually TOP-split and hence by \([10]\) \( M \) is TOP-split. This completes the proof of Theorem 1.1.

References


