Splitting a 4-manifold with infinite cyclic fundamental group, revised in a definite case

Akio KAWAUCHI

Osaka City University Advanced Mathematical Institute

Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

kawauchi@sci.osaka-cu.ac.jp

ABSTRACT

A sufficient condition that a closed connected definite 4-manifold with infinite cyclic fundamental group is TOP-split is given. By this condition, it is shown that every closed connected definite smooth 4-manifold with infinite cyclic fundamental group is TOP-split. By combining with an earlier result, it is confirmed that every closed connected oriented smooth 4-manifold with infinite cyclic fundamental group is TOP-split. This also implies that every smooth sphere-knot in a closed simply connected smooth 4-manifold is topologically unknotted if the fundamental group of the complement is infinite cyclic.

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1. Introduction

A closed connected oriented 4-manifold M is called a Z^{π_1}*-manifold* if the fundamental group $\pi_1(M)$ is isomorphic to Z, and a Z^{H_1}*-manifold* if the first homology group $H_1(M;Z)$ is isomorphic to Z. A Z^{π_1}-manifold M is *TOP-split* if M is homeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for a simply connected closed 4-manifold M_1, and *virtually* TOP-split if a finite covering of M is TOP-split. A Z^{H_1}-manifold M is *definite* if the rank of the intersection form

$$\mathrm{Int}^M : H_2(M;Z) \times H_2(M;Z) \to Z$$

is equal to the absolute value of the signature, and *positive definite* if, furthermore, the signature is positive. A definite Z^{H_1}-manifold is positive definite, if necessary, by changing an orientation of M.

In this paper, a sufficient condition for a definite Z^{π_1}-manifold to be TOP-split is given in a study following [10] of the revision of the author's earlier paper [6] of a TOP-split Z^{π_1}-manifold. This revision was needed because a non-TOP-split, positive definite and non-smoothable Z^{π_1}-manifold was given by Hambleton-Teichner in [5] and Friedl, Hambleton, Melvin and Teichner in [4] (see also [7, 8, 9] for some discussions).

To explain our main result, some observations and terminologies are needed. It is not always assumed that a closed 4-manifold is a smooth or piecewise-linear manifold, but smooth and piecewise-linear techniques can be used for it because a punctured manifold of it is smoothable (see Freedman-Quinn [3]). Let X be a Z^{H_1}-manifold, and V a leaf of X. Let S be a closed oriented surface (embedded) in X lifting trivially to the infinite cyclic covering $\tilde{X}$ of X. Then we can assume that the intersection $L = S \cap V$ is a closed oriented possibly disconnected 1-manifold unless it is empty. Let D_i $(i = 1, 2, \ldots, r)$ be the connected regions of S divided by L. Let α_{ij} be an oriented arc in S joining an interior point p_i of D_i to an interior point p_j of D_j. The absolute value $|\mathrm{Int}^S(\alpha_{ij}, L)|$ of the intersection number $\mathrm{Int}^S(\alpha_{ij}, L)$ is independent of any choices of p_i, p_j and α_{ij}, whose maximal number for all i, j is called the *winding index* of the surface S in X with respect to the leaf V and denoted by $\delta(S, V; X)$. Given a positive definite Z^{π_1}-manifold M, it is shown in [10] that the connected sum $X = \overline{\mathbf{C}P}^2 \# M$ is TOP-split because $\mathrm{sign}(\overline{\mathbf{C}P}) = -1$ and hence X is an indefinite Z^{π_1}-manifold. Let S^3 be a 3-sphere leaf of X. For the sphere $\overline{\mathbf{C}P}^1$ in $\overline{\mathbf{C}P}^2$, the winding index $\delta(\overline{\mathbf{C}P}^1, S^3; X)$ is simply called a *winding index on M*. We note that there are infinitely many winding indexes on M by deforming the leaf S^3 in X isotopically. We shall show the following theorem:

Theorem 1.1. A definite Z^{π_1}-manifold M is TOP-split if for any given winding index δ on M there is a Z-basis x_i $(i = 1, 2, \ldots, n)$ of the second homology $H_2(M^{(m)}; Z)$ of an m-fold covering $M^{(m)}$ of M with $m \geqq \delta$ such that the intersection number $\mathrm{Int}^{M^{(m)}}(x_i, x_i)$ has

$$1 \leqq |\mathrm{Int}^{M^{(m)}}(x_i, x_i)| \leqq 2$$

for all i. In particular, a Z^{π_1}-manifold M is TOP-split if every finite covering of M has an intersection matrix which is a block sum of copies of E_8[1] and/or (1).

If a positive definite Z^{π_1}-manifold M is smooth, then the intersection form

$$\mathrm{Int}^{M^{(m)}} : H_2(M^{(m)}; Z) \times H_2(M^{(m)}; Z) \to Z$$

of the m-fold covering $M^{(m)}$ of M for any $m > 0$ is standard by Donaldson's theorem in [1], because $M^{(m)}$ is a positive definite Z^{π_1}-manifold which is seen by using the Euler characteristic identity $\chi(M^{(m)}) = m\chi(M)$, the signature identity $\mathrm{sign}(M^{(m)}) = m\mathrm{sign}(M)$ and the Poincaé duality on $M^{(m)}$ and the intersection form $\mathrm{Int}^{M^{(m)}}$ is isomorphic to the intersection form

$$\mathrm{Int}^{M_1^{(m)}} : H_2(M_1^{(m)}; Z) \times H_2(M_1^{(m)}; Z) \to Z$$

of a simply connected smooth 4-manifold $M_1^{(m)}$ obtained from $M^{(m)}$ by surgery killing $\pi_1(M^{(m)}) = Z$. Thus, there is a Z-basis x_i $(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)}; Z)$ such that $\mathrm{Int}^{M^{(m)}}(x_i, x_i) = 1$ for all i and hence by Theorem 1.1 M is TOP-split. Since it is shown in [10] that every indefinite Z^{π_1}-manifold is TOP-split, we have the following corollary.

Corollary 1.2. Every smooth Z^{π_1}-manifold is TOP-split.

This result answers affirmatively a question thought so by the author himself in [8, p.209] and also confirms affirmatively a conjecture given by Friedl, Hambleton, Melvin and Teichner in [4]. We note that there is a smooth Z^{π_1}-manifold M which is not diffeomorphic to the connected sum $S^1 \times S^3 \# M_1$ for any simply connected smooth 4-manifold M_1 (see Fintushel and Stern [2]).

In terms of S^2-knot theory, Corollary 1.2 implies the following corollary by a similar reason as the proof of [10, Corollary 1.5].

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[1] This matrix is a square matrix (a_{ij}) of size 8 whose non-zero entries are given by $a_{ii} = 2$ $(i = 1, 2, \ldots, 8)$, $a_{14} = a_{41} = 1$, and $a_{jj+1} = a_{j+1j} = 1$ $(j = 2, 3, \ldots, 7)$.

Corollary 1.3. Let M_1 be a closed simply connected smooth 4-manifold. A smooth S^2-knot K in M_1 is topologically unknotted if we have one of the following two conditions:

(1) The fundamental group $\pi_1(M_1 \backslash K)$ is isomorphic to Z.

(2) The S^2-knot K is of Dehn's type in M_1, namely if there is a map f from the 3-disk D^3 to M_1 such that the image $f(\partial D^3) = K$ and the singular set $\Sigma(f) \subset \mathrm{int} D^3$.

2. Proof of Theorem 1.1

We first observe the following lemma.

Lemma 2.1. Let X be a Z^{H_1}-manifold, V a connected leaf of X and S a closed connected oriented surface in X lifting trivially to the infinite cyclic covering $\tilde{X}$ of X. Let V' and S' be connected lifts of V and S, respectively, to any m-fold covering $X^{(m)}$ of X with $m \geqq \delta(S, V; X)$. Then we have the winding index

$$0 \leqq \delta(S', V'; X^{(m)}) \leqq 1.$$

Proof. Let X_V be a 4-manifold obtained from X by cutting X along V whose boundary ∂M_V is given as the disjoint union $V^+ \cup V^-$ of two copies $V^{\pm}$ of V. Then the m-fold covering $X^{(m)}$ of X is obtained from the m copies X_V^i $(i = 1, 2, \ldots, m)$ of X_V by pasting V_i^+ to V_{i+1}^- $(i = 1, 2, \ldots, m)$ with $m+1 = 1)$ for the m copies $V_i^+ \cup V_i^-$ $(i = 1, 2, \ldots, m$ $)$ of $V^+ \cup V^-$. For any $m \geqq \delta(S, V; X)$, it is seen from the definition of a winding index that $0 \leqq \delta(S', V'; X^{(m)}) \leqq 1$. □

Throughout the remainder of this section, the proof of Theorem 1.1 is done. We assume that the Z^{π_1}-manifold M is positive definite by a choice of an orientation of M. It suffices to prove that M is virtually TOP-split since "virtually TOP-split"implies "TOP-split"by [10]. For any given winding index δ on M, there is an m-fold covering $M^{(m)}$ of M with $m \geqq \delta$ such that there is a Z-basis x_i $(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)}; Z)$ with $1 \leqq \mathrm{Int}^{M^{(m)}}(x_i, x_i) \leqq 2$ for all i. Since $\delta = \delta(\overline{\mathbf{C}P}^1, S^3; X)$ for $X = \overline{\mathbf{C}P}^2 \# M$, we see from Lemma 2.1 that any connected lift S_1^3 of the leaf S^3 of X to the m-fold cyclic covering $X^{(m)} = (m\overline{\mathbf{C}P}^2) \# M^{(m)}$ of X has the winding index

$$0 \leqq \delta(\overline{\mathbf{C}P}_k^1, S_1^3; X^{(m)}) \leqq 1$$

for all the connecting lifts $\overline{\mathbf{C}P}_k^1$ $(k = 1, 2, \ldots, m)$ of $\overline{\mathbf{C}P}^1$ to $X^{(m)}$. Let $L_k = \overline{\mathbf{C}P}_k^1 \cap S_1^3$ $(k = 1, 2, \ldots, m)$ be oriented links (possibly empty) with the orientations determined by the orientations of $\overline{\mathbf{C}P}_k^1$, S_1^3 and $X^{(m)}$. Represent x_i by a closed connected oriented surface F_i (embedded) in the connected summand $M^{(m)}$ of $X^{(m)}$. The intersection $K_i = F_i \cap S^3_1$ is assumed to be an oriented knot (possibly empty), if necessary, by doing a surgery of F_i along 1-handles on F_i in a collar $S^3_1 \times [-1, 1]$ of S^3_1 in $X^{(m)}$. We show that the linking number $\mathrm{Link}(K_i, L_k) = 0$ in S^3_1 for all k. Suppose that $\mathrm{Link}(K_i, L_k) = \ell_{i,k} \neq 0$ for some k. By the fact that every meridian of the sphere $\overline{\mathbf{C}P}^1$ in the complex projective plane $\overline{\mathbf{C}P}^2$ bounds a disk disjointly parallel to $\overline{\mathbf{C}P}^1$, we can construct a compact connected oriented surface D_i with $\partial D_i = K_i$ in the union U_m of $S^3_1 \times [-1, 1]$ and the connected summand $m\overline{\mathbf{C}P}^2$ of $X^{(m)}$ so that $D_i \cap m\overline{\mathbf{C}P}^1 = \emptyset$. As an important note, the surface D_i has the intersection number $-a_i$ with respect to a Seifert framing of K_i in S^3_1 for $a_i = \sum_{k=1}^{m} \ell^2_{i,k} > 0$. The surface D_i is regarded as a surface in the Z^{π_1}-manifold $M^{(m)}$ obtained from $X^{(m)}$ by replacing a normal disk bundle of $m\overline{\mathbf{C}P}^1$ in $m\overline{\mathbf{C}P}^2$ with the 4-disks mD^4. Let S_i be an immersed closed connected oriented surface obtained from F_i by a surgery along D_i, namely take

$$S_i = \mathrm{cl}(F_i \backslash K_i \times [-1, 1]) \cup D_i^{-1} \cup D_i^{+1}$$

for a collar $K_i \times [-1, +1]$ of K_i in F_i and isotopically deformed surfaces $D_i^{\pm 1}$ of D_i such that the boundaries $\partial D_i^{\pm 1}$ are deformed into $K_i \times \pm 1$ through F_i, respectively. Then we have $[S_i] = [F_i] = x_i \neq 0$ in $H_2(M^{(m)}; Z)$. Let S'_i be a connected lift of S_i to the double covering $X^{(2m)}$ of $X^{(m)}$ which is the $2m$-fold covering of X. Since U_m lifts trivially to $X^{(2m)}$, the self-intersection number of S'_i in $X^{(2m)}$ is computed as follows:

$$\mathrm{Int}^{X^{(2m)}}([S'_i], [S'_i]) = \mathrm{Int}^{X^{(m)}}(x_i, x_i) - 2a_i \leqq 2 - 2a_i \leqq 0.$$

Since the surface S'_i is in the connected summand $M^{(2m)}$ of $X^{(2m)}$, we have

$$\mathrm{Int}^{M^{(2m)}}([S'_i], [S'_i]) \leqq 0 \quad (i = 1, 2, \dots, n).$$

Since $M^{(2m)}$ is a positive definite Z^{π_1}-manifold, there is a real basis e_j $(j = 1, 2, \dots, 2n)$ of the real extension $H_2(M^{(2m)}; \mathbf{R})$ of the integral homology group $H_2(M^{(2m)}; Z)$ such that the real intersection number $\mathrm{Int}^{M^{(2m)}}_{\mathbf{R}}(e_j, e_{j'}) = \delta_{jj'}$ (the Kronecker's delta) for all j, j'. The covering projection $M^{(2m)} \to M^{(m)}$ induces a homomorphism $H_2(M^{(2m)}; Z) \to H_2(M^{(m)}; Z)$ sending the homology class $[S'_i]$ to $x_i \neq 0$, so that $[S'_i] \in H_2(M^{(2m)}; \mathbf{R})$ is written as

$$[S'_i] = \sum_{j=1}^{2n} c_{ij} e_j$$

with $c_{ij} \neq 0$ for some j and hence

$$\mathrm{Int}^{M^{(2m)}}([S'_i], [S'_i]) = \mathrm{Int}^{M^{(2m)}}_{\mathbf{R}}([S'_i], [S'_i]) = \sum_{j=1}^{2n} c^2_{ij} > 0,$$

which is a contradiction. Thus, we have

$$\mathrm{Link}(K_i, L_k) = 0$$

for all k. Then K_i bounds a compact connected oriented surface D_i^* embedded in U_m such that $D_i^* \cap m\overline{\mathbf{C}P}^1 = \emptyset$ and D_i^* has the self-intersection number 0 with respect to a Seifert framing of K_i in S_1^3. The surface D_i^* is regarded as a surface in the Z^{π_1}-manifold $M^{(m)}$ obtained from $X^{(m)}$ by replacing a normal disk bundle of $m\overline{\mathbf{C}P}^1$ in $m\overline{\mathbf{C}P}^2$ with the 4-disks mD^4. Then a closed connected orientable surface S_i^* embedded in $M^{(m)}$ is obtained from F_i by a surgery along a collar $D_i^* \times [-1,1]$ of D_i^* with $(D_i^* \times [-1,1]) \cap F_i = K_i \times [-1,+1]$. This modification can be done for all i. Since U_m lifts trivially to the infinite cyclic covering $\tilde{X}^{(m)}$ of $X^{(m)}$, we see that the Z-basis $x_i = [S_i^*]$ $(i = 1, 2, \ldots, n)$ of $H_2(M^{(m)}; Z)$ regarded as a direct summand of $H_2(X^{(m)}; Z)$ lifts to a set of elements $\tilde{x}_i = [\tilde{S}_i^*]$ $(i = 1, 2, \ldots, n)$ of $H_2(\tilde{X}^{(m)}; Z)$ for a connected lift $\tilde{S}_i^*$ of S_i^* to $\tilde{X}^{(m)}$ such that the Λ-intersection form

$$\mathrm{Int}_\Lambda^{\tilde{X}^{(m)}} : H_2(\tilde{X}^{(m)}; Z) \times H_2(\tilde{X}^{(m)}; Z) \to \Lambda$$

has

$$\mathrm{Int}_\Lambda^{\tilde{X}^{(m)}}(\tilde{x}_i, \tilde{x}_j) = \mathrm{Int}^{X^{(m)}}(x_i, x_j) \in Z$$

for all i, j, where $\Lambda = Z[Z] = Z[t, t^{-1}]$. Since the surfaces $\tilde{S}_i^*$ $(i = 1, 2, \ldots, n)$ belong to the infinite cyclic covering $\tilde{M}^{(m)}$ of $M^{(m)}$, the elements $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ are regarded as elements of $H_2(\tilde{M}^{(m)}; Z)$ with

$$\mathrm{Int}_\Lambda^{\tilde{M}^{(m)}}(\tilde{x}_i, \tilde{x}_j) = \mathrm{Int}_\Lambda^{\tilde{X}^{(m)}}(\tilde{x}_i, \tilde{x}_j)$$

for all i, j. Using that $H_2(\tilde{M}^{(m)}; Z)$ is a Λ-free module of rank n, we see from the non-singularity of the Λ-intersection matrix $(\mathrm{Int}_\Lambda^{\tilde{M}^{(m)}}(\tilde{x}_i, \tilde{x}_j))$ that the elements $\tilde{x}_i$ $(i = 1, 2, \ldots, n)$ form a Λ-basis for $H_2(\tilde{M}^{(m)}; Z)$. This implies that the Z^{π_1}-manifold $M^{(m)}$ is exact (see [8, 9]), so that by [8, Corollary 3.4] $M^{(m)}$ is TOP-split. Thus, M is virtually TOP-split and hence by [10] M is TOP-split. This completes the proof of Theorem 1.1.

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