A tabulation of 3-manifolds via Dehn surgery

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Abstract

We show that every well-order of the set of lattice points induces an embedding from the set of closed connected orientable 3-manifolds into the set of links which is a right inverse of the 0-surgery map and this embedding induces further two embeddings from the set of closed connected orientable 3-manifolds into the well-ordered set of lattice points and into the set of link groups. In particular, the set of closed connected orientable 3-manifolds is a well-ordered set by a well-order inherited from the well-ordered set of lattice points and the homeomorphism problem on the 3-manifolds can be in principle replaced by the isomorphism problem on the link groups. To determine the embedding images of every 3-manifold, we propose a tabulation program on the well-ordered set of 3-manifolds which can be carried out inductively until a concrete pair of indistinguishable 3-manifolds occurs (if there is such a pair). As a demonstration, we tabulate 3-manifolds corresponding to the lattice points of lengths up to 7.

1. Introduction

There are two fundamental problems in the theory of 3-manifolds, that is, the homeomorphism problem and the classification problem (see J. Hempel [11, p.169]). The homeomorphism problem is the problem of giving an effective procedure for determining whether two given 3-manifolds are homeomorphic and the classification problem is the problem effectively generating a list containing exactly one 3-manifold from every (unoriented) type of 3-manifolds. In this paper, we consider the classification problem on closed connected orientable 3-manifolds by establishing an embedding from the set of closed connected orientable 3-manifolds into the set of links in the 3-sphere S^3 which is a right inverse of the 0-surgery map. For this purpose, let $\mathbb{Z}$ be the set of integers, and $\mathbb{Z}^n$ the product of n copies of $\mathbb{Z}$ whose elements $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ we will call *lattice points* of *length* $\ell(\mathbf{x}) = n$. The *set* $\mathbb{X}$ *of lattice points* is the disjoint union of $\mathbb{Z}^n$ for all $n = 1, 2, 3, \dots$. Let Ω be any well-order in $\mathbb{X}$, although we define in §2 the canonical order[1] Ω_c, a particular well-order in $\mathbb{X}$ such that we have $\mathbf{x} < \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ with $\ell(\mathbf{x}) < \ell(\mathbf{y})$. We are particularly interested in the delta set Δ, a special subset of $\mathbb{X}$ defined in §3 such that the lattice points of Δ smaller than any given $\mathbf{x} \in \mathbb{X}$ in Ω_c form a finite set. The class of oriented links L' in S^3 such that there is a homeomorphism $h : S^3 \to S^3$ sending L to L' is called the *unoriented link type* $[L]$ of an oriented link

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[1]The present definition is modified from the definition made in earlier research announcements to make an enumeration of lattice points easier.

L in S^3, and the *oriented link type* $\langle L\rangle$ of L if moreover h preserves the orientation of S^3 and the orientations of L and L'. Let $\mathbb{L}$ and $\overrightarrow{\mathbb{L}}$ be the sets of unoriented link types and oriented link types in S^3, respectively. *A link type will be identified with a link belonging to the link type unless confusion might occur.* Thus, $\mathbb{L}$ and $\overrightarrow{\mathbb{L}}$ are understood as the sets of unoriented links and oriented links in S^3, respectively. We have a canonical surjection

$$\mathrm{cl}\beta_\iota : \mathbb{X} \xrightarrow{\mathrm{cl}\beta} \overrightarrow{\mathbb{L}} \xrightarrow{\iota} \mathbb{L}$$

sending a lattice point to the closure of the associated braid (see §2 for details), where $\iota : \overrightarrow{\mathbb{L}} \to \mathbb{L}$ denotes the forgetful surjection, which simply ignores the orientations of S^3 and links. On the other hand, every well-order Ω in $\mathbb{X}$ induces an injection

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X}$$

which is a right inverse of $\mathrm{cl}\beta_\iota$, so that Ω defines a well-order in $\mathbb{L}$, also denoted by Ω. This construction of σ is done in §2. In §3, we show that in the case of $\Omega = \Omega_c$ the image $\sigma(L)$ of a prime link L belongs to Δ. In §4, we define the concept of a π-minimal link (depending on a choice of a well-order Ω in $\mathbb{X}$). Let $\mathbb{L}^\pi$ be the subset of $\mathbb{L}$ consisting of π-minimal links. Then we see that the restriction

$$\sigma|_{\mathbb{L}^\pi} : \mathbb{L}^\pi \longrightarrow \mathbb{X}$$

is an embedding (see Lemma 4.4). Since a π-minimal link is a prime link by definition, we see in the case of $\Omega = \Omega_c$ that $\sigma(\mathbb{L}^\pi) \subset \Delta$ and every initial segment of $\mathbb{L}^\pi$ is a finite set. The *link group* of a link L in S^3 is the fundamental group $\pi_1 E(L)$ of the exterior $E(L) = \mathrm{cl}(S^3 - N(L))$ of L with $N(L)$ a tubular neighborhood of L in S^3. Let $\mathbb{G}$ be the set of the isomorphism types of the link groups for links in $\mathbb{L}$. *The isomorphism type of a group will be identified with a group belonging to the isomorphism type unless confusion might occur.* An *Artin presentation* is a finite group presentation

$$(x_1, x_2, \dots, x_n \,|\, x_i = w_i x_{p(i)} w_i^{-1}, i = 1, 2, \dots, n)$$

where $p(1), p(2), \dots, p(n)$ are a permutation of $1, 2, \dots, n$ and w_i $(i = 1, 2, \dots, n)$ are words in $x_1, x_2, \dots, x_n$ which satisfy the identity

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} w_i x_{p(i)} w_i^{-1}$$

in the free group F on the letters $x_1, x_2, \dots, x_n$. Then we have a braid $b \in B_n$ corresponding to the automorphism φ of F defined by

$$\varphi(x_i) = w_i x_{p(i)} w_i^{-1} \quad (i = 1, 2, \dots, n),$$

from which we see that the set $\mathbb{G}$ is characterized as the set of groups with Artin presentation (see for example [15; p.83] as well as J. S. Birman [2;p.46]). If the closure $\mathrm{cl}(b)$ is prime or π-minimal, then we say that the Artin presentation is *prime* or *π-minimal*, respectively. For the map

$$\pi : \mathbb{L} \longrightarrow \mathbb{G}$$

sending every link L to the link group $\pi_1 E(L)$, we also see that the restriction

$$\pi|_{\mathbb{L}^\pi} : \mathbb{L}^\pi \longrightarrow \mathbb{G}$$

is an embedding (see Lemma (4.4)). Let $\mathbb{M}$ and $\vec{\mathbb{M}}$ be the sets of unoriented types and oriented types of closed connected oriented 3-manifolds, respectively. *The type of a closed connected oriented 3-manifold will be identified with a 3-manifold belonging to the type unless confusion might occur.* We define the map $\chi_0 : \mathbb{L} \to \mathbb{M}$ by $\chi_0(L) = \chi(L, 0)$, where $\chi(L, 0)$ denotes the 0-surgery manifold of L. The following result is our main theorem which is proved in §5:

Theorem (1.1) Every well-order Ω of $\mathbb{X}$ induces an embedding

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}^\pi \subset \mathbb{L}$$

and hence two embeddings

$$\sigma_\alpha = \sigma\alpha : \mathbb{M} \longrightarrow \mathbb{X},$$
$$\pi_\alpha = \pi\alpha : \mathbb{M} \longrightarrow \mathbb{G}$$

which satisfy properties (1) and (2) below:

(1) $\chi_0 \alpha = 1$.
(2) If a lattice point $\sigma_\alpha(M) \in \mathbb{X}$ is given, then the π-minimal link $\alpha(M) \in \mathbb{L}$ with a braid presentation, the 3-manifold $M \in \mathbb{M}$ with a 0-surgery description along a π-minimal link and the link group $\pi_\alpha(M) \in \mathbb{G}$ with a π-minimal Artin presentation are determined.

Furthermore, when $\Omega = \Omega_c$, we have $\sigma_\alpha(\mathbb{M}) \subset \Delta$ and the properties (3) and (4) below are obtained:

(3) If a group $\pi_\alpha(M)$ with a prime Artin presentation is given, then the lattice point $\sigma_\alpha(M)$ is determined assuming a solution of the following problem:

Problem. Let $\mathbf{x} \in \mathbb{X}$ be a lattice point induced from the prime Artin presentation of $\pi_\alpha(M)$, and $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_n$ the lattice points in Δ smaller than or equal to $\mathbf{x}$. Then find the smallest index i such that the link $\mathrm{cl}\beta(\mathbf{x}_i)$ is prime and there is an isomorphism $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}_i)) \to \pi_\alpha(M)$.

(4) If a 3-manifold M with the 0-surgery description along a π-minimal link L is given, then the lattice point $\sigma_\alpha(M)$ is determined assuming a solution of the following problem:

Problem. Let $\mathbf{x} \in \mathbb{X}$ be a lattice point induced from a π-minimal link L, and $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_n$ the lattice points in Δ smaller than or equal to $\mathbf{x}$. Then find the smallest index i such that the link $\mathrm{cl}\beta(\mathbf{x}_i)$ is π-minimal and the 0-surgery manifold $\chi(\mathrm{cl}\beta(\mathbf{x}_i), 0)$ is $\chi(L, 0)$.

The embedding σ_α makes the set $\mathbb{M}$ a well-ordered set by a well-order, inherited from the well-order Ω of $\mathbb{L}$ and denoted also by Ω. The *length* of a 3-manifold $M \in \mathbb{M}$ is the length of the lattice point $\sigma_\alpha(M) \in \mathbb{X}$. In §6, to determine the images $\alpha(M)$, $\sigma_\alpha(M)$ and $\pi_\alpha(M)$ of every $M \in \mathbb{M}$, we take the canonical order Ω_c and propose a classification program on $\mathbb{M}$ based on Theorem 1.1 which we can carry out inductively until a concrete pair of indistinguishable 3-manifolds occurs (if there is such a pair). As a demonstration, we carry out this classification for 3-manifolds with lengths up to 7. The embedding π_α implies that *two 3-manifolds $M_i \in \mathbb{M}$ $(i = 1, 2)$ are homeomorphic if and only if the groups $\pi_\alpha(M_i)$ $(i = 1, 2)$ are isomorphic*, and thus the homeomorphism problem on $\mathbb{M}$ can be in principle replaced by the isomorphism problem on $\mathbb{G}$ (see Remark (5.5)), although it appears difficult to calculate the group $\pi_\alpha(M)$ of any given 3-manifold $M \in \mathbb{M}$ apart from the classification program. A lifting of the embedding α to the oriented version is discussed in §7 together with an observation on a relationship between oriented 3-manifold invariants and oriented link invariants.

This paper is a grow up version of a part of the research announcement "*Link corresponding to closed 3-manifold*". A version of the remaining part will appear in [16] (see http://www.sci.osaka-cu.ac.jp/~kawauchi/index.htm). The author is grateful to Dr. Ikuo Tayama for finding errors from an earlier version of this paper and to the referees for finding further errors and for helpful comments.

2. Representing links in the set of lattice points

For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of length n, we denote the lattice points $(x_n, \ldots, x_2, x_1)$ and $(|x_1|, |x_2|, \ldots, |x_n|)$ by $\mathbf{x}^T$ and $|\mathbf{x}|$, respectively. Let $|\mathbf{x}|_N$ be a permutation $(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ of the coordinates $|x_j|$ $(j = 1, 2, \ldots, n)$ of $|\mathbf{x}|$ such that $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$. For convenience, we use k^n for the lattice point of length n with k for every coordinate and $-k^n$ for $(-k)^n$. The integers $\min_{1 \leqq i \leqq n} |x_i|$ and $\max_{1 \leqq i \leqq n} |x_i|$ are also denoted by $\min |\mathbf{x}|$ and $\max |\mathbf{x}|$, respectively. Further, we define the *dual* lattice point $\delta(\mathbf{x}) = (x'_1, x'_2, \ldots, x'_n)$ of $\mathbf{x}$ by

$$x'_i = \begin{cases} \mathrm{sign}(x_i)(\max |\mathbf{x}| + 1 - |x_i|) & x_i \neq 0 \\ 0 & x_i = 0. \end{cases}$$

Defining $\delta^0(\mathbf{x}) = \mathbf{x}$ and $\delta^n(\mathbf{x}) = \delta(\delta^{n-1}(\mathbf{x}))$ inductively, we note that $\delta^2(\mathbf{x}) \neq \mathbf{x}$ in general, but $\delta^{n+2}(\mathbf{x}) = \delta^n(\mathbf{x})$ for all $n \geqq 1$. For example, taking $\mathbf{x} = (2^3, 3, -2, 3)$, we have $\delta^{2m-1}(\mathbf{x}) = (2^3, 1, -2, 1)$ and $\delta^{2m}(\mathbf{x}) = (1^3, 2, -1, 2)$ for all $m \geqq 1$. For a lattice point $\mathbf{y} = (y_1,\ y_2, \ldots, y_m)$ of length m, we denote by $(\mathbf{x}, \mathbf{y})$ the lattice point $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ of length $n+m$. Let $\vec{\mathbb{L}}$ be the set of oriented links. By the Alexander theorem (see J. S. Birman [2]), every oriented link L is represented by the closure $\mathrm{cl}(b)$ of an s-string braid $b \in B_s$ for some $s \geqq 1$. The braiding algorithm of S. Yamada [23] would be useful to deform a link into a closed braid form. Let σ_i $(i = 1, 2, \ldots, s-1)$ be the standard generators of the s-string braid group B_s. By convention, we regard the sign of the crossing point of the diagram σ_i as $+1$. We consider that every braid b in B_s is written as a word on the letters σ_i $(i = 1, 2, \ldots, s-1)$. When b is not written as 1, we write

$$b = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_r}^{\epsilon_r}, \quad \epsilon_i = \pm 1 \quad (i = 1, 2, \ldots, r).$$

Then we define the *lattice point* $\mathbf{x}(b)$ *of the braid* b by the identity

$$\mathbf{x}(b) = (\epsilon_1 i_1, \epsilon_2 i_2, \ldots, \epsilon_r i_r) \in \mathbb{Z}^r \subset \mathbb{X}.$$

When b is written as 1, we understand that $\mathbf{x}(b) = 0 \in \mathbb{Z} \subset \mathbb{X}$. For a non-zero lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{X}$, let x_{i_j} $(j = 1, 2, \ldots, m; i_1 < i_2 < \cdots < i_m)$ be the set of the non-zero integers in the coordinates x_i $(i = 1, 2, \ldots, n)$ of $\mathbf{x}$. Then the lattice point $\tilde{\mathbf{x}} = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ is called the *core* of $\mathbf{x}$. When $\mathbf{x}$ is a zero lattice point, we understand the core $\tilde{\mathbf{x}} = 0$. We note that for every non-zero lattice point $\mathbf{x}$, there is a unique braid $b \in B_s$ for every $s \geqq \max|\mathbf{x}| + 1$ such that $\mathbf{x}(b) = \tilde{\mathbf{x}}$. The braid b is called *the associated braid with index* s of $\mathbf{x}$ and denoted by $\beta^{(s)}(\mathbf{x})$, and in particular for $s = \max|\mathbf{x}| + 1$, called *the associated braid* of $\mathbf{x}$ and denoted by $\beta(\mathbf{x})$. The associated braid with index s of any zero lattice point of $\mathbb{X}$ is understood as $1 \in B_s$, and in particular the associated braid as $1 \in B_1$. Taking the closure $\mathrm{cl}\beta(\mathbf{x})$ of the braid $\beta(\mathbf{x})$, we obtain a surjection

$$\mathrm{cl}\beta : \mathbb{X} \longrightarrow \vec{\mathbb{L}}.$$

Then every well-order Ω in $\mathbb{X}$ defines an injection (which is a right inverse of the map $\mathrm{cl}\beta$)

$$\vec{\sigma} : \mathbb{L} \longrightarrow \mathbb{X}$$

by sending to a link L to the initial element of the subset $\{\mathbf{x} \in \mathbb{X} | \mathrm{cl}\beta(\mathbf{x}) = L\}$ of $\mathbb{X}$ indicated by Ω. By definition, the closed braid $\mathrm{cl}\beta^{(s)}(\mathbf{x})$ with $s > \max|\mathbf{x}| + 1$ is obtained from the closed braid $\mathrm{cl}\beta(\mathbf{x})$ by adding a trivial link of $(s - \max|\mathbf{x}| - 1)$ components. We introduce an equivalence relation $\sim$ in $\mathbb{X}$ as follows:

Definition (2.1) Two lattice points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{X}$ are related as $\mathbf{x} \sim \mathbf{y}$ if we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links.

Clearly the relation $\sim$ is an equivalence relation in $\mathbb{X}$. Let $\mathbb{X}/\sim$ be the quotient set of $\mathbb{X}$ by $\sim$, and $\langle \mathbf{x} \rangle$ the equivalence class of a lattice point $\mathbf{x} \in \mathbb{X}$ by $\sim$. The quotient map

$$\vec{\sigma}_\sim : \vec{\mathbb{L}} \longrightarrow \mathbb{X}/\sim$$

has the identity $\vec{\sigma}_\sim(\mathrm{cl}(b)) = \langle \mathbf{x}(b) \rangle$ and is a bijection from the quotient set of $\vec{\mathbb{L}}$ modulo split additions of trivial links onto $\mathbb{X}/\sim$. In particular, $\vec{\sigma}_\sim$ is independent of a choice of Ω. We can describe the equivalence relation $\sim$ only in terms of $\mathbb{X}$ by using the braid group presentation and the Markov theorem (see J. S. Birman [2]), as stated in the following lemma:

Lemma (2.2) The relations (1)-(6) below are in the equivalence relation $\sim$ in $\mathbb{X}$. Concersely, if we have $\mathbf{x} \sim \mathbf{y}$ in $\mathbb{X}$, then $\mathbf{y}$ is obtained from $\mathbf{x}$ by applying the relations (1)-(6) finitely often.

(1) $(\mathbf{x}, 0) \sim \mathbf{x},\ \mathbf{x} \sim (\mathbf{x}, 0)$ for all $\mathbf{x} \in \mathbb{X}$,
(2) $(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) \sim \mathbf{x},\ \mathbf{x} \sim (\mathbf{x}, \mathbf{y}, -\mathbf{y}^T)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}$,
(3) $(\mathbf{x}, y) \sim \mathbf{x},\ \mathbf{x} \sim (\mathbf{x}, y)$ for all $\mathbf{x} \in \mathbb{X}$ and $y \in \mathbb{Z}$ such that $|y| > \max|\mathbf{x}|$,
(4) $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \sim (\mathbf{x}, \mathbf{z}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{X}$ such that $\min|\mathbf{y}| > \max|\mathbf{z}| + 1$ or $\min|\mathbf{z}| > \max|\mathbf{y}| + 1$,
(5) $(\mathbf{x}, \varepsilon y, y+1, y) \sim (\mathbf{x}, y+1, y, \varepsilon(y+1))$ for all $\mathbf{x} \in \mathbb{X}$ and $y \in \mathbb{Z}$ such that $y(y+1) \neq 0$ and $\varepsilon = \pm 1$,
(6) $(\mathbf{x}, \mathbf{y}) \sim (\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}$.

Proof. The relation (1) is in $\sim$ since $\beta(\mathbf{x}, 0) = \beta(\mathbf{x})$. For (2), we take $\beta^{(s)}(\mathbf{x})$ and $\beta^{(s)}(\mathbf{y})$ in B_s for some s. Then we have

$$\beta^{(s)}(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{y})^{-1} = \beta^{(s)}(\mathbf{x})$$

in B_s and hence

$$\mathrm{cl}\beta(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) = \mathrm{cl}\beta(\mathbf{x})$$

in $\vec{\mathbb{L}}$ modulo split additions of trivial links, showing that the relation (2) is in $\sim$. For (3), let $s = |y| + 1$. Then by the Markov theorem,

$$\mathrm{cl}\beta(\mathbf{x}, y) = \mathrm{cl}\beta^{(s)}(\mathbf{x})$$

in $\vec{\mathbb{L}}$ and the last link is equal to $\mathrm{cl}\beta(\mathbf{x})$ modulo split additions of trivial links, showing that the relation (3) is in $\sim$. For (4), we take $\beta^{(s)}(\mathbf{x})$, $\beta^{(s)}(\mathbf{y})$ and $\beta^{(s)}(\mathbf{z})$ in B_s for some s. By the assumption on $\mathbf{y}$ and $\mathbf{z}$, we have

$$\beta^{(s)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{z}) = \beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{z})\beta^{(s)}(\mathbf{y}) = \beta^{(s)}(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

in B_s which shows that

$$\mathrm{cl}\beta(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathrm{cl}\beta(\mathbf{x}, \mathbf{z}, \mathbf{y})$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links. Thus, the relation (4) is in $\sim$. For (5), consider $\beta^{(s)}(\mathbf{x})$ and $\sigma_j^{\varepsilon'}$ ($j = |y|$, $\varepsilon' = \mathrm{sign}(y)$) in B_s for some s. Let $\varepsilon' = +1$. Then

$$\beta^{(s)}(\mathbf{x}, \varepsilon y, y+1, y) = \beta^{(s)}(\mathbf{x})\sigma_j^{\varepsilon}\sigma_{j+1}\sigma_j$$

and the last braid is equal to

$$\beta^{(s)}(\mathbf{x})\sigma_{j+1}\sigma_j\sigma_{j+1}^{\varepsilon} = \beta^{(s)}(\mathbf{x}, y+1, y, \varepsilon(y+1))$$

in B_s by a well-known braid relation. Hence we have

$$\mathrm{cl}\beta(\mathbf{x}, \varepsilon y, y+1, y) = \mathrm{cl}\beta(\mathbf{x}, y+1, y, \varepsilon(y+1))$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links, showing that the relation (5) is in $\sim$. For $\varepsilon' = -1$, a similar argument gives the desired result since $\mathrm{sign}(y+1) = -1$ by assumption. For (6), let $\beta^{(s)}(\mathbf{x})$ and $\beta^{(s)}(\mathbf{y})$ in B_s for some s. Then we have

$$\mathrm{cl}\beta^{(s)}(\mathbf{x})\beta^{(s)}(\mathbf{y}) = \mathrm{cl}\beta^{(s)}(\mathbf{y})\beta^{(s)}(\mathbf{x})$$

by the Markov theorem and hence

$$\mathrm{cl}\beta(\mathbf{x}, \mathbf{y}) = \mathrm{cl}\beta(\mathbf{y}, \mathbf{x})$$

in $\overrightarrow{\mathbb{L}}$ modulo split additions of trivial links, showing that the relation (6) is in $\sim$.

Next, we assume $\mathbf{x} \sim \mathbf{y}$. By the relations (1) and (6), we assume $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{y}} = \mathbf{y}$. Let $b = \beta(\mathbf{x})$ and $b' = \beta(\mathbf{y})$ be the associated braids. We show that if $b = b'$ in B_s for an index s, then we can change $\mathbf{x}$ into $\mathbf{y}$ by finitely many applications of the relations (2), (4), (5) and (6). We use the group presentation of B_s with generators σ_i ($i = 1, 2, \ldots, s-1$) and relators

$$\text{(i) } \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \ (|i-j| \geqq 2) \quad \text{and} \quad \text{(ii) } \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \ (1 \leqq i \leqq s-2)$$

(see [2]). Let F be the free group on the letters σ_i ($i = 1, 2, \ldots, s-1$). If $b = b'$ in F, then the solution of the word problem on F guarantees us to change $\mathbf{x}$ into $\mathbf{y}$ by finitely many applications of the relations (2) and (6). If $b = b'$ in B_s, then the word $b(b')^{-1}$ is written in the form

$$b(b')^{-1} = \prod_{k=1}^{n} R_k^{\varepsilon_k W_k}$$

in F, where $R_k^{\varepsilon_k W_k} = W_k R_k^{\varepsilon_k} W_k^{-1}$ for $\varepsilon_k = \pm 1$ and R_k denotes a relator of the type (i) or (ii) and W_k is a word in F written on the letters σ_i ($i = 1, 2, \ldots, s-1$). Thus, $(\mathbf{x}, -\mathbf{y}^T)$ is changed into

$$\mathbf{a} = (\mathbf{x}(R_1^{\varepsilon_1 W_1}), \mathbf{x}(R_2^{\varepsilon_2 W_2}), \ldots, \mathbf{x}(R_n^{\varepsilon_n W_n}))$$

by finitely many applications of the relations (2) and (6). Since we can change $\mathbf{x}$ into $(\mathbf{x}, -\mathbf{y}^T, \mathbf{y}) = (\mathbf{a}, \mathbf{y})$ by the relation (2), we may consider $b(b')^{-1}b' = \beta(\mathbf{a}, \mathbf{y})$ instead of $b = \beta(\mathbf{x})$. We note that

$$\mathbf{x}(R_k) = (i, j, -i, -j), \quad \mathbf{x}(R_k^{-1}) = (j, i, -j, -i)$$

for the relator (i) and

$$\begin{aligned} \mathbf{x}(R_k) &= (i, i+1, i, -(i+1), -i, -(i+1)), \\ \mathbf{x}(R_k^{-1}) &= (i+1, i, i+1, -i, -(i+1), -i) \end{aligned}$$

for the relator (ii). Since

$$\mathbf{x}(R_k^{\varepsilon_k W_k}) = (\mathbf{x}(W_k), \mathbf{x}(R_k^{\varepsilon_k}), -\mathbf{x}(W_k)^T),$$

we see that $(\mathbf{a}, \mathbf{y})$ is changed into $\mathbf{y}$ by finitely many applications of the relations (2), (4), (5) and (6). Thus, in the case that $b = b'$ in B_s for an index s, we showed that $\mathbf{x}$ can be changed into $\mathbf{y}$ by finitely many applications of the relations (2), (4), (5) and (6).

Now we consider the general case of b and b'. Applying the relation (3) to $\mathbf{x}$ or $\mathbf{y}$, we can assume that $\mathrm{cl}(b) = \mathrm{cl}(b')$ in $\vec{\mathbb{L}}$. Then the Markov theorem says that we have $b = b'$ in B_s with a suitable index s after finitely many applications of the Markov equivalences:

$$\begin{aligned} b_1 b_2 &\leftrightarrow b_2 b_1 \quad (b_1, b_2 \in B_m), \\ b\sigma_m^{\pm 1} &\leftrightarrow b \quad (b \in B_m \subset B_{m+1}) \end{aligned}$$

for any m. This is equivalent to saying that $b = b' \in B_s$ after finitely many applications of the relations (3) and (6) besides the relations (2), (4), (5) and (6) to $\mathbf{x}$ and $\mathbf{y}$. Thus, $\mathbf{x}$ is changed into $\mathbf{y}$ by finitely many applications of the relations (2), (3), (4), (5) and (6). □

Composing the forgetful surjection $\iota : \vec{\mathbb{L}} \to \mathbb{L}$ to the map $\mathrm{cl}\beta$, we obtain a canonical surjection

$$\mathrm{cl}\beta_\iota : \mathbb{X} \to \mathbb{L}$$

and an injection which is a right inverse of $\mathrm{cl}\beta_\iota$

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X}$$

sending an unoriented link L to the initial element of the subset $\{\mathbf{x} | \mathrm{cl}\beta_\iota(\mathbf{x}) = L\}$ of $\mathbb{X}$ indicated by Ω. The *length* of a link $L \in \mathbb{L}$ is the length of the lattice point $\sigma(L)$. By the rule that $L_1 < L_2$ if and only if $\sigma(L_1) < \sigma(L_2)$, a well-order in $\mathbb{L}$ is defined. Since the map σ is induced from Ω, we may say that this well-order in $\mathbb{L}$ is *induced from* Ω and denoted also by Ω. We also introduce an equivalence relation $\approx$ in $\mathbb{X}$ more relaxed than $\sim$.

Definition (2.3) Two lattice points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{X}$ are related as $\mathbf{x} \approx \mathbf{y}$ if we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\mathbb{L}$ modulo split additions of trivial links.

It is straightforward to see that the relation $\approx$ is an equivalence relation in $\mathbb{X}$. The quotient map

$$\sigma_{\approx} : \mathbb{L} \longrightarrow \mathbb{X}/\approx$$

is independent of a choice of Ω and induces a bijection from the quotient set of $\mathbb{L}$ modulo split additions of trivial links onto $\mathbb{X}/\approx$. For the natural surjection $\mathbb{X}/\sim \to \mathbb{X}/\approx$ also denoted by ι, we have the following commutative square:

$$\begin{array}{ccc} \vec{\mathbb{L}} & \xrightarrow{\vec{\sigma}_{\sim}} & \mathbb{X}/\sim \\ \iota\downarrow & & \downarrow\iota \\ \mathbb{L} & \xrightarrow{\sigma_{\approx}} & \mathbb{X}/\approx . \end{array}$$

In this diagram, we denote $\iota\langle\mathbf{x}\rangle$ by $[\mathbf{x}]$. Then we have the identity $\sigma_{\approx}(\mathrm{cl}(b)) = [\mathbf{x}(b)]$. To determine the class $[\mathbf{x}] \in \mathbb{X}/\approx$, it is desired to describe the equivalence relation $\approx$ only in terms of $\mathbb{X}$. At present, only what we can say about $\approx$ is the following lemma:

Lemma (2.4) We have the following (1) and (2):

(1) For any $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ such that $\mathbf{x} \sim \mathbf{y}$, we have $\mathbf{x} \approx \mathbf{y}$.
(2) For all $\mathbf{x} \in \mathbb{X}$, we have $\mathbf{x} \approx \mathbf{x}^T \approx -\mathbf{x} \approx -\mathbf{x}^T$.

Proof. (1) follows directly from the surjection $\iota : \mathbb{X}/\sim \to \mathbb{X}/\approx$. For (2), let $-L$ denote the inverse of an oriented link L, and $\pm\bar{L}$ the mirror image of $\pm L$. Then we have $L = -L = \bar{L} = -\bar{L}$ in $\mathbb{L}$. Taking $L = \mathrm{cl}(b)$ for a braid b, we have

$$\vec{\sigma}_{\sim}(L) = \langle\mathbf{x}(b)\rangle,\ \vec{\sigma}_{\sim}(-L) = \langle\mathbf{x}(b)^T\rangle,\ \vec{\sigma}_{\sim}(\bar{L}) = \langle-\mathbf{x}(b)\rangle,\ \vec{\sigma}_{\sim}(-\bar{L}) = \langle-\mathbf{x}(b)^T\rangle.$$

Then the commutative square preceding to Lemma (2.4) shows (2). $\square$

The following remark means that (1) and (2) of Lemma (2.4) are sufficient to characterize the equivalence relation $\approx$ in the set of knots:

Remark (2.5) Let $\mathbb{X}_1$ be the subset of $\mathbb{X}$ consisting of lattice points $\mathbf{x}$ such that $\mathrm{cl}\beta(\mathbf{x})$ is a knot. Then every relation $\mathbf{x} \approx \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{X}_1$ is generated by the equivalence relation $\sim$ and the relations in (2) of Lemma (2.4). In fact, let $K = \mathrm{cl}\beta(\mathbf{x})$ and $K' = \mathrm{cl}\beta(\mathbf{y})$. If $\mathbf{x} \approx \mathbf{y}$, then we have $[K] = [K']$ modulo split additions of trivial links. Then there is an oriented knot K'' which is one of the knots $\pm K$ or $\pm\bar{K}$ such that $K'' = K'$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links. Thus, we have $\mathbf{z} \sim \mathbf{y}$ for a lattice point $\mathbf{z}$ which is one of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$. More generally, for oriented links L, L' in S^3, we have $L = L'$ in $\mathbb{L}$ modulo split additions of trivial links if and only if we have $L = L'$ in $\vec{\mathbb{L}}$ modulo split additions of trivial links after a suitable choice of orientations of L and S^3. By Lemma (2.4), this implies that in order to know the class $\sigma_{\approx}(L) \in \mathbb{X}/\approx$ of an oriented link L in S^3 with $r(\geqq 2)$-components K_i $(i = 1, 2, \ldots, r)$, it suffices to know a braid presentation of the link $(-L') \cup (L \setminus L')$ for every sublink L' of L with $1 \leqq \#L' \leqq \frac{r}{2}$ besides a braid presentation of L, where $\#L'$ denotes the number of components of L'.

We now define the *canonical order* Ω_c in $\mathbb{X}$. We define a well-order in $\mathbb{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$ and extend it to a well-order in $\mathbb{Z}^n$ for every $n \geqq 2$ as follows: Namely, for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^n$ we define $\mathbf{x}_1 < \mathbf{x}_2$ if we have one of the following conditions (1)-(3):

(1) $|\mathbf{x}_1|_N < |\mathbf{x}_2|_N$ by the lexicographic order (on the natural number order).
(2) $|\mathbf{x}_1|_N = |\mathbf{x}_2|_N$ and $|\mathbf{x}_1| < |\mathbf{x}_2|$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}_1| = |\mathbf{x}_2|$ and $\mathbf{x}_1 < \mathbf{x}_2$ by the lexicographic order on the well-order of $\mathbb{Z}$ defined above.

Finally, for any two lattice points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$ with $\ell(\mathbf{x}_1) < \ell(\mathbf{x}_2)$, we define $\mathbf{x}_1 < \mathbf{x}_2$.

Then *this order* Ω_c *makes* $\mathbb{X}$ *a well-ordered set.* In fact, let S be any non-empty subset of $\mathbb{X}$. Let S_ℓ be the subset of S consisting of lattice points with the smallest length, say n. Since $\mathbb{Z}^n$ is a well-ordered set as defined above, we can find the initial lattice point of S_ℓ which is the initial lattice point of S by definition. The following lemma is useful in an actual tabulation of prime links.

Lemma (2.6) Let L be a link without a splittable component of the trivial knot. Then in the canonical order Ω_c, the lattice point $\sigma(L)$ is the initial elemnent of the equivalence class $[\sigma(L)] \in \mathbb{X}/\approx$. In particular, we have $\mathrm{cl}\beta(\sigma(L)) = L$.

Proof. Let $\mathbf{x}$ be the initial element of $[\sigma(L)]$. Suppose that $\mathrm{cl}\beta(\mathbf{x})$ has a splittable component of the trivial knot O. If a crossing point of the closed braid diagram $\mathrm{cl}\beta(\mathbf{x})$ is in O, then there is a shorter length lattice point $\mathbf{x}'$ such that $\mathrm{cl}\beta(\mathbf{x}')$ is obtained from the diagram $\mathrm{cl}\beta(\mathbf{x})$ by removing the component O, contradicting the minimality of $\mathbf{x}$. If there are no crossing point in O, then we see from the definition of β that there is a lattice point $\mathbf{x}'$ with $\mathbf{x}' < \mathbf{x}$ such that $\mathrm{cl}\beta(\mathbf{x}')$ is obtained from $\mathrm{cl}\beta(\mathbf{x})$ by removing the component O, contradicting the minimality of $\mathbf{x}$. Thus, we have $\mathrm{cl}\beta(\mathbf{x}) = L$. By definition, we have $\sigma(L) = \mathbf{x}$. □

3. The range of prime links in the canonical order

In this section, we consider $\mathbb{X}$ ordered by the canonical order Ω_c unless otherwise stated. A lattice point $\mathbf{x} \in \mathbb{X}$ is *minimal* if $\mathbf{x}$ is the initial element of the class $[\mathbf{x}]$ in Ω_c. A *prime* link is a link which is neither a splittable link nor a connected sum of two non-trivial links. Let $\mathbb{L}^{\mathrm{p}}$ be the subset of $\mathbb{L}$ consisting of prime links. By Lemma (2.6), the lattice point $\sigma(L)$ is minimal for every prime link L. The following relations are consequences of the relations in Lemma (2.2) and useful in finding minimal lattice points :

Lemma (3.1)

(1) (Duality relation) For any lattice point $\mathbf{x}$, we have $\mathbf{x} \sim \delta(\mathbf{x})$.
(2) (Flype relation) For any lattice points $\mathbf{x}$, $\mathbf{y}$ with $\min|\mathbf{x}| \geqq 2$, $\min|\mathbf{y}| \geqq 2$, any integer $m \geqq 1$ and $\varepsilon', \varepsilon = \pm 1$, we have $(\varepsilon^m, \mathbf{x}, \varepsilon', \mathbf{y}) \sim (\varepsilon^m, \mathbf{y}, \varepsilon', \mathbf{x})$.
(3) For any lattice points $\mathbf{x}, \mathbf{z}$, any integers $m, y \in \mathbb{Z}$ with $m \geqq 1$, $y(y+1) \neq 0$ and $\varepsilon = \pm 1$, we have

$$(\mathbf{x}, \varepsilon y^m, y+1, y, \mathbf{z}) \sim (\mathbf{x}, y+1, y, \varepsilon(y+1)^m, \mathbf{z}),$$
$$(\mathbf{x}, y, \varepsilon(y+1)^m, -y, \mathbf{z}) \sim (\mathbf{x}, -(y+1), \varepsilon y^m, y+1, \mathbf{z}).$$

Proof. For (1), we note that the lattice point $\delta(\mathbf{x})$ is obtained by changing the usual indices $1, 2, \ldots, m$ of the strings of the associated braid $b = \beta(\mathbf{x})$ into $m, m-1, \ldots, 1$ and then overturning the braid diagram, where $m = \max|\mathbf{x}|+1$ by definition. Since this deformation does not change the link type of $\mathrm{cl}(b)$ in $\overrightarrow{\mathbb{L}}$, we have $\mathbf{x} \sim \delta(\mathbf{x})$ by Definition (2.1). For (2), the closed braid diagrams of the lattice points $(\mathbf{y}, \varepsilon^m, \mathbf{x}, \varepsilon')$ and $(\mathbf{y}, \varepsilon', \mathbf{x}, \varepsilon^m)$ are in the braid-preserving flype relation (see J. S. Birman-W. W. Menasco [3]) [To understand this easier, we number the strings of the closed braid diagram so that the most inside string is 1]. Hence they are related by the relation $\sim$. Since these lattice points are related to $(\varepsilon^m, \mathbf{x}, \varepsilon', \mathbf{y})$ and $(\varepsilon^m, \mathbf{y}, \varepsilon', \mathbf{x})$ respectively by a relation in Lemma (2.2), the desired relation is obtained. For (3), the first equivalence is proved by induction on m using (5),(6) of Lemma (2.2). The second equivalence follows from (2),(6) of Lemma (2.2) and the first equivalence as follows:

$$\begin{aligned}(\mathbf{x}, y, \varepsilon(y+1)^m, -y, \mathbf{z}) &\sim (\mathbf{x}, -(y+1), y+1, y, \varepsilon(y+1)^m, -y, \mathbf{z}) \\ &\sim (\mathbf{x}, -(y+1), \varepsilon y^m, y+1, y, -y, \mathbf{z}) \\ &\sim (\mathbf{x}, -(y+1), \varepsilon y^m, y+1, \mathbf{z}). \quad \square\end{aligned}$$

To limit the image $\sigma(\mathbb{L}^{\mathrm{p}}) \subset \mathbb{X}$, we introduce the delta set Δ as follows:

Definition (3.2) The *delta set* Δ is the subset of $\mathbb{X}$ consisting of

$$0 (\in \mathbb{Z}), \quad 1^n (n \geqq 2)$$

and all the lattice points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ $(n \geqq 4)$ which satisfy all the following conditions (1)-(8):

(1) $x_1 = 1$, $|x_n| \geqq 2$, $n/2 \geqq \max|\mathbf{x}| \geqq 2$ and $\min|\mathbf{x}| \geqq 1$.
(2) For every integer k with $1 < k < \max|\mathbf{x}|$, there is an index i such that $|x_i| = k$.
(3) Every lattice point obtained from $\mathbf{x}$ by permuting the coordinates of $\mathbf{x}$ cyclically is not of the form $(\mathbf{x}', \mathbf{x}'')$ where $1 \leqq \max|\mathbf{x}'| < \min|\mathbf{x}''|$.
(4) If $|x_i| > |x_{i+1}|$, then $|x_i| - 1 = |x_{i+1}|$.
(5) If $|x_i| = |x_{i+1}|$, then $\mathrm{sign}(x_i) = \mathrm{sign}(x_{i+1})$.
(6) If $|(x_i, x_{i+1}, \ldots, x_{i+m+1})| = (k, (k+1)^m, k)$, $(k^m, k+1, k)$ or $(k, k+1, k^m)$ for some $k, m \geqq 1$ and $|x_j| \neq k$ for all $j < i$ and $j > i+m+1$, then $(x_i, x_{i+1}, \ldots, x_{i+m+1})$ is equal to $\pm(k, -\varepsilon(k+1)^m, k)$, $\pm(\varepsilon k^m, -(k+1), k)$ or $\pm(k, -(k+1), \varepsilon k^m)$ for some $\varepsilon = \pm 1$, respectively. Further, if $m = 1$, then we have $\varepsilon = 1$.
(7) If $|(x_i, x_{i+1}, \ldots, x_{i+m+1})|$ is of the form $(k+1, k^m, k+1)$ for some $k, m \geqq 1$, then $(x_i, x_{i+1}, \ldots, x_{i+m+1}) = \pm(k+1, \varepsilon k^m, k+1)$ for some $\varepsilon = \pm 1$. Further if $m = 1$, then we have $\varepsilon = -1$.
(8) $\mathbf{x}$ is the initial element (in the canonical order Ω_c) of the set of the lattice points obtained from every lattice point of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$, $\pm\delta(\mathbf{x})$ and $\pm\delta(\mathbf{x})^T$ by permuting the coordinates cyclically.

See Example (6.2) for some small length lattice points in Δ. It follows directly from the definition of Ω_c that the lattice points in Δ smaller than any given lattice point $\mathbf{x} \in \mathbb{X}$ form a finite set. To analyze the image $\sigma(L) \in \mathbb{X}$ of a prime link $L \in \mathbb{L}^{\mathrm{p}}$, we use the following notion:

Definition (3.3) A lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is *reducible* if satisties one of the following conditions:
(1) $\min|\mathbf{x}| = 0$ and $\ell(\mathbf{x}) > 1$.
(2) There is an integer k such that $\min|\mathbf{x}| < k < \max|\mathbf{x}|$ and $k \neq |x_i|$ for all i.
(3) There is a lattice point of the form $(\mathbf{x}', \mathbf{x}'')$ obtained from $\mathbf{x}$ by permuting the coordinates of $\mathbf{x}$ cyclically where $1 \leqq \max|\mathbf{x}'| < \min|\mathbf{x}''|$.

Otherwise, $\mathbf{x}$ is *irreducible*.

In Definition (3.3), we note the following points: In (1), the core $\tilde{\mathbf{x}}$ of $\mathbf{x}$ has a shorter length. In (2), the link $L = \mathrm{cl}\beta(\mathbf{x})$ is split. In (3), the closed braid diagram $L = \mathrm{cl}\beta(\mathbf{x})$ is a connected sum of two closed braid diagrams. Thus, L is a non-prime link or we have a shorter length lattice point $\mathbf{x}'$ with $\mathbf{x}' \sim \mathbf{x}$.

Since $\min|\mathbf{x}| = 0$ if and only if $\mathbf{x} = 0 \in \mathbb{Z}$ in Δ, we see from (2) and (3) of Definition (3.2) that every lattice point in Δ is irreducible. The following lemma is important to our argument:

Lemma (3.4) The lattice point $\sigma(L) \in \mathbb{X}$ of any prime link $L \in \mathbb{L}^{\mathrm{p}}$ belongs to Δ.

Proof. By Lemma (2.6), $\sigma(L) = \mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a minimal lattice point and $L = \mathrm{cl}\beta(\mathbf{x})$. If $n = 1$, then $\mathbf{x} = 0 \in \Delta$ (and hence L is a trivial knot). In fact, if $\mathbf{x} \neq 0$, then

$$\mathbf{x} \sim (\mathbf{x}, 0) \sim (0, \mathbf{x}) \sim 0$$

by (1),(3) and (6) of Lemma (2.2), contradicting that $\mathbf{x}$ is minimal. Assume that $n > 1$. If $\mathbf{x}$ is reducible, then we see from the remarks following Definition (3.3) that we have a shorter length lattice point $\mathbf{x}'$ with $\mathbf{x}' \sim \mathbf{x}$ because L is a prime link except the trivial knot, a contradiction. Hence $\mathbf{x}$ is irreducible. By the duality relation, we have $\mathbf{x}' \leqq \mathbf{x}$ with $\mathbf{x}' \sim \mathbf{x}$ and $\min|\mathbf{x}'| = 1$. Since $\mathbf{x}$ is minimal, we have $\mathbf{x}' = \mathbf{x}$ and $\min|\mathbf{x}| = 1$. By Lemmas (2.2) and (2.4), we must have $x_1 = 1$. If $\max|\mathbf{x}| = 1$, then $x_i = 1$ for all i, since otherwise $\mathbf{x}$ has a shorter length lattice point $\mathbf{x}'$ with $\mathbf{x}' \sim \mathbf{x}$, a contradiction. Let $\max|\mathbf{x}| > 1$. We show that $\mathbf{x}$ has the properties (1)-(8) of Definition (3.2). Using that $\mathbf{x}$ is irreducible, we see that $\mathbf{x}$ has (1), (2), (3) except that $|x_n| \geqq 2$. Suppose $|x_n| = 1$. Then by Lemma (2.2), there is a smaller lattice point $\mathbf{x}'$ with $\mathbf{x}' \sim \mathbf{x}$, a contradiction. Thus, the condition $|x_n| \geqq 2$ is also satisfied. If $|x_i| - 1 > |x_{i+1}|$, then the lattice point $\mathbf{x}'$ obtained from $\mathbf{x}$ by interchanging x_i and x_{i+1} has $\mathbf{x}' < \mathbf{x}$ and $\mathbf{x}' \sim \mathbf{x}$ by Lemma (2.2), a contradiction. Hence we have (4). We have also (5) since otherwise $\mathbf{x}$ has a shorter lattice point $\mathbf{x}'$ with $\mathbf{x}' \sim \mathbf{x}$ by Lemma (2.2). For (6), first let $(x_i, x_{i+1}, \ldots, x_{i+m+1}) = (\varepsilon k^m, \varepsilon'(k+1), \varepsilon'' k)$. When $\varepsilon'' = \varepsilon'$, we obtain from (3) of Lemma (3.1)

$$\mathbf{x} \sim \mathbf{x}' = (x_1', x_2', \ldots, x_n')$$

where $(x_i', x_{i+1}', \ldots, x_{i+m+1}') = (\varepsilon'(k+1)], \varepsilon' k, \varepsilon(k+1)^m)$ and $x_j' = x_j$ for all $j < i$ and $j > i+m+1$. Since $|x_j'| \neq k$ for all $j < i$ and $j > i+m+1$, we see that $\mathbf{x}'$ is reducible, contradicting to the minimality of $\mathbf{x}$. Hence $\varepsilon'' = -\varepsilon'$. For $(x_i, x_{i+1}, \ldots, x_{i+m+1}) = (\varepsilon' k, \varepsilon''(k+1), \varepsilon k^m)$ or $(\varepsilon'' k, \varepsilon(k+1)^m, -\varepsilon' k)$, we see that $\varepsilon'' = -\varepsilon'$ by a similar argument using (3) of Lemma (3.1). In particular when $m = 1$, we have also $\varepsilon' = \varepsilon$. Thus, we have (6). For (7), we take $(x_i, x_{i+1}, \ldots, x_{i+m+1}) = (\varepsilon'(k+1), \varepsilon k^m, \varepsilon''(k+1))$. When $\varepsilon'' = -\varepsilon'$, we obtain from (3) of Lemma (3.1)

$$\mathbf{x} \sim \mathbf{x}' = (x_1', x_2', \ldots, x_n')$$

where $(x_i', x_{i+1}', \ldots, x_{i+m+1}') = (-\varepsilon' k, \varepsilon(k+1)^m, \varepsilon' k)$ and $x_j' = x_j$ for all $j < i$ and $j > i+m+1$. Then $\mathbf{x}' < \mathbf{x}$, a contradiction. Hence $\varepsilon'' = \varepsilon'$. When $m = 1$ and $\varepsilon'' = \varepsilon' = \varepsilon$, we have

$$\mathbf{x} \sim \mathbf{x}' = (x_1', x_2', \ldots, x_n'),$$

where $(x_i', x_{i+1}', x_{i+2}') = \varepsilon(k, k+1, k)$ and $x_j' = x_j$ for $j \neq i, i+1, i+2$. Then $\mathbf{x}' < \mathbf{x}$, a contradiction. Hence $\varepsilon' = \varepsilon'' = -\varepsilon$ and we have (7). Since $\mathbf{x}$ is minimal, we have (8). Thus, $\mathbf{x} = \sigma(L)$ is in Δ. □

We see from Lemma (2.6) that the length of a prime link (or more generally, a link without a splittable component of the trivial knot) L in Ω_c is nothing but the minimal crossing number among the crossing numbers of the closed braid diagrams representing L, so that there are only finitely many prime links with the same length. This property also holds for every well-order Ω of $\mathbb{X}$ such that $\ell(\mathbf{x}) < \ell(\mathbf{y})$ means $\mathbf{x} < \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{X}$. There are long histories on making a table of knots and links, for example, by C. F. Gauss, T. P. Kirkman, P. G. Tait, C. N. Little, M. G. Haseman, J. W. Alexander-B. G. Briggs, K. Reidemeister for earlier studies (see [15] for references) and by J. H. Conway [5], D. Rolfsen [21], G. H. Dowker-M. B. Thistlethwaite [7], H. H. Doll- M. J. Hoste [6] and Y. Nakagawa [20] for relatively recent studies. In comparison with these tabulations, our tabulation method has three points which may be noted. The first point is that every prime link has a unique expression in canonically ordered lattice points, because $\mathbb{L}^{\mathrm{p}}$ is canonically identified with a subset of the well-ordered set Δ by σ. J. H. Conway's expression in [5] using basic polyhedra and algebraic tangles is excellent for enumerating knots and links together with some global features except for ordering them in a canonical way. C. H. Dowker and M. B. Thistlethwaite in [7] (for knots) and H. H. Doll- M. J. Hoste in [6] (for links) assigned integer sequences to oriented, ordered knot and link diagrams for a tabulation via computer use. As the second point, we can state in the context of their works that we can specify a unique integer sequence among lots of integer sequences representing every prime link, because our method specifies a unique closed braid diagram for every prime link. Using a result of R. W. Ghrist [9], Y. Nakagawa [20] defined an injection ϕ from the set of oriented knots into the set of positive integers so that the value $\phi(K)$ reconstructs K. Then the third point is that we can have a similar result for $\mathbb{L}^{\mathrm{p}}$ by our argument. In fact, in the forthcoming paper [17] (see [18]), we establish an embedding ζ from Δ into the set $\mathbb{Q}_+$ of positive rational numbers so that the value $\zeta(\mathbf{x})$ reconstructs $\mathbf{x}$. Thus, we can identify $\mathbb{L}^{\mathrm{p}}$ with a subset of $\mathbb{Q}_+$ in the sense that the value $\zeta\sigma(L) \in \mathbb{Q}_+$ reconstructs L. In §6, we explain how to make the table of prime links graded by the canonical order Ω_c, and as a demonstration, we make the table for the prime links with lengths up to 7.

4. π-minimal links

Let $K_i (i = 1, 2, \ldots, r)$ be the components of an oriented link L in S^3. A *coloring* f of L is a function

$$f : \{K_i |\, i = 1, 2, \ldots, r\} \longrightarrow \mathbb{Q} \cup \{\infty\}.$$

By a *meridian-longitude system* of L on $N(L)$, we mean a pair of a meridian system $m(L) = \{m_i |\, i = 1, 2, \ldots, r\}$ and a longitude system $\ell(L) = \{\ell_i |\, i = 1, 2, \ldots, r\}$ on $N(L)$ such that (m_i, ℓ_i) is the meridian-longitude pair of K_i on $N(K_i)$ for every i. We can specify the orientations of $m(L)$ and $\ell(L)$ from those of L and S^3 uniquely. Let $f(K_i) = \frac{a_i}{b_i}$ for coprime integers a_i, b_i for every i where we take $a_i = \pm 1$ and $b_i = 0$ when $f(K_i) = \infty$. Then we have a (unique up to isotopies) simple loop s_i on $\partial N(K_i)$ with $[s_i] = a_i[m_i] + b_i[\ell_i]$ in the first integral homology $H_1(\partial N(K_i))$.

We note that if the different choice $f(K_i) = \frac{-a_i}{-b_i}$ is made, then only the orientation of s_i is changed. The *Dehn surgery manifold* of a colored link (L, f) is the oriented 3-manifold

$$\chi(L, f) = E(L) \bigcup_{i=1}^{r} {}_{s_i = 1\times\partial D_i^2} S^1 \times D_i^2$$

with the orientation induced from $E(L) \subset S^3$, where $\bigcup_{s_i=1\times\partial D_i^2}$ denotes a pasting of $S^1 \times \partial D_i^2$ to $\partial N(K_i)$ so that s_i is identified with $1 \times \partial D_i^2$. In this construction, the 3-manifold $\chi(L, f) \in \mathbb{M}$ is uniquely determined from the colored link (L, f). In this paper, we are particularly interested in the 0-surgery manifold, that is, $\chi(L, f)$ with $f = 0$. For every link $L \in \mathbb{L}$, we consider the subset

$$\{L\}_\pi = \{L' \in \mathbb{L} |\ \pi_1 E(L') = \pi_1 E(L)\}$$

of $\mathbb{L}$. Here are some examples on $\{L\}_\pi$.

Example (4.1) (1) For every prime knot $K \in \mathbb{L}$, we have $\{K\}_\pi = \{K\}$ by the Gordon-Luecke theorem [10] and W. Whitten [22]. However, for example if K is the trefoil knot, then $\{K\#K\}_\pi = \{K\#K, K\#\bar{K}\}$ where $\bar{K}$ denotes the mirror image of K.

(2) Let L be the Whitehead link obtained from the Hopf link $O\cup O'$ by replacing O' with the untwisted double D of O': $L = O \cup D$. Further, let L_m be the link obtained by replacing D with the m-full twist D_m of D along O for every $m \in \mathbb{Z}$ where we take $L_0 = L$. Then we have

$$\{L\}_\pi = \{L_m \mid m \in \mathbb{Z}\}.$$

To see (2), let $L' \in \{L\}_\pi$. Since $E(L)$ is a hyperbolic 3-manifold and hence $\pi_1 E(L) = \pi_1 E(L')$ means $E(L) = E(L')$ (see W. Jaco [12]), the meridian system on L' indicates a coloring f of L. Since the linking numer of O and D is 0, we have $f(O) = \frac{1}{m}$ and $f(D) = \frac{1}{n}$ for some integers $m, n \in \mathbb{Z}$. If m or n is not 0, then we can assume that $m \neq 0$ since the components O and D are interchangeable. If $m \neq 0$, then we obtain L_m by taking m full twists along O. Since any twisted doubled knot K' is non-trivial and $\chi(K', \frac{1}{n}) \neq S^3$ for $n \neq 0$, we must have $n = 0$, giving the desired result. On this example, one may note that since the linking numer of L_m is 0, the longitude system of L_m coincides with the longitude system of L in $\partial E(L)$, so that $\chi(L_m, 0) = \chi(L, 0)$ for every m.

We consider $\mathbb{L}$ as a well-ordered set by the well-order Ω (defined from the well-order Ω of $\mathbb{X}$ in §2). The following definition is needed to choose exactly one link in the set $\{L\}_\pi$ for a link $L \in \mathbb{L}$:

Definition (4.2) A link $L \in \mathbb{L}$ is *π-minimal* if L is the initial element of the set $\{L\}_\pi \cap \mathbb{L}^{\mathrm{p}}$ in the well-order Ω.

The following remark gives a reason why we restrict ourselves to a link in S^3:

Remark (4.3) For a certain torus knot $L \in \mathbb{L}$, there are homotopy torus knot spaces E', not the exterior of any knot in S^3, such that $\pi_1(E') = \pi_1 E(L)$ (see J. Hempel [11,p.152]).

Let $\mathbb{L}^\pi$ be the subset of $\mathbb{L}$ consisting of π-minimal links. We note that

$$\mathbb{L}^\pi \subset \mathbb{L}^{\mathrm{p}} \subset \mathbb{L}.$$

For the map $\pi : \mathbb{L} \to \mathbb{G}$ sending a link to the link group, we have the following lemma:

Lemma (4.4) The restriction $\pi|_{\mathbb{L}^\pi} : \mathbb{L}^\pi \longrightarrow \mathbb{G}$ is injective.

Proof. For $L, L' \in \mathbb{L}^\pi$, assume that $\pi_1 E(L) = \pi_1 E(L')$. Since both L and L' are π-minimal in $\{L\}_\pi = \{L'\}_\pi$, we have $L \leqq L'$ and $L \geqq L'$ by definition. Hence $L = L'$. $\square$

The following question is related to determining when a given prime link is π-minimal:

Question (4.5) For $L, L' \in \mathbb{L}^{\mathrm{p}}$, does $\pi_1 E(L) = \pi_1 E(L')$ mean $E(L) = E(L')$?

This question is known to be yes for a large class of prime links, including all prime knots by W. Whitten [22], and prime links L such that $E(L)$ does not contain any essential embedded annulus, in particular, hyperbolic links, by the Johannson Theorem (see W. Jaco [12]). Here is another class of links.

Proposition 4.6 For links $L, L' \in \mathbb{L}$, assume that $E(L)$ is a special Seifert manifold (that is, a Seifert manifold without essential embedded torus) and there is an isomorphism $\pi_1 E(L) \to \pi_1 E(L')$. Then there is a homeomorphism $E(L) \to E(L')$.

Proof. By a classification result of G. Burde-K. Murasugi [4], the Seifert structure of $E(L)$ comes from a Seifert structure on S^3. By [12], the orbit surface of the Seifert manifold $E(L)$ is

(i) the disk with at most two exceptional fibers,
(ii) the annulus with at most one exceptional fiber, or
(iii) the disk with two holes and no exceptional fibers.

In particular, $\pi_1 E(L)$ and hence $\pi_1 E(L')$ are groups with non-trivial centers, so that $E(L')$ is also a special Seifert fibered manifold with the same orbit data as $E(L)$. In the case (i), both L and L' are torus knots and $\pi_1 E(L) \cong \pi_1 E(L')$

implies $L = L'$ (confirmed for example by the Alexander polynomials) and hence $E(L) = E(L')$. In the cases of (ii) without exceptional fiber and (iii), we have $E(L) = E(L') = S^1 \times C$ for the annulus or the disk with two holes C. Assume that $E(L)$ and $E(L')$ are in the case of (ii) with one exceptional fiber. Let (p, q) and (r, s) be the types of the exceptional fibers of $E(L)$ and $E(L')$, respectively, where $p, r \geqq 2$, $(p, q) = 1$, $(r, s) = 1$. Let

$$\pi_1 E(L) = (t, a, b | ta = at, tb = bt, t^q = a^p) \quad \text{and}$$
$$\pi_1 E(L') = (t, a, b | ta = at, tb = bt, t^s = a^r)$$

be the fundamental group presentations of $E(L)$ and $E(L')$, respectively, obtained from $S^1 \times C$ with C the disk with two holes by adjoining a fibered solid torus around the exceptional fiber. Let $\psi : \pi_1 E(L) \to \pi_1 E(L')$ be an isomorphism. Considering the central group which is the infinite cyclic group generated by t, we see that $\psi(t) = t^{\pm 1}$. Replacing $-s$ with s if necessary, we may have $\psi(t) = t$. In the quotient groups, ψ induces an isomorphism

$$\psi_* : (a | a^p = 1) * (b | -) \cong (a | a^r = 1) * (b | -).$$

Hence $p = r$ and $\psi(a) = t^m a^\varepsilon$ for some integer m and $\varepsilon = \pm 1$. Then

$$t^q = \psi(a^p) = t^{mp} a^{\varepsilon p} = t^{mp} a^{\varepsilon r} = t^{mp + \varepsilon s}$$

and hence $q \equiv \pm s \pmod p$, which shows the types (p, q) and (r, s) are equivalent. Thus, there is a fiber-preserving homeomorphism $E(L) \to E(L')$. □

Here is a remark on π-minimal links.

Remark (4.7) Let L be the 2-fold connected sum of the Hopf link, and L' the $(3, 3)$-torus link. Then we have $\sigma(L) = (1^2, 2^2)$ and $\sigma(L') = (1^2, 2, 1^2, 2)$ in the canonical order Ω_c (cf. §6). Although $E(L) = E(L')$ and $L < L'$, the link L' is a π-minimal link. We note that $\chi(L, 0) = S^1 \times S^2$ and $\chi(L', 0) = P^3$ (the projective 3-space).

5. Proof of Theorem(1.1)

The following lemma is a folklore result obtained by the Kirby calculus (see R. Kirby [19]):

Lemma (5.1) The map $\chi_0 : \mathbb{L} \to \mathbb{M}$ defined by $\chi_0(L) = \chi(L, 0)$ is a surjection.

Proof. For every $M \in \mathbb{M}$, we have a colored link (L, f) with components K_i ($i = 1, 2, \ldots, r$) such that $\chi(L, f) = M$ and $f(K_i) = m_i$ is an even integer for all i (see S. J. Kaplan [13]). We show that there is a link L'_2 with $r + 2$ components such that

$\chi(L_2', 0) = \chi(L, f)$. Let $L_2 = L \cup L_H$ be the split union of the oriented link L and an oriented Hopf link $L_H = O_1 \cup O_2$ with linking number $\mathrm{Link}(O_1, O_2) = -1$. Let f_2 be the coloring of L_2 obtained from f and the 0-coloring of L_H. If $m_i \neq 0$, then we take a fusion knot K_i' of K_i and $\frac{|m_i|}{2}$ parallels of $\mathrm{sign}(m_i)O_1$ and one parallel copy of O_2 in the 0-framings. If $m_i = 0$, then we take $K_i' = K_i$. Doing these operations for all i, we obtain from (L_2, f_2) a colored link (L_2', f_2') with $L_2' = (\cup_{i=1}^r K_i') \cup L_H$, a link with $r + 2$ components and a coloring f_2' such that

$$f_2'(K_i') = f_2(K_i) + 2\mathrm{Link}(\frac{m_i}{2} O_1, O_2) = m_i - m_i = 0.$$

Since $f_2'|_{L_H} = f_2|_{L_H} = 0$, we have $f_2' = 0$. By the Kirby calculus on handle slides ([19], [15,p.245]), we have $\chi(L_2', 0) = \chi(L_2, f_2) = M$. $\square$

Let $\mathbb{L}^\pi(M)$ be the subset of $\mathbb{L}^\pi$ consisting of π-minimal links L such that $\chi(L, 0) = M$. When we consider a prime link $L \in \mathbb{L}$ with $\chi(L, 0) = M$ to find a π-minimal link in $\mathbb{L}^\pi(M)$ for a given $M \in \mathbb{M}$, the following points should be noted: If we take the initial element L_0 of the set $\{L\}_\pi$, then the link L_0 need not be a prime link, as it is noted in Remark (4.7). If L_0 is the initial element of the prime link subset of $\{L\}_\pi$, then L_0 is a π-minimal link in $\mathbb{L}^\pi(\chi(L_0, 0))$, but in general we cannot guarantee that $\chi(L_0, 0) = M$, as we note in the following example:

Example (5.2) There are hyperbolic links $L, L' \in \mathbb{L}$ such that $E(L) = E(L')$, $\chi(L, 0) \neq \chi(L', 0)$ and $\{L\}_\pi = \{L'\}_\pi = \{L, L'\}$. Thus, if $L < L'$ in the well-order Ω, then the link L is π-minimal, but L is not in $\mathbb{L}(\chi(L', 0))$. To obtain this example, let $L_H = O_1 \cup O_2$ be the Hopf link with coloring f such that $f(O_1) = 0, f(O_2) = 1$. Then $\chi(L_H, f) = S^3$ and the dual colored link (L_H', f') of (L_H, f) is given by $L_H' = L_H$ and $f'(O_1) = -1$ and $f'(O_2) = 0$. By Remark 4.7 of [16], we have a normal imitation $q : (S^3, L_H^*) \to (S^3, L_H)$ with $\chi(L_H^*, fq) = S^3$ and a dual normal imitation $q' : (S^3, L_H'^*) \to (S^3, L_H')$, that is a normal imitation such that $E(L_H^*) = E(L_H'^*)$, $q'|_{E(L_H^*)} = q|_{E(L_H^*)}$ and $(L_H'^*, f'q')$ is the dual colored link of (L_H^*, fq). As it is stated in Remark 4.7 of [16], we can impose on these normal imitations the following additional properties: namely, L_H^* and $L_H'^*$ are totally hyperbolic, componentwise distinct links, and every homeomorphism $h : E(L'') \to E(L_H^*)$ extends to a homeomorphism $h^+ : (S^3, L'') \to (S^3, L_H^*)$ or $h^{+\prime} : (S^3, L'') \to (S^3, L_H'^*)$. On the other hand, we see that $\chi(L_H', 0) = S^3$ and the dual colored link (L_H, f'') of $(L_H', 0)$ is given by $f''(O_1) = -1$ and $f'(O_2) = \infty$. Further, we can assume from Theorem 4.1(2) of [16] that $\chi(L_H^*, 0)$ and $\chi(L_H^*, f''q) = \chi(L_H'^*, 0)$ are distinct because 0 and f'' are distinct from ∞, f. Thus, we can take L_H^* and $L_H'^*$ as L and L', respectively. (We note that $\chi(L_H^*, 0)$ and $\chi(L_H'^*, 0)$ are homology 3-spheres, because they are normal imitations of $\chi(L_H, 0) = \chi(L_H', 0) = S^3$.)

In spite of Example (5.2), we can show the following lemma:

Lemma (5.3) For every $M \in \mathbb{M}$, the set $\mathbb{L}^\pi(M)$ is an infinite set.

Proof. By Lemma (5.1), we take a disconnected link L in S^3 such that $\chi(L,0) = M$. Let $M \neq S^3$. By a result of [16], there are infinitely many normal imitations

$$q_i : (S^3, L_i^*) \longrightarrow (S^3, L) \quad (i = 1, 2, 3, \dots)$$

such that

(1) $\chi(L_i^*, 0) = \chi(L, 0) = M$,
(2) L_i^* is (totally) hyperbolic, and
(3) every homeomorphism $h : E(L_i^*) \to E(L')$ for a link L' in S^3 extends to a homeomorphism $h^+ : (S^3, L_i^*) \to (S^3, L')$.

Then L_i^* is π-minimal by (2) and (3), so that $L_i^* \in \mathbb{L}^\pi(M)$, $i = 1, 2, 3, \dots$. For $M = S^3$, let L be a Hopf link. Then $\chi(L, 0) = S^3$ and the dual link L' of the Dehn surgery is also the Hopf link. By Remark 4.7 of [16], there are infinitely many pairs of normal imitations

$$\begin{aligned} q_i :&(S^3, L_i^*) \longrightarrow (S^3, L), \\ q_i' :&(S^3, L_i'^*) \longrightarrow (S^3, L') \quad (i = 1, 2, 3, \dots) \end{aligned}$$

such that

(1) $\chi(L_i^*, 0) = \chi(L, 0) = S^3 = \chi(L', 0) = \chi(L_i'^*, 0)$,
(2) $E(L_i^*) = E(L_i'^*)$,
(3) L_i^* and $L_i'^*$ are (totally) hyperbolic,
(4) every homeomorphism $h : E(L_i^*) \to E(L'')$ for a link L'' in S^3 extends to a homeomorphism $h^+ : (S^3, L_i^*) \to (S^3, L'')$ or $h'^+ : (S^3, L_i'^*) \to (S^3, L'')$.

Thus, $\{L_i^*\}_\pi = \{L_i^*, L_i'^*\}$ for every i, and we can take a π-minimal link, say L_i^* in $\{L_i^*\}_\pi$ for every i, so that $L_i^* \in \mathbb{L}^\pi(S^3)$, $i = 1, 2, 3, \dots$. □

We are in a position to prove the first half of Theorem (1.1).

Proof of Theorem (1.1). Since $\mathbb{L}^\pi(M) \neq \emptyset$ by Lemma (5.3), we can take the initial element L_M of $\mathbb{L}^\pi(M)$ for every $M \in \mathbb{M}$. Using that the set $\mathbb{L}^\pi(M)$ is uniquely determined by M and Ω, we see that the well-order Ω of $\mathbb{X}$ induces a map

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}^\pi \subset \mathbb{L}$$

sending a 3-manifold M to the link L_M. This map α must be injective, because the 0-surgery manifold $\chi(\alpha(M), 0) = M$. Combining this result with Lemma (4.4), we obtain the embeddings σ_α and π_α. If a lattice point $\mathbf{x} = \sigma_\alpha(M)$ is given, then we obtain the link $\alpha(M) = \mathrm{cl}\beta(\mathbf{x})$ with braid presentation, the 3-manifold $M = \chi(\mathrm{cl}\beta(\mathbf{x}), 0)$ with 0-surgery description and the link group $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}))$ with Artin presentation associated with the braid $\beta(\sigma_\alpha(M))$, completing the proof of the first half. If a link group $G = \pi_\alpha(M)$ with a prime Artin presentation is given, then we have a braid b such that G is the link group of the prime link $\mathrm{cl}(b)$. Let $\mathbf{x}_i \in \Delta$ $(i = 1, 2, \ldots, n)$ be the lattice points smaller than or equal to the lattice point $\mathbf{x}(b)$. By Lemma (3.4), there is a lattice point $\mathbf{x}_i$ with $\mathbf{x}_i \approx \mathbf{x}(b)$. By using a solution of the problem in (3), let $\mathbf{x}_{i_0}$ be the smallest lattice point such that $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is a prime link and there is an isomorphism $\pi_1 E(\mathrm{cl}\beta(\mathbf{x}_{i_0})) \to G$ among $\mathbf{x}_i$ $(i = 1, 2, \ldots, n)$. Then the link $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is π-minimal by this construction. Thus, the desired lattice point $\sigma_\alpha(M) = \mathbf{x}_{i_0}$ is obtained, proving (3). If a π-minimal link L with $\chi(L, 0) = M$ is given, we take a braid b representing L. Let $\mathbf{x}_i \in \Delta$ $(i = 1, 2, \ldots, n)$ be the lattice points smaller than or equal to $\mathbf{x}(b)$. By Lemma (3.4), there is a lattice point $\mathbf{x}_i$ with $\mathbf{x}_i \approx \mathbf{x}(b)$. By using a solution of the problem in (4), we take the smallest lattice point $\mathbf{x}_{i_0}$ such that the link $\mathrm{cl}\beta(\mathbf{x}_{i_0})$ is a π-minimal link and $\chi(\mathrm{cl}\beta(\mathbf{x}_{i_0}), 0) = M$. Thus, the desired lattice point $\sigma_\alpha(M) = \mathbf{x}_{i_0}$ is obtained, proving (4). □

As a matter of fact, we can construct many variants of the embedding $\alpha : \mathbb{M} \to \mathbb{L}$. Here are remarks on constructing some other embeddings α.

Remark (5.4) Let $\mathbb{L}^{\mathrm{h}} \subset \mathbb{L}$ be the subset consisting of hyperbolic links L (possibly with infinite volume) such that L is determined by the exterior $E(L)$ (that is, $E(L) = E(L')$ for a link L' means $L = L'$), and $\mathbb{L}^{\mathrm{h}}(M) = \{L \in \mathbb{L}^{\mathrm{h}} \,|\, \chi(L, 0) = M\}$. Then we still have an embedding $\alpha : \mathbb{M} \to \mathbb{L}^{\mathrm{h}} \subset \mathbb{L}$ with $\chi_0\alpha = 1$ such that σ_α and π_α are embeddings by the proof of Theorem (1.1) using $\mathbb{L}^{\mathrm{h}}(M)$ instead of $\mathbb{L}^{\pi}(M)$. (For this proof, we use that $\mathbb{L}^{\mathrm{h}}(S^3)$ contains the Hopf link and the set $\mathbb{L}^{\mathrm{h}}(M)$ for $M \neq S^3$ is infinite by Lemma (5.3).) In this case, the links $\alpha(S^1 \times S^2)$, $\alpha(S^3)$ and $\alpha(M)$ for every $M \neq S^1 \times S^2, S^3$ are the trivial knot, the Hopf link and a hyperbolic link of finite volume, respectively. If we take the subset $\mathbb{L}(M) \subset \mathbb{L}$ consisting of all links L with $\chi(L, 0) = M$, then the proof of Theorem (1.1) using $\mathbb{L}(M)$ instead of $\mathbb{L}^{\pi}(M)$ shows the existence of an embedding $\alpha : \mathbb{M} \to \mathbb{L}$ with $\chi_0\alpha = 1$. However, in this case, the map π_α is no longer injective in the canonical order Ω_c. In fact, if $K\#K$ is the granny knot and $K\#\bar{K}$ is the square knot where K is a trefoil knot, then we see that $\alpha(\chi(K\#K, 0)) = K\#K$ and $\alpha(\chi(K\#\bar{K}, 0)) = K\#\bar{K}$. Then we have $\pi_\alpha(\chi(K\#K, 0)) = \pi_\alpha(\chi(K\#\bar{K}, 0))$, although $\chi(K\#K, 0) \neq \chi(K\#\bar{K}, 0)$ (see [14, Example 3.2]).

Remark (5.5) The subsets $\mathbb{L}^{\mathrm{h}}(M) \subset \mathbb{L}^{\pi}(M) \subset \mathbb{L}(M)$ of $\mathbb{L}$ are defined up to automorphisms of M, but the Kirby calculus of [19] enables us to make "automorphism-free" definitions of them. In fact, for a given link L, let $\mathbb{L}(L)$ the set of links L' such that the 0-colored link $(L', 0)$ is obtained from the 0-colored link $(L, 0)$ or $(\bar{L}, 0)$ by a finite number of Kirby moves, and then we define $\mathbb{L}^{\mathrm{h}}(L)$ and $\mathbb{L}^{\pi}(L)$ to be the restrictions of $\mathbb{L}(L)$ to the hyperbolic links determined by the exteriors and the π-minimal links, respectively. R. Kirby's theorem in [19] shows that for a link L

with $\chi(L,0)=M$ we have the identities

$$\mathbb{L}(L)=\mathbb{L}(M), \quad \mathbb{L}^h(L)=\mathbb{L}^h(M) \quad \text{and} \quad \mathbb{L}^\pi(L)=\mathbb{L}^\pi(M),$$

where the right hand sides are the sets defined on before for M. Thus, the embedding α is defined "automorphism-freely". In particular, in any use of $\mathbb{L}^{\mathrm{h}}(M)$ or $\mathbb{L}^\pi(M)$, the embedding π_α is defined "automorphism-freely". This is the precise meaning of that the homeomorphism problem on $\mathbb{M}$ can be in principle replaced by the isomorphism problem on $\mathbb{G}$, stated in the introduction.

6. A classification program

In this section, we take the canonical order Ω_c unless otherwise stated. We consider the following mutually related three embeddings already established in Theorem (1.1):

$$\begin{aligned}\alpha :&\mathbb{M} \longrightarrow \mathbb{L},\\ \sigma_\alpha :&\mathbb{M} \longrightarrow \mathbb{X},\\ \pi_\alpha :&\mathbb{M} \longrightarrow \mathbb{G}.\end{aligned}$$

Since $\sigma_\alpha(\mathbb{M})\subset\Delta$ and every initial segment of Δ is a finite set, we can attach (without overlapping) to every 3-manifold M in $\mathbb{M}$ a label (n,i) where n denotes the length of M and i denotes that M appears as the ith 3-manifold of length n, so that we have

$$M_{n,1}<M_{n,2}<\cdots<M_{n,m_n}$$

for a positive integer $m_n<\infty$. Let

$$\alpha(M_{n,i})=L_{n,i}\in\mathbb{L}, \quad \pi_\alpha(M_{n,i})=G_{n,i}\in\mathbb{G} \quad \text{and} \quad \sigma_\alpha(M_{n,i})=\mathbf{x}_{n,i}\in\Delta.$$

Our classification program is to enumerate the 3-manifolds $M_{n,i}$ for all $n=1,2,\dots$ and $i=1,2,\dots,m_n$ together with the data $L_{n,i},G_{n,i}$ and $\mathbf{x}_{n,i}$, but by (2) of Theorem (1.1) it is sufficient to give the lattice point $\mathbf{x}_{n,i}$, because we can easily construct $L_{n,i}$, $M_{n,i}$ and $G_{n,i}$ by $L_{n,i}=\mathrm{cl}\beta(\mathbf{x}_{n,i})$, $M_{n,i}=\chi(L_{n,i},0)$ and $G_{n,i}=\pi_1E(L_{n,i})$. We proceed the argument by induction on the length n. Since the lattice points of lengths $1,2,3$ in Δ are 0, 1^2 and 1^3, we can do the classification of $\mathbb{M}$ with lengths $1,2,3$ as follows (where $T^2\times_A S^1$ denotes the torus bundle over S^1 with monodromy matrix A):

length 1: $m_1=1,\ M_{1,1}=S^1\times S^2,\ L_{1,1}=O$ (the trivial knot),
$G_{1,1}=\mathbb{Z},\ \mathbf{x}_{1,1}=0.$

length 2: $m_2=1,\ M_{2,1}=S^3,\ L_{2,1}=2_1^1$ (the Hopf link),
$G_{2,1}=\mathbb{Z}\oplus\mathbb{Z},\ \mathbf{x}_{2,1}=1^2.$

length 3: $m_3=1,\ M_{3,1}=T^2\times_A S^1,\ A=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix},$
$L_{3,1}=3_1$ (the trefoil knot), $G_{3,1}=(x,y|\,xyx=yxy),\ \mathbf{x}_{3,1}=1^3.$

To explain our classification of $\mathbb{M}$ with any length $n \geqq 4$, we assume that the classification of $\mathbb{M}$ with lengths $\leqq n-1$ is done. Let Δ_n be the subset of Δ consisting of lattice points of length n. The first step of our classification program is as follows:

Step 1. Make an ordered list $\Delta_n^* \subset \Delta_n$ containing all the minimal lattice points in Δ_n.

If we take the list Δ_n^* smaller, then our work will be simpler. It is recommended to make first the ordered list $|\Delta_n^*| = \{|\mathbf{x}| \,|\, \mathbf{x} \in \Delta_n^*\}$ counting the property of Ω_c that we have $\mathbf{x} < \mathbf{y}$ if we have one of the following three conditions: (i) $\ell(\mathbf{x}) < \ell(\mathbf{y})$, (ii) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$, and (iii) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$. To establish Step 1, we use the following notion:

Definition (6.1) A lattice point $\mathbf{x} \in \mathbb{X}$ is *locally-minimal* if it is the initial element of the subset of $[\mathbf{x}]$ consisting of the lattice points obtained from $\mathbf{x}$ by the duality relation, the flype relation and the moves in Lemmas (2.2) and (2.4) except the length-increasing moves.

Every minimal lattice point is locally-minimal, but the converse is not true. It is realistic to make as Δ_n^* a list containing all the locally-minimal lattice points of Δ_n. The following list is such a list for Step 1.

Example (6.2) The following list contains all the minimal lattice points of lengths $\leqq 7$ in Δ:

$$
\begin{aligned}
\Delta_1^*:\ & 0,\\
\Delta_2^*:\ & 1^2,\\
\Delta_3^*:\ & 1^3,\\
\Delta_4^*:\ & 1^4, (1,-2,1,-2),\\
\Delta_5^*:\ & 1^5, (1^2,2,-1,2), (1^2,-2,1,-2),\\
\Delta_6^*:\ & 1^6, (1^3,2,-1,2), (1^3,-2,1,-2), (1^2,2,1^2,2),\\
& (1^2,2,(-1)^2,2), (1^2,-2,1^2,-2), (1^2,-2,1,(-2)^2),\\
& (1,-2,1,-2,1,-2), (1,-2,1,3,-2,3),\\
\Delta_7^*:\ & 1^7, (1^4,2,-1,2), (1^4,-2,1,-2),\\
& (1^3,2,1^2,2), (1^3,2,(-1)^2,2), (1^3,-2,1^2,-2),\\
& (1^3,-2,(-1)^2,-2), (1^3,2,-1,2^2), (1^3,-2,1,(-2)^2),\\
& (1^2,-2,1^2,(-2)^2), (1^2,-2,1,-2,1,-2),\\
& (1^2,2,-1,-3,2,-3), (1^2,-2,1,3,-2,3), (1,-2,1,-2,3,-2,3),\\
& (1,-2,1,3,2^2,3), (1,-2,1,3,(-2)^2,3).
\end{aligned}
$$

Let $\mathbb{L}^{\mathrm{p}}_n$ be the subset of $\mathbb{L}^{\mathrm{p}}$ consisting of prime links of length n. Let $\mathbb{D}^*_n$ be the set consisting of the link diagrams $\mathrm{cl}\beta(\mathbf{x})$ for all $\mathbf{x} \in \Delta^*_n$. By Lemma (3.4), we observe that *if* $L = cl\beta(\mathbf{x}) \in \mathbb{L}^{\mathrm{p}}_n$ *for a lattice point* $\mathbf{x} \in \mathbb{X}$, *then there is a minimal lattice point* $\mathbf{x}' \in \Delta_n$ *with* $\mathbf{x}' \leqq \mathbf{x}$ *such that* $L = cl\beta(\mathbf{x}')$. This implies that the set $\mathbb{L}^{\mathrm{p}}_n$ consists of the prime links represented by link diagrams of $\mathbb{D}^*_n$ not belonging to $\mathbb{L}^{\mathrm{p}}_j$ $(j = 1, 2, \ldots, n-1)$ (which are assumed to have already constructed by our inductive hypothesis). Step 2 is the following procedure:

Step 2. Construct $\mathbb{L}^{\mathrm{p}}_n$ from $\mathbb{D}^*_n$.

The link $\mathrm{cl}\beta(\mathbf{x})$ of a lattice point $\mathbf{x}$ of length n such that $\tilde{\mathbf{x}} = \mathbf{x}$ admits a braided link diagram with crossing number n. Thus, if a list of prime links with crossing numbers up to n is available, then this procedure would not be so difficult. In the following example, our main work is only to identify the lattice points of length $n \leqq 7$ in Example (6.2) with prime links in Rolfsen's table [21].

Example (6.3) The following list gives the elements of the sets $\mathbb{L}^{\mathrm{p}}_n$ for $n \leqq 7$ together with the corresponding lattice points.

$\mathbb{L}^{\mathrm{p}}_1:\ O \qquad \sigma(O) = 0.$

$\mathbb{L}^{\mathrm{p}}_2:\ 2^2_1 \qquad \sigma(2^2_1) = 1^2.$

$\mathbb{L}^{\mathrm{p}}_3:\ 3_1 \qquad \sigma(3_1) = 1^3.$

$\mathbb{L}^{\mathrm{p}}_4:\ 4^2_1 < 4_1$
$\sigma(4^2_1) = 1^4,$
$\sigma(4_1) = (1, -2, 1, -2).$

$\mathbb{L}^{\mathrm{p}}_5:\ 5_1 < 5^2_1$
$\sigma(5_1) = 1^5,$
$\sigma(5^2_1) = (1^2, -2, 1, -2).$

$\mathbb{L}^{\mathrm{p}}_6:\ 6^2_1 < 5_2 < 6_2 < 6^3_3 < 6^3_1 < 6_3 < 6^3_2 < 6^2_3$
$\sigma(6^2_1) = 1^6,$
$\sigma(5_2) = (1^3, 2, -1, 2),$
$\sigma(6_2) = (1^3, -2, 1, -2),$
$\sigma(6^3_3) = (1^2, 2, 1^2, 2),$
$\sigma(6^3_1) = (1^2, -2, 1^2, -2),$
$\sigma(6_3) = (1^2, -2, 1, (-2)^2),$
$\sigma(6^3_2) = (1, -2, 1, -2, 1, -2),$
$\sigma(6^2_3) = (1, -2, 1, 3, -2, 3).$

$\mathbb{L}_7^{\mathrm{p}}:\ 7_1 < 6_2^2 < 7_1^2 < 7_7^2 < 7_8^2 < 7_4^2 < 7_2^2 < 7_5^2 < 7_6^2 < 6_1 < 7_6 < 7_7 < 7_1^3$

$\sigma(7_1) = 1^7,$
$\sigma(6_2^2) = (1^4, 2, -1, 2),$
$\sigma(7_1^2) = (1^4, -2, 1, -2),$
$\sigma(7_7^2) = (1^3, 2, 1^2, 2),$
$\sigma(7_8^2) = (1^3, 2, (-1)^2, 2),$
$\sigma(7_4^2) = (1^3, -2, 1^2, -2),$
$\sigma(7_2^2) = (1^3, -2, 1, (-2)^2),$
$\sigma(7_5^2) = (1^2, -2, 1^2, (-2)^2),$
$\sigma(7_6^2) = (1^2, -2, 1, -2, 1, -2),$
$\sigma(6_1) = (1^2, 2, -1, -3, 2, -3),$
$\sigma(7_6) = (1^2, -2, 1, 3, -2, 3),$
$\sigma(7_7) = (1, -2, 1, -2, 3, -2, 3)$
$\sigma(7_1^3) = (1, -2, 1, 3, (-2)^2, 3).$

The following lattice points of Example (6.2)

$$(1^2, 2, -1, 2),\ (1^2, 2, (-1)^2, 2),\ (1^3, -2, (-1)^2, -2),\ (1^3, 2, -1, 2^2),\ (1, -2, 1, 3, 2^2, 3)$$

are removed from the list, since these links are seen to be 4_1^2, 6_3^3, 7_7^2, 6_3^2, 6_3^3, respectively. The links $7_2, 7_3, 7_4, 7_5, 7_3^2$ in Rolfsen's table of [21] are also excluded from the list since these links turn out to have lengths greater than 7. In Steps 3 and 4, powers of low dimensional topology techniques will be seriously tested.

Step 3. Construct the subset $\mathbb{L}_n^{\pi} \subset \mathbb{L}_n^{\mathrm{p}}$ by removing every link $L \in \mathbb{L}_n^{\mathrm{p}}$ such that there is a link $L' \in \mathbb{L}_j^{\mathrm{p}}$ $(j \leqq n)$ with $L' < L$ and $\pi_1 E(L) = \pi_1 E(L')$.

From construction, we see that the set $\mathbb{L}_n^{\pi}$ consists of π-minimal links of length n. Among the links in Example (6.3), we see that $E(4_1^2) = E(7_7^2)$ and $E(5_1^2) = E(7_8^2)$ by taking one full twist along a component and that except these identities, all the links have mutually distinct link groups by using the following lemma on the Alexander polynomials:

Lemma (6.4) Let $A(t_1, t_2, \dots, t_r)$ and $A'(t_1, t_2, \dots, t_r)$ be the Alexander polynomials of oriented links L and L' with r components. If there is a homeomorphism $E(L) \to E(L')$, then there is an automorphism ψ of the multiplicative free abelian group $\langle t_1, t_2, \dots, t_r \rangle$ with basis t_i $(i = 1, 2, \dots, r)$ such that

$$A'(t_1, t_2, \dots, t_r) = \pm t_1^{s_1} t_2^{s_2} \dots t_r^{s_r} A(\psi(t_1), \psi(t_2), \dots, \psi(t_r))$$

for some integers s_i $(i = 1, 2, \dots, r)$.

The proof of this lemma is direct from the definition of Alexander polynomial(see [15]). Thus, we obtain the following example:

Example (6.5) We have $\mathbb{L}^\pi_n = \mathbb{L}^{\mathrm{p}}_n$ for $n \leqq 6$ and

$$\mathbb{L}^\pi_7: \quad 7_1 < 6^2_2 < 7^2_1 < 7^2_4 < 7^2_2 < 6_1 < 7^2_5 < 7^2_6 < 7_6 < 7_7 < 7^3_1.$$

Let $\mathbb{M}_n$ be the subset of $\mathbb{M}$ consisting of 3-manifolds of length n, and $\mathbb{L}^{\mathbb{M}}_n$ the subset of $\mathbb{L}^\pi_n$ by removing a π-minimal link $L \in \mathbb{L}^\pi_n$ such that there is a π-minimal link $L' \in \mathbb{L}^\pi_j$ $(j \leqq n)$ with $L' < L$ and $\chi(L,0) = \chi(L',0)$. The following step is the final step of our classification program:

Step 4. Construct the set $\mathbb{L}^{\mathbb{M}}_n$.

Let L_i $(i = 1,2,\ldots,r)$ be the π-minimal links in the set $\mathbb{L}^{\mathbb{M}}_n$, ordered by Ω_c. Then we have $M_{n,i} = \chi(L_i,0)$, $\alpha(M_{n,i}) = L_i$ $(i = 1,2,\ldots,r)$. An important notice is that every 3-manifold in $\mathbb{M}$ appears once as $M_{n,i}$ without overlaps. As we shall show later, the 0-surgery manifolds of the π-minimal links in Example (6.5) are mutually non-homeomorphic, so that we have the complete list of 3-manifolds in $\mathbb{M}$ with length $\leqq 7$ as it is stated in Example (6.6).

Example (6.6)
$M_{1,1} = \chi(O,0)$, $\quad \mathbf{x}_{1,1} = 0$,

$M_{2,1} = \chi(2^2_1,0)$, $\quad \mathbf{x}_{2,1} = 1^2$,

$M_{3,1} = \chi(3_1,0)$, $\quad \mathbf{x}_{3,1} = 1^3$,

$M_{4,1} = \chi(4^2_1,0)$, $\quad \mathbf{x}_{4,1} = 1^4$,
$M_{4,2} = \chi(4_1,0)$, $\quad \mathbf{x}_{4,2} = (1,-2,1,-2)$,

$M_{5,1} = \chi(5_1,0)$, $\quad \mathbf{x}_{5,1} = 1^5$,
$M_{5,2} = \chi(5^2_1,0)$, $\quad \mathbf{x}_{5,2} = (1^2,-2,1,-2)$,

$M_{6,1} = \chi(6^2_1,0)$, $\quad \mathbf{x}_{6,1} = 1^6$,
$M_{6,2} = \chi(5_2,0)$, $\quad \mathbf{x}_{6,2} = (1^3,2,-1,2)$,
$M_{6,3} = \chi(6_2,0)$, $\quad \mathbf{x}_{6,3} = (1^3,-2,1,-2)$,
$M_{6,4} = \chi(6^3_3,0)$, $\quad \mathbf{x}_{6,4} = (1^2,2,1^2,2)$,
$M_{6,5} = \chi(6^3_1,0)$, $\quad \mathbf{x}_{6,5} = (1^2,-2,1^2,-2)$,
$M_{6,6} = \chi(6_3,0)$, $\quad \mathbf{x}_{6,6} = (1^2,-2,1,(-2)^2)$,
$M_{6,7} = \chi(6^3_2,0)$, $\quad \mathbf{x}_{6,7} = (1,-2,1,-2,1,-2)$,
$M_{6,8} = \chi(6^2_3,0)$, $\quad \mathbf{x}_{6,8} = (1,-2,1,3,-2,3)$.

$M_{7,1} = \chi(7_1,0)$, $\quad \mathbf{x}_{7,1} = 1^7$,
$M_{7,2} = \chi(6^2_2,0)$, $\quad \mathbf{x}_{7,2} = (1^4,2,-1,2)$,

$M_{7,3} = \chi(7_1^2, 0)$, $\mathbf{x}_{7,3} = (1^4, -2, 1, -2)$,
$M_{7,4} = \chi(7_4^2, 0)$, $\mathbf{x}_{7,4} = (1^3, -2, 1^2, -2)$,
$M_{7,5} = \chi(7_2^2, 0)$, $\mathbf{x}_{7,5} = (1^3, -2, 1, (-2)^2)$,
$M_{7,6} = \chi(7_5^2, 0)$, $\mathbf{x}_{7,6} = (1^2, -2, 1^2, (-2)^2)$,
$M_{7,7} = \chi(7_6^2, 0)$, $\mathbf{x}_{7,7} = (1^2, -2, 1, -2, 1, -2)$,
$M_{7,8} = \chi(6_1, 0)$, $\mathbf{x}_{7,8} = (1^2, 2, -1, -3, 2, -3)$,
$M_{7,9} = \chi(7_6, 0)$, $\mathbf{x}_{7,9} = (1^2, -2, 1, 3, -2, 3)$,
$M_{7,10} = \chi(7_7, 0)$, $\mathbf{x}_{7,10} = (1, -2, 1, -2, 3, -2, 3)$,
$M_{7,11} = \chi(7_1^3, 0)$, $\mathbf{x}_{7,11} = (1, -2, 1, 3, (-2)^2, 3)$.

To see that the 3-manifolds in Example (6.6) are mutually non-homeomorphic, we first check the first integral homology. It is computed as follows:

(1) $H_1(M) = \mathbb{Z}$ for $M = M_{1,1}$, $M_{3,1}$, $M_{4,2}$, $M_{5,1}$, $M_{6,2}$, $M_{6,3}$, $M_{6,6}$, $M_{7,1}$, $M_{7,8}$, $M_{7,9}$, $M_{7,10}$.
(2) $H_1(M) = \mathbb{Z} \oplus \mathbb{Z}$ for $M = M_{5,2}$, $M_{7,4}$, $M_{7,7}$.
(3) $H_1(M) = \mathbb{Z}_2$ for $M = M_{6,4}$, $M_{6,5}$, $M_{7,11}$.
(4) $H_1(M) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ for $M = M_{6,7}$.
(5) $H_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $M = M_{4,1}$, $M_{6,8}$, $M_{7,6}$.
(6) $H_1(M) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ for $M = M_{6,1}$, $M_{7,2}$.
(7) $H_1(M) = 0$ for $M = M_{2,1}$, $M_{7,3}$, $M_{7,5}$.

For (1), since the Alexander polynomial of a knot K is an invariant of the homology handle $\chi(K, 0)$, we see that the homology handles of (1) are mutually distinct. For (2), since the Alexander polynomial of an oriented link L with all the linking numbers 0 is also an invariant of $\chi(L, 0)$ in the sense of Lemma (6.4), these 3-manifolds are mutually distinct. For (3), we note that $M_{6,4} = P^3$ the projective 3-space, $M_{6,5} = \chi(3_1, -2)$ (where we take 3_1 the positive trefoil knot) and $M_{7,11} = \chi(4_1, -2)$. We take the connected double covering spaces $\tilde{M}$ of $M = M_{6,4}$, $M_{6,5}$ and $M_{7,11}$. The homology $H_1(\tilde{M})$ for $M = M_{6,4}$, $M_{6,5}$ or $M_{7,11}$ is respectively computed as 0, $\mathbb{Z}_3$, $\mathbb{Z}_5$, showing that these 3-manifolds are mutually distinct. For (4), we have nothing to prove. Note that $M_{6,7} = T^3$. For (5), we compare the first integral homologies of the three kinds of connected double coverings of every $M = M_{4,1}$, $M_{6,8}$, $M_{7,6}$. For $M = M_{4,1}$, it is the quaternion space Q and we have $H_1(\tilde{M}) = \mathbb{Z}_4$ for every connected double covering space $\tilde{M}$ of M. For $M = M_{6,8}$, we have $H_1(\tilde{M}; \mathbb{Z}_3) = \mathbb{Z}_3$ for every connected double covering space $\tilde{M}$ of M. On the other hand, for $M = M_{7,6}$, we have $H_1(\tilde{M}) = \mathbb{Z}_{20}$ and $H_1(\tilde{M}; \mathbb{Z}_3) = 0$ for some connected double covering space $\tilde{M}$ of M. Thus, these 3-manifolds are mutually distinct. For(6), we use the following lemma:

Lemma (6.7) Let $H = \mathbb{Z}_p \oplus \mathbb{Z}_p$ for an odd prime $p > 1$. If the linking form $\ell : H \times H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is hyperbolic, then the hyperbolic $\mathbb{Z}_p$-basis e_1, e_2 of H is unique up to unit multiplications of $\mathbb{Z}_p$.

Proof. Let e'_1, e'_2 be another hyperbolic $\mathbb{Z}_p$-basis of H. Let $e'_i = a_{i1}e_1 + a_{i2}e_2$. Then

$$0 = \ell(e'_i, e'_i) = \frac{2a_{i1}a_{i2}}{p} \pmod 1,$$
$$\frac{1}{p} = \ell(e'_1, e'_2) = \frac{a_{11}a_{22} + a_{12}a_{21}}{p} \pmod 1.$$

By these identities, we have either $e'_1 = a_{11}e_1$ and $e'_2 = a_{22}e_2$ with $a_{11}a_{22} = 1$ in $\mathbb{Z}_p$ or $e'_1 = a_{12}e_2$ and $e'_2 = a_{21}e_1$ with $a_{12}a_{21} = 1$ in $\mathbb{Z}_p$. □

By Lemma (6.7), there are just two connected $\mathbb{Z}_3$-coverings $\tilde{M}$ of every $M = M_{6,1}$, $M_{7,2}$ associated with a hyperbolic direct summand $\mathbb{Z}_3$ of $H_1(M) = \mathbb{Z}_3 \oplus \mathbb{Z}_3$. In other words, the covering $\tilde{M}$ is associated with a $\mathbb{Z}_3$-covering covering of the exterior $E(L)$ lifting one torus boundary component trivially, where $L = 6^2_1, 6^2_2$. Since the link L is interchangeable, it is sufficient to check one covering for each M. For $M = M_{6,1}$ we have $H_1(\tilde{M}) = \mathbb{Z}_9 \oplus \mathbb{Z}_3$ and for $M = M_{7,2}$ we have $H_1(\tilde{M}) = \mathbb{Z} \oplus \mathbb{Z}$. Thus, these 3-manifolds are distinct. For (7), the Dehn surgery manifolds $\chi(7^2_1, 0)$ and $\chi(7^2_2, 0)$ are the boundaries of Mazur manifolds (which are normal imitations of S^3) and identified with the Brieskorn homology 3-spheres $\Sigma(2,3,13)$, $\Sigma(2,5,7)$ by S. Akbult and R. Kirby [1]. Hence, we have $M_{2,1} = S^3$, $M_{7,3} = \Sigma(2,3,13)$ and $M_{7,5} = \Sigma(2,5,7)$ and these 3-manifolds are mutually distinct. Thus, we see that the 3-manifolds of Example (6.6) are mutually distinct.

For the Poincaré homology 3-sphere $\Sigma = \Sigma(2,3,5)$, the prime link $\alpha(\Sigma)$ must have at least 10 components. [To see this, assume that $\alpha(\Sigma)$ has r components. Using that Σ is a homology 3-sphere and $\Sigma = \chi(\alpha(\Sigma), 0)$, we see that Σ bounds a simply connected 4-manifold W with an $r \times r$ non-singular intersection matrix whose diagonal entries are all 0. Since the Rochlin invariant of Σ is non-trivial, it follows that the signature of W is an odd multiple of 8 and r is even. Hence $r \geqq 8$. If $r = 8$, then the intersection matrix is a positive or negative definite matrix which is not in our case. Thus, we have $r \geqq 10$.] Since $\chi(3_1, 1) = \Sigma$ for the positive trefoil knot 3_1, an answer to the following question on Kirby calculus (see [13, 19,21]) will help in understanding the link $\alpha(\Sigma)$:

Question (6.8) How is Ω_c understood among colored links ?

We note that the cardinal numbers $l_n = \#\mathbb{L}^{\mathrm{p}}_n$ and $m_n = \#\mathbb{M}_n$ are independent of a choice of any well-order Ω of $\mathbb{X}$ with the condition that any lattice points $\mathbf{x}, \mathbf{y}$ with $\ell(\mathbf{x}) < \ell(\mathbf{y})$ has the order $\mathbf{x} < \mathbf{y}$. A sequence of non-negative integers k_n $(n = 1, 2, \dots)$ is a *polynomial growth sequence* if there is an integral polynomial $f(x)$ in one variable x such that $k_n \leqq f(n)$ for all n. Concerning the classifications of $\mathbb{L}^{\mathrm{p}}$ and $\mathbb{M}$, the following question may be interesting:

Question (6.9) Are the sequences l_n and m_n $(n = 1, 2, \dots)$ polynomial growth sequences ?

Let p_n be the number of prime links with the crossing number n. C. Ernst and D.W. Sumners [8] showed that the sequence p_n $(n = 0, 1, 2, \dots)$ is not any polynomial growth sequence by counting the numbers of 2-bridge knots and links.

7. Notes on the oriented version and oriented 3-manifold invariants

Let $\vec{\mathbb{M}}$ be the set of closed connected oriented 3-manifolds. Using the injection $\vec{\sigma} : \vec{\mathbb{L}} \to \mathbb{X}$, we have a well-order in $\vec{\mathbb{L}}$ induced from a well-order Ω in $\mathbb{X}$ and also denoted by Ω. Writing

$$\vec{\mathbb{L}}^{\pi} = \iota^{-1}\mathbb{L}^{\pi} \subset \vec{\mathbb{L}},$$

we can show that the embedding $\alpha : \mathbb{M} \to \mathbb{L}$ in Theorem (1.1) lifts to an embedding

$$\vec{\alpha} : \vec{\mathbb{M}} \longrightarrow \vec{\mathbb{L}}$$

such that $\chi_0\vec{\alpha} = 1$ and $\vec{\alpha}(-M) = -\overline{\vec{\alpha}(M)}$ for every $M \in \vec{\mathbb{M}}$, where the map $\chi_0 : \vec{\mathbb{L}} \to \vec{\mathbb{M}}$ denotes the oriented vbersion of the 0-surgery map $\chi_0 : \mathbb{L} \to \mathbb{M}$. To see this, for every $M \in \vec{\mathbb{M}}$, we note that the link $L_0 = \mathrm{cl}\beta\sigma_\alpha(M)$ is canonically oriented and $\chi(L_0, 0) = \pm M$, where $-M$ denotes the same M but with the orientation reversed. If $M = -M$, then we define $\vec{\alpha}(M) = L_0$. If $M \neq -M$, then we define $\vec{\alpha}(M)$ so as to satisfy

$$\{\vec{\alpha}(M), \vec{\alpha}(-M)\} = \{L_0, -\bar{L}_0\} \quad \text{and} \quad \chi(\vec{\alpha}(M), 0) = M.$$

As a related question, it would be interesting to know *whether or not there is an oriented link* $L \in \vec{\mathbb{L}}$ *with* $L = -\bar{L}$ and $\chi(L, 0) = M$ *for every* $M \in \vec{\mathbb{M}}$ *with* $M = -M$.

For an algebraic system Λ, an *oriented* 3-*manifold invariant* in Λ is a map $\vec{\mathbb{M}} \to \Lambda$ and *an oriented link invariant* in Λ is a map $\vec{\mathbb{L}} \to \Lambda$. Let $\mathrm{Inv}(\vec{\mathbb{M}}, \Lambda)$ and $\mathrm{Inv}(\vec{\mathbb{L}}, \Lambda)$ be the sets of oriented 3-manifold invariants and oriented link invariants in Λ, respectively. Then we have $\chi_0\vec{\alpha} = 1$. We consider the following sequence

$$\mathrm{Inv}(\vec{\mathbb{M}}, \Lambda) \xrightarrow{\chi_0^{\#}} \mathrm{Inv}(\vec{\mathbb{L}}, \Lambda) \xrightarrow{\vec{\alpha}^{\#}} \mathrm{Inv}(\vec{\mathbb{M}}, \Lambda)$$

of the dual maps $\vec{\alpha}^{\#}$ and $\chi_0^{\#}$ of $\vec{\alpha}$ and χ_0. Since the composite $\vec{\alpha}^{\#}\chi_0^{\#} = 1$, we see that $\chi_0^{\#}$ is injective and $\vec{\alpha}^{\#}$ is surjective, both of which imply that *every oriented* 3-*manifold invariant can be obtained from an oriented link invariant.* More precisely, if I is an oriented 3-manifold invariant, then $\chi_0^{\#}(I)$ is an oriented link invariant which takes the same value $I(M)$ on the subset $\vec{\mathbb{L}}(M) = \{L \in \vec{\mathbb{L}} | \chi(L, 0) = M\}$ for every $M \in \vec{\mathbb{M}}$. Conversely, if J is an oriented link invariant, then $\vec{\alpha}^{\#}(J)$ is an oriented 3-manifold invariant and every oriented 3-manifold invarint is obtained in this way. Here is an example creating an oriented 3-manifold invariant from an oriented link invariant when we use the right inverse $\vec{\alpha}$ of χ_0, defined by the canonical order Ω_c.

Example (7.1) We denote by V a Seifert matrix associated with a connected Seifert surface of the link (see [15]). Then the signature $\mathrm{sign}(V+V')$ and the determinant $\det(tV-V')$ give respectively oriented link invariants, that is, the signature invariant $\lambda \in \mathrm{Inv}(\vec{\mathbb{L}}, \mathbb{Z})$ and the (one variable) Alexander polynomial $A \in \mathrm{Inv}(\vec{\mathbb{L}}, \mathbb{Z}[t, t^{-1}])$ (an oriented link invariant up to multiples of $\pm t^m$, $m \in \mathbb{Z}$). For the right inverse $\vec{\alpha}$ of χ_0 using the canonical order Ω_c, we have the oriented 3-manifold invariants

$$\lambda_{\vec{\alpha}} = \vec{\alpha}^{\#}(\lambda) \in \mathrm{Inv}(\vec{\mathbb{M}}, \mathbb{Z}) \quad \text{and} \quad A_{\vec{\alpha}} = \vec{\alpha}^{\#}(A) \in \mathrm{Inv}(\vec{\mathbb{M}}, \mathbb{Z}).$$

For some 3-manifolds, these invariants are calculated as follows:

(7.1.1) $\lambda_{\vec{\alpha}}(S^1 \times S^2) = 0$, $A_{\vec{\alpha}}(S^1 \times S^2) = 1$.
(7.1.2) $\lambda_{\vec{\alpha}}(S^3) = -1$, $A_{\vec{\alpha}}(S^3) = t-1$.
(7.1.3) $\lambda_{\vec{\alpha}}(\pm Q) = \mp 3$, $A_{\vec{\alpha}}(\pm Q) = (t-1)(t^2+1)$ (we note that $Q \neq -Q$).
(7.1.4) $\lambda_{\vec{\alpha}}(P^3) = -4$, $A_{\vec{\alpha}}(P^3) = (t-1)^2$.
(7.1.5) $\lambda_{\vec{\alpha}}(T^3) = 0$, $A_{\vec{\alpha}}(T^3) = (t-1)^4$.

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