A CLASS OF TORUS MANIFOLDS WITH NONCONVEX ORBIT SPACE

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ABSTRACT. We study a class of smooth torus manifolds whose orbit space has the structure of a simple polytope with holes. We prove that these manifolds have stable almost complex structure and give combinatorial formula for some of their Hirzebruch genera. They have (invariant) almost complex structure if they admit positive omniorientation. In dimension four, we calculate the homology groups, construct symplectic structure on a large class of these manifolds, and give a family which is symplectic but not complex.

1. INTRODUCTION

The moment polytope of the Hamiltonian action of the real torus $\mathbb{T}^n$ on a smooth projective toric variety (toric manifold) may be identified with the orbit space of the action. The moment polytope (Delzant polytope) is rather rigid with severe integrality constraints, see [Sil01]. In 1991 Davis and Januskiewicz [DJ91] introduced a generalization of toric manifolds, now known as quasitoric manifolds, which may be obtained as identification spaces of $P \times \mathbb{T}^n$ where P is a simple n-dimensional polytope. In general these spaces do not have algebraic or invariant symplectic structure, but they still have a lot of remarkable properties; see the survey [BP02]. In this article we study a class of even dimensional manifolds which may be obtained as identification space of $P \times \mathbb{T}^n$ where P is not convex, but a simple polytope with holes which are also simple polytopes. In [Mas99] and [HM03], Masuda and Hattori introduced the notion of torus manifold which is an even dimensional manifold with effective action of the half dimensional torus such that the fixed point set is nonempty. The manifolds considered by us are a special class of torus manifolds. As in the case of quasitoric manifolds, the torus action on these manifolds is *locally standard*, i.e. locally equivalent to the natural action, up to automorphism, of $U(1)^n$ on $\mathbb{C}^n$.

We describe the combinatorial construction of these manifolds in section 2. However, these manifolds are also obtained by gluing quasitoric manifolds along deleted neighborhoods of principal torus orbits (Lemma 2.1). We refer to this as the fiber sum construction. This is used to impart the manifolds with smooth structure (Lemma 2.1) and stable complex structure (Lemma 4.1). We give a combinatorial formula for the χ_y genus of these manifolds (Theorem 4.4) following the work of Panov [Pan01] in quasitoric case. These manifolds admit almost complex structure if they admit a positive omniorientation (Lemma 4.2 and Theorem 4.3). Positive omniorientation is also a necessary condition if we require the almost complex structure to be invariant.

These manifolds cannot admit an invariant symplectic structure if the orbit space has at least one hole (Lemma 5.1). In dimension four, we use the symplectic fiber sum technique to construct symplectic structure on the manifolds obtained by gluing nonsingular toric varieties (Theorem 5.2). We then give examples of symplectic torus manifolds that are

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not complex. It would be interesting to know if any of these torus manifolds is indeed complex.

A lot is known about the topological invariants of these manifolds from the works [Mas99] and [HM03]. However as they often have nontrivial homology in odd degrees, explicit formulas for their cohomology ring (or homology groups) are not known. In section 3, we give a combinatorial formula for the homology groups when dimension is four. We also describe a method for computing the cohomology ring for the four dimensional manifolds.

2. Construction and smooth structure

2.1. **Polytope with holes.**

A polytope is the convex hull of a finite set of points in $\mathbb{R}^n$. An n-dimensional polytope is said to be simple if every vertex is the intersection of exactly n codimension one faces. Let P_0 be an n-dimensional simple polytope in $\mathbb{R}^n$. Let $P_1, P_2, \dots, P_s$ be a disjoint collection of simple polytopes belonging to the interior of P_0. Let

$$P = P_0 - \bigcup_{k=1}^{s} P_k^{\circ}. \tag{2.1}$$

We call P an n-dimensional *polytope with simple holes*. The polytopes $P_1, P_2, \dots, P_s$ are called holes of P. The faces of P are the faces of P_k, $k = 0, \dots, s$.

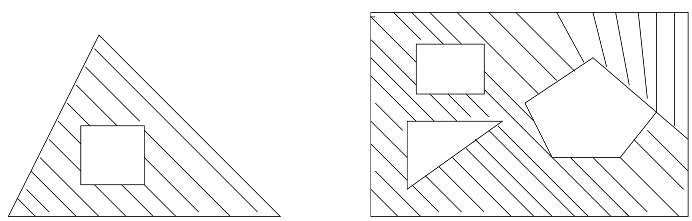

Figure 1. Polytopes with simple holes in $\mathbb{R}^2$.

2.2. **Combinatorial construction.**

Let P be an n-dimensional simple polytope with s simple holes. Let $\mathcal{F}(P) = \{F_1, F_2, \dots, F_m\}$ be the set of all codimension one faces (*facets*) of P. Note that $\mathcal{F}(P) = \bigcup_{k=0}^{s} \mathcal{F}(P_k)$. Also, if F is a nonempty face of P of codimension k then F is the intersection of a unique collection of k facets of P. The following definition is a straightforward generalization of the notion of characteristic function for a simple polytope, which is a crucial concept for studying quasitoric manifolds [DJ91, BP02].

Definition 2.1. *A function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}^n$ is called a characteristic function if it satisfies the following condition: Whenever $F = \bigcap_{j=1}^{k} F_{i_j}$ is a $(n-k)$-dimensional face of P, the span of the vectors $\lambda(F_{i_1}), \lambda(F_{i_2}), \dots, \lambda(F_{i_k})$ is a k-dimensional direct summand of $\mathbb{Z}^n$. We will denote $\lambda(F_i)$ by λ_i for simplicity and call it the characteristic vector of F_i.*

For any face $F = \bigcap_{j=1}^{k} F_{i_j}$ of P, let $N(F)$ be the submodule of $\mathbb{Z}^n$ generated by $\lambda_{i_1}, \dots, \lambda_{i_k}$. The module $N(F)$ defines a sub-torus G_F of $\mathbb{T}^n = \mathbb{Z}^n \otimes \mathbb{R}/\mathbb{Z}^n = \mathbb{R}^n/\mathbb{Z}^n$ as follows.

$$G_F := (N(F) \otimes \mathbb{R})/N(F). \tag{2.2}$$

Define an equivalence relation $\sim$ on the product space $\mathbb{T}^n \times P$ by

$$(t, x) \sim (u, y) \text{ if } x = y \text{ and } u^{-1}t \in G_{F(x)} \tag{2.3}$$

where $F(x)$ is the unique face of P whose relative interior contains x.

We denote the quotient space as follows.

$$M = M(P, \lambda) := (\mathbb{T}^n \times P)/\sim . \tag{2.4}$$

The space M is a $2n$-dimensional manifold. The proof of this is analogous to the quasitoric case [DJ91]. The $\mathbb{T}^n$ action on $(\mathbb{T}^n \times P)$ induces a natural effective action of $\mathbb{T}^n$ on M, which is locally standard (see [DJ91]). Let $\pi\colon M \to P$ be the projection or orbit map defined by $\pi([(t,x)]) = x$.

The fixed point set corresponds bijectively to the set of vertices of P. Hence M is a torus manifold (cf. [HM03]). We say that M is the torus manifold derived from the *characteristic pair* (P, λ).

Observe that the sets $\{X_i := \pi^{-1}(F_i) : i = 1, \ldots, m\}$ are the *characteristic submanifolds* [HM03] of M. Each X_i is a $2(n-1)$-dimensional quasitoric manifold.

2.3. **Fiber sum construction.**

Lemma 2.1. The torus manifold $M(P, \lambda)$ is smooth and orientable.

Proof. By induction it is sufficient to prove that $M(P, \lambda)$ has a smooth structure when P is a polytope with one hole, that is, $P = P_0 - P_1^0$. Let $\mathcal{F}(P_0)$ and $\mathcal{F}(P_1)$ be the set of facets of P_0 and P_1 respectively. The restrictions λ_0 and λ_1 of λ on $\mathcal{F}(P_0)$ and $\mathcal{F}(P_1)$ are characteristic functions on P_0 and P_1 respectively. Let M_0 and M_1 be the quasitoric manifolds associated to the characteristic pairs (P_0, λ_0) and (P_1, λ_1) respectively. These manifolds, being quasitoric, have smooth structure.

Let $\pi_k : M_k \to P_k,\ k = 0, 1$ be the orbit maps. Fix points $x_k \in P_k^\circ$. Let

$$L_k = \pi_k^{-1}(x_k). \tag{2.5}$$

Let $U_k \subset M_k$ be a $\mathbb{T}^n$ invariant neighborhood of L_k such that

$$B_k := \pi_k(U_k) \subset P_k \tag{2.6}$$

is diffeomorphic to an open ball in $\mathbb{R}^n$.

The quasitoric manifolds M_k are orientable. An orientation on M_k is determined by orientations on P_k and $\mathbb{T}^n$. We fix an orientation on $\mathbb{T}^n$ for once and for all, corresponding to the standard orientation on its Lie algebra. We also induce orientations on each P_k from the standard orientation on $\mathbb{R}^n$.

By (2.6) there exist equivariant orientation preserving diffeomorphisms

$$f_k : U_k \to \mathbb{T}^n \times B, \tag{2.7}$$

where B is the unit n-ball centered at the origin. Denote the punctured unit n-ball, $B - \{0\}$, by B^-.

Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be the standard Cartesian and angular coordinates on $\mathbb{R}^n$ and $\mathbb{T}^n$ respectively. Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. Define

$$r := |\mathbf{p}| \quad \text{and} \quad \Theta = (\theta_1, \ldots, \theta_n) := \frac{\mathbf{p}}{r}. \tag{2.8}$$

The space $M(P, \lambda)$ can be obtained from $M_0 - L_0$ and $M_1 - L_1$ by identifying $U_0 - L_0$ and $U_1 - L_1$ as follows. Let $g : B^- \to B^-$ be the orientation preserving involution,

$$g(\mathbf{p}) = \frac{\sqrt{1 - r^2}}{r}(p_1, \ldots, p_{n-1}, -p_n). \tag{2.9}$$

In other words, $g(r, \Theta) = (\sqrt{1-r^2}, \theta_1, \ldots, \theta_{n-1}, -\theta_n)$. Define

$$h = f_0^{-1} \circ (Id \times g) \circ f_1. \tag{2.10}$$

Identify $U_0 - L_0$ with $U_1 - L_1$ by the orientation preserving equivariant diffeomorphism h. □

REMARK 2.2. We refer to the above gluing construction as fiber sum construction because of its similarity to the symplectic fiber sum construction.

REMARK 2.3. The sign of the characteristic vectors do not affect the equivariant diffeomorphism type of M. This follows from similar observation for quasitoric manifolds, see [DJ91, BR01].

2.4. **Omniorientation.** We fix an orientation for $M(P, \lambda)$ as above by choosing standard orientations on $\mathbb{T}^n$ and $\mathbb{R}^n$. Also each characteristic submanifold X_i is quasitoric and hence orientable.

Definition 2.2. *An omniorientation is an assignment of orientation for $M(P, \lambda)$ as well as for each X_i. Given such an assignment, we say that $M(P, \lambda)$ is omnioriented.*

Given the above choice of orientation for M, the characteristic function λ determines a natural omniorientation on M as follows: The characteristic vector λ_i determines a fiberwise S^1 action on the normal bundle of X_i, corresponding to the isotropy group G_{F_i}. This equips the normal bundle with a complex structure and therefore an orientation. This, together with the orientation on M, induces an orientation on X_i. We will refer to this omniorientation as the *characteristic omniorientation*.

Consider an omniorientation on M. Let $v \in M$ be a fixed point of the $\mathbb{T}^n$ action (or corresponding vertex of P). If the orientation of $T_v(M)$ determined by the orientation on M and the orientations of characteristic submanifolds containing v coincide then the *sign* $\sigma(v)$ is defined to be 1, otherwise $\sigma(v)$ is -1.

Definition 2.3. *An omniorientation is called positive if $\sigma(v) = 1$ for each fixed point v.*

For the characteristic omniorientation, the sign of a vertex v may be computed as follows [BP02]. Suppose $v = F_{i_1} \cap \ldots \cap F_{i_n}$. To each codimension one face F_{i_k} assign the unique edge E_k such that $E_k \cap F_{i_k} = v$. Let e_k be a vector along E_k with origin at v. Order (rename) the e_ks so that $e_1, \ldots, e_n$ is a positively oriented basis for $\mathbb{R}^n$. Consider the corresponding matrix $\Lambda_{(v)} = [\lambda_{i_1} \ldots \lambda_{i_n}]$. Then

$$\sigma(v) = \det \Lambda_{(v)}. \tag{2.11}$$

REMARK 2.4. It is also evident that the oriented intersection number of the submanifolds $X_{i_1}, \ldots, X_{i_n}$ is $\sigma(v)$.

3. Calculations in dimension four

Let $\pi : M(P, \lambda) \to P$ be a 4-dimensional torus manifold, where P is a polytope with s simple holes. We give a CW structure on $M(P, \lambda)$ and compute the homology groups.

First assume that P has only one hole. Then $P = P_0 - P_1^0$, where P_0 and P_1 are simple 2-dimensional polytopes with vertices $\{v_1, \ldots, v_{l_0}\}$ and $\{u_1, \ldots, u_{l_1}\}$ respectively. Assume that $dist(v_1 u_1) \leq dist(v_1 u_j)$ for all $j = 1, \ldots, l_1$. Let E_{v_i} and E_{u_j} be the edges of P joining the vertices $\{v_i, v_{i+1}\}$ and $\{u_j, u_{j+1}\}$ respectively for $i = 1, \ldots, l_0$; $j = 1, \ldots, l_1$. Here assume $v_{l_0+1} = v_1$ and $u_{l_1+1} = u_1$. Let $E_{v_1 u_1}$ be the line segment joining v_1 and u_1.

We construct the i-th skeleton X_i of $M(P,\lambda)$ as follows. Let $X_0 = \{v_1, \dots, v_{l_0-1}, u_1, \dots, u_{l_1}\}$. Define

$$\begin{aligned} &e^1_i = (\{(1,1)\} \times E_{v_i})/\sim && \text{for } i = 1, \dots, l_0 - 2 \\ &e^1_{l_0-1} = (\{(1,1)\} \times E_{v_1u_1})/\sim \\ &e^1_{l_0+j-1} = (\{(1,1)\} \times E_{v_j})/\sim && \text{for } j = 1, \dots, l_1 \\ &X_1 = \cup_{i=1}^{l_0+l_1-1} \overline{e^1_i}. \end{aligned} \tag{3.1}$$

Define

$$\begin{aligned} &e^2_i = ((\mathbb{T}^2 \times E_{v_i})/\sim) - \overline{e^1_i} && \text{for } i = 1, \dots, l_0 - 2 \\ &e^2_{l_0-1} = ((\{1\} \times S^1 \times E_{v_1u_1})/\sim) - \overline{e^1_{l_0-1}} \\ &e^2_{l_0} = ((S^1 \times \{1\} \times E_{v_1u_1})/\sim) - \overline{e^1_{l_0-1}} \\ &e^2_{l_0+j} = ((\mathbb{T}^2 \times E_{v_j})/\sim) - \overline{e^1_{l_0+j-1}} && \text{for } j = 1, \dots, l_1 \\ &X_2 = \cup_{i=1}^{l_0+l_1} \overline{e^2_i}. \end{aligned} \tag{3.2}$$

Define

$$\begin{aligned} &e^3 = ((\mathbb{T}^2 \times E_{v_1u_1})/\sim) - (\overline{e^2_{l_0-1}} \cup \overline{e^2_{l_0}}) \\ &X_3 = \overline{e^3} \cup X_2. \end{aligned} \tag{3.3}$$

Define

$$U^4 = P - \{E_{v_1} \cup \ldots \cup E_{v_{l_0-2}} \cup \partial P_1 \cup E_{v_1u_1}\}. \tag{3.4}$$

Clearly U^4 is homeomorphic to $\mathbb{R}^2_{\geq 0}$. So

$$(\mathbb{T}^2 \times U^4)/\sim \,\cong B^4 = \{x \in \mathbb{R}^4 : |x| < 1\}. \tag{3.5}$$

Define

$$e^4 = (\mathbb{T}^2 \times U^4)/\sim \ \text{ and } X_4 = \overline{e^4} \tag{3.6}$$

For the above CW structure, by reasons of either dimension or orientation, the cellular boundary maps d_2, d_3, d_4 are zero. Since X_1 is homotopic to a circle, we get the following result.

Theorem 3.1. *Suppose P is a 2-polytope with one hole. Then*

$$H_i(M(P,\lambda), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{l_0+l_1} & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 0, 1, 3, 4 \\ 0 & \text{if } i > 4. \end{cases}$$

We can give a similar CW structure on $M(P,\lambda)$ when P is a 2-polytope with multiple holes. The figure 2 gives a representation of the 1-skeleton of such a structure when there are two holes.

Corollary 3.2. *Suppose P is a 2-polytope with m vertices and s simple holes. Then*

$$H_i(M(P,\lambda), \mathbb{Z}) = \begin{cases} \mathbb{Z}^{m+2s-2} & \text{if } i = 2 \\ \mathbb{Z}^s & \text{if } i = 1, 3, \\ \mathbb{Z} & \text{if } i = 0, 4 \\ 0 & \text{if } i > 4. \end{cases}$$

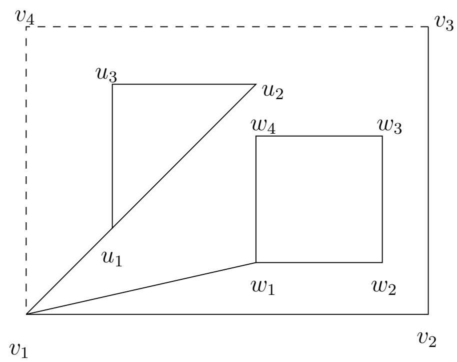

FIGURE 2. 1-skeleton for a 2-polytope with two holes.

3.1. **Cohomology ring.** Assume that M has the characteristic omniorientation. In dimension four it is possible to compute the cohomology ring by using Poincaré duality and intersection product. To illustrate, we consider the case when there is one hole. Let $x_k \in H_2(M)$ denote the homology class of the sphere associated to the 2-cell e^2_k. The products of two classes x_i and x_j when i and j are both less than $l_0 - 1$, is the same as obtained by considering them as classes in $H_*(M_0)$. This is because the homotopies needed to achieve transversality can be done away from a neighborhood of any given principal torus fiber. Similar remarks apply when i and j both exceed l_0. If $i < l_0 - 1$ and $j > l_0$, or vice versa, then the product is obviously zero. Now consider the class x_{l_0-1}.

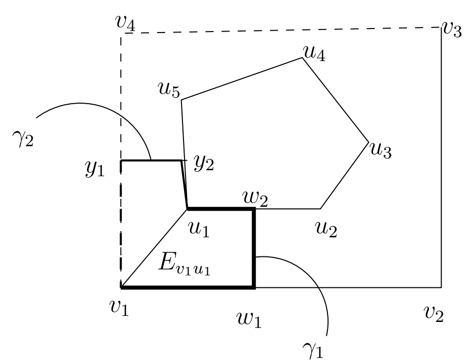

FIGURE 3. Homotopic copies of x_{l_0-1}, here $l_0 = 5$ and $l_1 = 4$.

To compute the self intersection $x^2_{l_0-1}$, we choose two different homotopy representatives, S^2_1 and S^2_2, of x_{l_0-1} which intersect only at v_1 and u_1. Let w_1, y_1 be points in the relative interior of the edges v_1v_2 and $v_{l_0}v_1$ respectively. Similarly let w_2, y_2 be points in the relative interior of the edges u_1u_2 and $u_{l_1}u_1$ respectively. Let $\gamma_1,\ \gamma_2$ be the piecewise linear paths $v_1w_1w_2u_1$ and $v_1y_1y_2u_1$ respectively. Let S^2_i be the homotopy sphere $(\{1\} \times S^1) \times \gamma_i/\sim$. The circle subgroup $\{1\} \times S^1$ corresponds to the submodule of $\mathbb{Z}^2$ generated by $(0, 1)$. It is possible to express $(0, 1)$ uniquely as an integral linear combination $a_1\lambda_1 + a_2\lambda_{l_0}$. Let $d = \det[\lambda_1, \lambda_{l_0}] = -\sigma(v_1)$. Near v_1, the sphere S^2_1 is homotopic to a_2d times the characteristic sphere over v_1v_2. Similarly S^2_2 is homotopic to $-a_1d$ times the characteristic sphere over $v_1v_{l_0}$. Therefore the contribution of v_1 to $x^2_{l_0-1}$ is $(-d)(-a_1d)(a_2d) = a_1a_2d$, see remark 2.4 . The contribution from the point u_1 may be calculated similarly.

Other intersection products of degree 2 classes may be calculated by using similar homotopies. For example, $x_1 \cdot x_{l_0-1} = (-d)(-a_1 d) = a_1$. Finally the intersection of the generators degree one and degree three homology classes is 1 up to sign.

4. Almost complex structure

In this section we prove three results: i) That every omniorientation of M determines a stable almost complex structure on it, ii) that if M admits a positive omniorientation and $\dim(M) = 4$, then there exists an almost complex structure on M which is equivalent to the associated stable complex structure, and iii) that there exists a $\mathbb{T}^n$-invariant almost complex structure on M if and only if M has a positive omniorientation. It is not known to us if the invariant almost complex structure is equivalent to the associated stable almost complex structure.

Lemma 4.1. Every omniorientation of the torus manifold $M(P, \lambda)$ determines a stable almost complex structure on it.

Proof. Let N_i be the normal bundle to the characteristic submanifold $X_i = \pi^{-1}(F_i)$. Suppose F_i is a codimension one face of P_j. Then there exists a complex line bundle ν_i on the quasitoric manifold M_j such that $\nu_i|_{X_i} = N_i$ and ν_i is trivial away from X_i, see [BR01] or [DJ91]. The complex structure on ν_i agrees with the one on N_i determined by the omniorientation. We extend ν_i trivially to a complex line bundle on M, which we denote by the same symbol.

Any point in M has a neighborhood U which can be thought of as belonging to one of the quasitoric manifolds M_j that glue to produce M. By a well-known result on stable complex structure on quasitoric manifolds, TU is stably isomorphic to $\bigoplus_k \nu_{j_k}|_U$ where the line bundles ν_{j_k}s correspond to the characteristic submanifolds of M_j. We may assume that the rest of the ν_is are trivial on U. Therefore $\bigoplus_{i=i}^m \nu_i|_U \cong TU \oplus U \times \mathbb{R}^{2m-2n}$. Consequently we have,

$$\bigoplus_{i=i}^{m} \nu_i \cong TM \oplus M \times \mathbb{R}^{2m-2n}. \tag{4.1}$$

Since the bundle $\bigoplus_{i=i}^m \nu_i$ is complex, the proof is complete. □

The total Chern class of $M(P, \lambda)$ associated to a stable complex structure admits the following product decomposition,

$$c(TM) = \prod_{i=1}^{m} (1 + c_1(\nu_i)). \tag{4.2}$$

Using standard localization formula or theorem 4.4, we obtain

$$c_n(TM) = \sum \sigma(v) \tag{4.3}$$

where the sum is over all vertices of P.

Lemma 4.2. If $M(P, \lambda)$ admits a positive orientation and $\dim(M) = 4$ then it admits an almost complex structure which is equivalent to the associated stable almost complex structure.

Proof. By Theorem 1.7 of [Tho67], the lemma holds if $c_2(TM) = e(TM)$. This follows from (4.3) and corollary 3.2. □

Theorem 4.3. *The torus manifold $M(P, \lambda)$ admits a $\mathbb{T}^n$-invariant almost complex structure if and only if it has a positive omniorientation.*

Proof. The necessity of positive omniorientation for existence of $\mathbb{T}^n$-invariant almost complex structure follows from similar argument as in quasitoric case, see [BP02].

To prove sufficiency, first assume that the number of holes is one. Note that a positive omniorientation of $M(P, \lambda)$ induces positive omniorientation on M_0 and M_1. Then by the work of Kustarev [Kus09], there exist $\mathbb{T}^n$-invariant orthogonal almost complex structures J_k on M_k, $k = 0, 1$. In particular, these structures are orientation preserving. We may assume that the complex structure J_k is locally constant in the normal direction near L_k, as explained below.

Recall the orientation preserving diffeomorphisms f_k in (2.7). Since $T(\mathbb{T}^n \times B)$ is trivial, df_k defines an isomorphism

$$df_k : TU_k \to \mathbb{T}^n \times B \times \mathbb{R}^{2n}. \tag{4.4}$$

Consider the almost complex structures

$$\widehat{J}_k = df_k \circ J_k \circ df_k^{-1} \tag{4.5}$$

on $\mathbb{T}^n \times B \times \mathbb{R}^{2n}$. Choose a smooth non-decreasing function $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma(t) = \begin{cases} 0 & \text{if } t \leq \epsilon_1 \\ t & \text{if } t \geq \epsilon_2 \end{cases} \tag{4.6}$$

where $0 < \epsilon_1 < \epsilon_2 < 1$ are small real numbers. Define

$$J'_k(\mathbf{q}, r, \Theta) = \widehat{J}_k(\mathbf{q}, \gamma(r), \Theta). \tag{4.7}$$

Replace J_k by $df_k^{-1} J'_k df_k$ on U_k. Denote the resulting almost complex structure on M_k by J_k without confusion. Note that these new almost complex structures are orientation preserving and $\mathbb{T}^n$-invariant.

Recall the orientation preserving diffeomorphism g in (2.9). Define

$$\phi_0 := f_0, \quad \phi_1 := (Id \times g) \circ f_1 : U_1 - L_1 \to \mathbb{T}^n \times B^-. \tag{4.8}$$

We have orientation preserving isomorphisms,

$$d\phi_k : T(U_k - L_k) \to \mathbb{T}^n \times B^- \times \mathbb{R}^{2n}. \tag{4.9}$$

Consider the almost complex structures

$$\widetilde{J}_k = d\phi_k \circ J_k \circ d\phi_k^{-1} \tag{4.10}$$

on $\mathbb{T}^n \times B^- \times \mathbb{R}^{2n}$. The space of orientation preserving almost complex structures on $\mathbb{R}^{2n}$ may be identified with $GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$. Since ϕ_k is orientation preserving, we can regard $\widetilde{J}_k$ as a map

$$\widetilde{J}_k : \mathbb{T}^n \times B^- \to GL^+(2n, \mathbb{R})/GL(n, \mathbb{C}). \tag{4.11}$$

Since J_k is locally constant in the normal direction near L_k, we may define

$$\widetilde{J}_0(\mathbf{q}, 0, \Theta) = \widetilde{J}_0(\mathbf{q}, \epsilon_1/2, \Theta), \quad \widetilde{J}_1(\mathbf{q}, 1, \Theta) := \widetilde{J}(\mathbf{q}, \sqrt{1 - (\epsilon_1/2)^2}, \Theta). \tag{4.12}$$

The space $GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ is path connected. Hence there exists path

$$F(t) : [0.4, 0.6] \to GL^+(2n, \mathbb{R})/GL(n, \mathbb{C}),\ F(0.4) = \widetilde{J}_1(\mathbf{1}, 1, \Theta),\ F(0.6) = \widetilde{J}_0(\mathbf{1}, 0, \Theta). \tag{4.13}$$

By $\mathbb{T}^n$-invariance, we construct a smooth family of paths $F(\mathbf{q},t) : \mathbb{T}^n \times [0.4, 0.6] \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$,

$$F(\mathbf{q},t) := d\mathbf{q}F(t)d\mathbf{q}^{-1}, \tag{4.14}$$

satisfying $F(\mathbf{q}, 0.4) = \widetilde{J}_1(\mathbf{q}, 1, \Theta),\ F(\mathbf{q}, 0.6) = \widetilde{J}_0(\mathbf{q}, 0, \Theta)$.
Choose a smooth non-decreasing function $\alpha : (0,1) \to [0,1)$ such that

$$\alpha(t) = \begin{cases} t & \text{if } t \geq 0.8 \\ 0 & \text{if } t \leq 0.6. \end{cases} \tag{4.15}$$

Choose another smooth non-decreasing function $\beta : (0,1) \to (0,1]$ such that

$$\beta(t) = \begin{cases} t & \text{if } t \leq 0.2 \\ 1 & \text{if } t \geq 0.4. \end{cases} \tag{4.16}$$

Define a map $\widetilde{J} : \mathbb{T}^n \times B^- \to GL^+(2n,\mathbb{R})/GL(n,\mathbb{C})$ by

$$\widetilde{J}(\mathbf{q}, r, \Theta) = \begin{cases} \widetilde{J}_0(\mathbf{q}, \alpha(r), \Theta) & \text{if } r > 0.6 \\ F(\mathbf{q}, r) & 0.6 \geq r \geq 0.4 \\ \widetilde{J}_1(\mathbf{q}, \beta(r), \Theta) & \text{if } r < 0.4. \end{cases} \tag{4.17}$$

Note that

$$\widetilde{J}(\mathbf{q}, r, \Theta) = \begin{cases} \widetilde{J}_0(\mathbf{q}, r, \Theta) & \text{if } r > 0.8 \\ \widetilde{J}_1(\mathbf{q}, r, \Theta) & \text{if } r < 0.2. \end{cases} \tag{4.18}$$

Define a $\mathbb{T}^n$-invariant almost complex structure $\overline{J}_k$ on $T(U_k - L_k)$ by

$$\overline{J}_k = d\phi_k^{-1} \circ \widetilde{J} \circ d\phi_k. \tag{4.19}$$

By construction, $\overline{J}_k$ agrees with J_k in a neighborhood of the outer boundary of $U_k - L_k$. Therefore $\overline{J}_k$ extends to a $\mathbb{T}^n$-invariant almost complex structure on $M_k - L_k$. Moreover $\overline{J}_0 \circ dh = dh \circ \overline{J}_1$ on $U_1 - L_1$ since $h = \phi_0^{-1} \circ \phi_1$, see (2.10) and (4.8). Therefore $\overline{J}_0$ and $\overline{J}_1$ glue to produce a $\mathbb{T}^n$-invariant almost complex structure $\overline{J}$ on M. Finally, note that we may apply induction when the number of holes is greater than one. □

4.1. χ_y **genus.** The Hirzebruch χ_y genus is an invariant of the complex cobordism class of the manifold and thus depends on the stable almost complex structure. We give a combinatorial formula of the χ_y genus of M, following Panovs work on quasitoric manifolds. The proofs are the same as in [Pan01].

Let E be an edge of P^n. The isotropy subgroup of $\pi^{-1}(E)$ is an $(n-1)$-dimensional torus generated by a submodule K of rank $(n-1)$ in $\mathbb{Z}^n$. A primitive vector μ in $(\mathbb{Z}^n)^*$ is called an edge vector corresponding to E if $\mu(\alpha) = 0$ for each $\alpha \in K$. The edge vector of E is therefore unique up to sign.

Let ν be a primitive vector in $\mathbb{Z}^n$ such that

$$\mu(\nu) \neq 0 \text{ for any edge vector } \mu. \tag{4.20}$$

Then the circle $S^1_\nu = (\mathbb{Z} < \nu > \otimes \mathbb{R})/\mathbb{Z} < \nu >$ acts smoothly on M with only isolated fixed points corresponding to the vertices of P.

We choose signs for each edge vector at a vertex v according to the characteristic omniorientation as follows. Order the codimension one faces meeting at v and corresponding edges E_ks as in subsection 2.4. Let μ_k be an edge vector corresponding to E_k. Let $M_{(v)}$ be the matrix, $M_{(v)} = [\mu_1, \ldots, \mu_k]$. Then choose sign for each μ_k such that $M^t_{(v)}\Lambda_{(v)} = I_n$.

Under this choice of signs the action of S^1_ν induces a representation of S^1 on the tangent space T_vM with weights $\mu_1(\nu), \ldots, \mu_n(\nu)$.

Definition 4.1. *Define the index of a vertex $v \in P$ as the number of negative weights of the S^1 representation on $T_v(M)$,*

$$\mathrm{ind}_\nu(v) = |\{k : \mu_k(\nu) < 0\}|.$$

Theorem 4.4. *For any vector ν satisfying (4.20),*

$$\chi_y(M) = \sum_v (-y)^{\mathrm{ind}_\nu(v)} \sigma(v).$$

Specializing the formula in theorem 4.4 to $y = -1$ and $y = 1$, respectively yield formulas for the top Chern number and the signature. Moreover following Theorem 3.4 of [Pan01] we obtain the following formula for Todd genus of M,

$$\mathrm{td}(M) = \sum_{\mathrm{ind}_\nu(v)=0} \sigma(v). \tag{4.21}$$

5. Symplectic structure

Lemma 5.1. The torus manifold $M(P, \lambda)$ does not support any invariant symplectic form.

Proof. When the dimension $2n > 4$, $M(P, \lambda)$ is simply connected. So any symplectic circle action is Hamiltonian. Therefore if $M(P, \lambda)$ supports a T^n-invariant symplectic form, then the action of T^n must be Hamiltonian. Then $M(P, \lambda)$ would be a symplectic toric manifold with a moment map whose image is a Delzant polytope. Then the orbit space of the T^n-action on $M(P, \lambda)$ would be a Delzant polytope, see Theorem 2.6.2 of [Sil01]. Therefore, as the orbit space of $M(P, \lambda)$ is not convex it cannot support an invariant symplectic form.

When $2n = 4$, a result of McDuff [McD88] states that a symplectic circle action on a compact four dimensional manifold is Hamiltonian if and only if it has fixed points. Therefore, again, if $M(P, \lambda)$ supports a T^n-invariant symplectic form, then the action of T^n must be Hamiltonian. We get a contradiction as above. □

However, when dim $= 4$, we can construct examples of $M(P, \lambda)$ having symplectic structure by using symplectic fiber sum method [Gro86, Gom95].

Theorem 5.2. *Suppose M_i, $i = 0, \ldots, s$, be toric manifolds of real dimension 4. Then the torus manifold M obtained from their fiber sum supports a symplectic structure.*

Proof. First assume that the number of holes, $s = 1$. We follow the notation of lemma 2.1. Consider the standard symplectic structures ω_k on M_k. Denote the Lagrangian torus $\pi_k^{-1}(x_k)$ by L_k. Let (p_1, p_2) and (q_1, q_2) be the standard coordinates on P_k and T^2 respectively. Then the form $\omega_k = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ in an ϵ-tubular neighborhood of L_k. Following an idea of Gompf [Gom95], we will modify this form to make L_k into a symplectic submanifold.

Choose a compactly supported smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi > 0$ on $(-\sqrt{\epsilon}, \sqrt{\epsilon})$ and zero outside this interval. In addition we require global bounds $\phi < a$ and $|\phi'| < b$ where a and b are suitably chosen small positive numbers.

We define a new form $\tilde{\omega}_k$ on M_k by

$$\tilde{\omega}_k = \omega_k + d(\phi(R^2) q_1) \wedge dq_2 \text{ where } R^2 = |(p_1, p_2) - x_k|^2. \tag{5.1}$$

Note that if (c, d) denotes the coordinates of x_k,

$$\tilde{\omega}_k - \omega_k = 2\phi'(R^2)((p_1 - c)q_1 dp_1 \wedge dq_2 + (p_2 - d)q_1 dp_2 \wedge dq_2) + \phi(R^2) dq_1 \wedge dq_2. \tag{5.2}$$

Since q_1 is globally bounded and $(p_1 - c)$ and $(p_2 - d)$ are bounded in the ϵ-neighborhood of L_k, it is possible to control the size of $\tilde{\omega}_k - \omega_k$ by controlling the sizes of ϕ and ϕ'. Since non-degeneracy is an open condition, $\tilde{\omega}_k$ is non-degenerate if $\tilde{\omega}_k - \omega_k$ is small. We achieve this by choosing a and b to be sufficiently small. By construction $\tilde{\omega}_k$ is closed. Thus it is a symplectic form on M_k. One easily checks from (5.2) that the torus L_k is symplectic with respect to $\tilde{\omega}_k$.

Note that the symplectic submanifold L_k still has trivial normal bundle in $(M_k, \tilde{\omega}_k)$. Assume that the tubular neighborhoods U_k (see (2.6)) correspond to $R^2 < \delta_k < \epsilon$. Applying Weinstein's symplectic neighborhood theorem [Wei71] and scaling the standard symplectic form on $\mathbb{T}^2 \times B$ by a constant factor, we may assume that the equivariant diffeomorphisms f_0 and f_1 (see (2.7))are symplectomorphisms. The map g (see (2.9)) is a symplectomorphism when $n = 2$. Therefore the gluing map h (see (2.10)) is also a symplectomorphism. This provides a symplectic structure on $M(P, \lambda)$. For $s > 1$, we just iterate the above construction. □

5.1. **Examples.** Consider the manifolds obtained by symplectic fiber sum of four dimensional toric manifolds. It may be argued using (4.2) and intersection theory, that c_1^2 and c_2 are additive with respect to the fiber sum operation. Hence all of them obey the Bogomolov-Miyaoka-Yau inequality: $c_1^2 \le 3c_2$. So we have to use further details from the Enrique-Kodaira classification of surfaces (see [BPV84]) to exhibit one that is not complex.

For example, suppose M is obtained by fiber summing a copies of $\mathbf{P}^2$ with b many Hirzebruch surfaces. A Hirzebruch surface has characteristic vectors $(1, 0)$, $(0, 1)$, $(-1, k)$ and $(0, -1)$, in that order. One may verify using remark 2.4 that any such complex surface has $c_1^2 = 8$, whereas $c_2 = 4$. Therefore $c_1(M)^2 = 9a + 8b$ and $c_2(M) = 3a + 4b$.

Now consider the case when there is only one hole, i.e. $a + b = 2$. Then by theorem 3.1, the first Betty number $b_1(M) = 1$. As Gompf points out in p. 560 of [Gom95], it follows from the the Enrique-Kodaira classification that no symplectic manifold with $b_1 = 1$ can be homotopic to a complex surface. Therefore none of these manifolds are complex if $a + b = 2$.

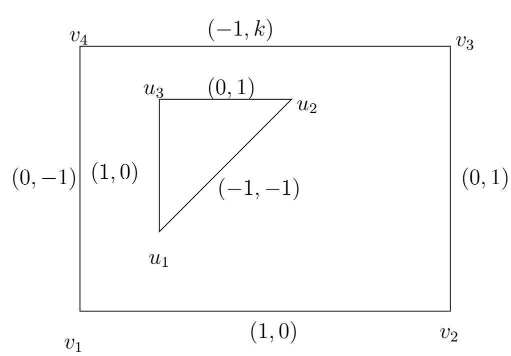

FIGURE 4. Some symplectic but non-complex torus manifolds

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