On the rational T-equivariant cohomology of the weighted Grassmannians

Hiraku Abe
Tokyo Metropolitan University

29 Nov. 2011

Contents

§0. Background

$\mathrm{Gr}(d, n)$: the Grassmannian manifold

§0. Background

Gr(d,n) : the Grassmannian manifold
$H^*(\mathrm{Gr}(d,n):\mathbb{Z})$

§0. Background

Gr(d,n) : the Grassmannian manifold

$H^*(\text{Gr}(d,n):\mathbb{Z})$

$\Omega_\lambda \subset \text{Gr}(d,n)$: Schubert cells

$$\text{Gr}(d,n) = \coprod_{\lambda} \Omega_\lambda$$

$$\overline{\Omega_\lambda} = \coprod_{\lambda \leq \mu} \Omega_\mu$$

§0. Background

Gr(d,n) : the Grassmannian manifold
$H^*(\mathrm{Gr}(d,n):\mathbb{Z})$

$\Omega_\lambda \subset \mathrm{Gr}(d,n)$: Schubert cells

$$\mathrm{Gr}(d,n) = \coprod_\lambda \Omega_\lambda$$

$$\overline{\Omega_\lambda} = \coprod_{\lambda \le \mu} \Omega_\mu$$

$H^*(\mathrm{Gr}(d,n):\mathbb{Z}) = \bigoplus_\lambda \mathbb{Z} S_\lambda$ ($S_\lambda := [\overline{\Omega_\lambda}]$: a Schubert class)

§0. Background

Gr(d,n) : the Grassmannian manifold

$H^*(\mathrm{Gr}(d,n):\mathbb{Z})$

$\Omega_\lambda \subset \mathrm{Gr}(d,n)$: Schubert cells

$$\mathrm{Gr}(d,n) = \coprod_\lambda \Omega_\lambda$$

$$\overline{\Omega_\lambda} = \coprod_{\lambda\leq\mu} \Omega_\mu$$

$H^*(\mathrm{Gr}(d,n):\mathbb{Z}) = \bigoplus_\lambda \mathbb{Z}S_\lambda$ ($S_\lambda := [\overline{\Omega_\lambda}]$: a Schubert class)

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$$

§0. Background

Gr(d,n) : the Grassmannian manifold

$H^*(\mathrm{Gr}(d,n):\mathbb{Z})$

$\Omega_\lambda \subset \mathrm{Gr}(d,n)$: Schubert cells

$$\mathrm{Gr}(d,n) = \coprod_{\lambda} \Omega_\lambda$$

$$\overline{\Omega_\lambda} = \coprod_{\lambda \le \mu} \Omega_\mu$$

$H^*(\mathrm{Gr}(d,n):\mathbb{Z}) = \bigoplus_\lambda \mathbb{Z} S_\lambda$ ($S_\lambda := [\overline{\Omega_\lambda}]$: a Schubert class)

$$S_\lambda S_\mu = \sum_{\nu} c_{\lambda\mu}^{\nu} S_\nu$$

Question : $c_{\lambda\mu}^{\nu} = ?$

$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$

Question : $c^\nu_{\lambda\mu} = ?$

$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$
Question : $c^\nu_{\lambda\mu} = ?$

$\exists$ answers from geometry, representation theory and combinatorics

$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$
Question : $c^\nu_{\lambda\mu} = ?$

$\exists$ answers from geometry, representation theory and combinatorics

Consider the standard T action on Gr(d, n).

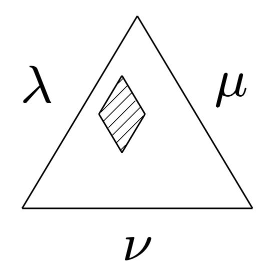
$\rightsquigarrow$ $\{c^\nu_{\lambda\mu}\}$ in $H_T{}^*(\mathrm{Gr}(d, n) : \mathbb{Z})$

$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu$
Question : $c^\nu_{\lambda\mu} = ?$

$\exists$ answers from geometry, representation theory and combinatorics

Consider the standard T action on Gr(d, n).
$\rightsquigarrow$ $\{c^\nu_{\lambda\mu}\}$ in $H_T{}^*(\mathrm{Gr}(d,n):\mathbb{Z})$

Knutson-Tao(2003) :

$c^\nu_{\lambda\mu}$ = the number of equivariant puzzles 

Knutson-Tao(2003) : $H_T(\mathsf{Gr}(d,n):\mathbb{Z})$

Knutson-Tao(2003) : $H_T(\mathrm{Gr}(d,n):\mathbb{Z})$

Tymoczko(2008) : a combinatorial approach to $H_T^*(\mathrm{w}\mathbb{P}^n:\mathbb{Z})$

Knutson-Tao(2003) : $H_T(\mathrm{Gr}(d,n):\mathbb{Z})$

Tymoczko(2008) : a combinatorial approach to $H_T^*(\mathrm{w}\mathbb{P}^n:\mathbb{Z})$

Aim

Compute similar "structure constants" in $H_T^*(\mathrm{wGr}(d,n):\mathbb{Q})$

§1. Definition and basic properties

§1. Definition and basic properties

Plücker embedding : $\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{C}^n)$

§1. Definition and basic properties

Plücker embedding : $\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{C}^n)$

The image of this embedding is the set of the points satisfying the Plücker relation.

§1. Definition and basic properties

Plücker embedding : $\mathrm{Gr}(d,n) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{C}^n)$

The image of this embedding is the set of the points satisfying the Plücker relation.

"Definition" (Corti-Reid, Grojnowski, 2002)

The weighted Grassmannian $\mathrm{wGr}(d,n)$ is defined by the Plücker relation in $\mathrm{w}\mathbb{P}(\wedge^d \mathbb{C}^n)$:

$$\mathrm{wGr}(d,n) \hookrightarrow \mathrm{w}\mathbb{P}(\wedge^d \mathbb{C}^n).$$

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$\mathbb{C}^{\times} \curvearrowright \mathbb{C}^n$ via $\theta \cdot (z_1, \cdots, z_n) = (\theta^{w_1} z_1, \cdots, \theta^{w_n} z_n)$

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^n$ via $\theta \cdot (z_1, \cdots, z_n) = (\theta^{w_1} z_1, \cdots, \theta^{w_n} z_n)$

$\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright \bigwedge^d \mathbb{C}^n$ via $\theta \cdot (x_1 \wedge \cdots \wedge x_d) = \theta(\theta x_1) \wedge \cdots \wedge (\theta x_n)$

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^n$ via $\theta \cdot (z_1, \cdots, z_n) = (\theta^{w_1} z_1, \cdots, \theta^{w_n} z_n)$

$\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright \bigwedge^d \mathbb{C}^n$ via $\theta \cdot (x_1 \wedge \cdots \wedge x_d) = \theta(\theta x_1) \wedge \cdots \wedge (\theta x_n)$

$\rightsquigarrow$ $\mathrm{w}\mathbb{P}(\bigwedge^d \mathbb{C}^n)$: the weighted projective space

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^n$ via $\theta \cdot (z_1, \cdots, z_n) = (\theta^{w_1} z_1, \cdots, \theta^{w_n} z_n)$

$\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright \bigwedge^d \mathbb{C}^n$ via $\theta \cdot (x_1 \wedge \cdots \wedge x_d) = \theta(\theta x_1) \wedge \cdots \wedge (\theta x_n)$

$\rightsquigarrow$ w$\mathbb{P}(\bigwedge^d \mathbb{C}^n)$: the weighted projective space

$\rightsquigarrow$ the Plücker polynomials : weighted-homogeneous

$w_1, \cdots, w_n \in \mathbb{Z}_{\geq 0}$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^n$ via $\theta \cdot (z_1, \cdots, z_n) = (\theta^{w_1} z_1, \cdots, \theta^{w_n} z_n)$

$\rightsquigarrow$ $\mathbb{C}^\times \curvearrowright \bigwedge^d \mathbb{C}^n$ via $\theta \cdot (x_1 \wedge \cdots \wedge x_d) = \theta(\theta x_1) \wedge \cdots \wedge (\theta x_n)$

$\rightsquigarrow$ w$\mathbb{P}(\bigwedge^d \mathbb{C}^n)$: the weighted projective space

$\rightsquigarrow$ the Plücker polynomials : weighted-homogeneous

$\rightsquigarrow$ wGr$(d, n) \subset$ w$\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ by the Plücker relation

§1. Definition and basic properties

Plücker embedding : $\text{Gr}(d, n) \hookrightarrow \mathbb{P}(\wedge^d \mathbb{C}^n)$

The image of this embedding is the points satisfying the Pücker relation.

"Definition" (Corti-Reid, Grojnowski, 2002)

The weighted Grassmannian $\text{wGr}(d, n)$ is defined by the Pücker relation in $\text{w}\mathbb{P}(\wedge^d \mathbb{C}^n)$:

$$\text{wGr}(d, n) \hookrightarrow \text{w}\mathbb{P}(\wedge^d \mathbb{C}^n).$$

A quasi-cell decomposition

wGr(d,n) : the weighted Grassmannian

A quasi-cell decomposition

wGr(d,n) : the weighted Grassmannian

Lemma

There exists a similar decomposition : wGr$(d,n) = \coprod_\lambda \mathrm{w}\Omega_\lambda$

with quasi-cells $\mathrm{w}\Omega_\lambda \cong \mathbb{C}^{d(n-d)-l(\lambda)}/G_\lambda$

A quasi-cell decomposition

wGr(d,n) : the weighted Grassmannian

Lemma

There exists a similar decomposition : wGr$(d,n) = \coprod_\lambda \mathrm{w}\Omega_\lambda$

with quasi-cells $\mathrm{w}\Omega_\lambda \cong \mathbb{C}^{d(n-d)-l(\lambda)}/G_\lambda$

G_λ : a finite subgroup of $S^1 \subset \mathbb{C}^\times$
acting on $\mathbb{C}^{d(n-d)-l(\lambda)}$ as an unitary representation

A quasi-cell decomposition

wGr(d,n) : the weighted Grassmannian

Lemma

There exists a similar decomposition : wGr$(d,n) = \coprod_\lambda$ wΩ_λ

with quasi-cells w$\Omega_\lambda \cong \mathbb{C}^{d(n-d)-l(\lambda)}/G_\lambda$

G_λ : a finite subgroup of $S^1 \subset \mathbb{C}^\times$
acting on $\mathbb{C}^{d(n-d)-l(\lambda)}$ as an unitary representation

$H^i($wGr$(d,n):\mathbb{Q}) \cong H^i(Gr(d,n):\mathbb{Q})$ as vector spaces over $\mathbb{Q}$

§2. A torus action and GKM graph

A torus action on wGr(d, n)

§2. A torus action and GKM graph

A torus action on wGr(d, n)

$T = (\mathbb{C}^\times)^n$: the complex torus

§2. A torus action and GKM graph

A torus action on wGr(d, n)

$T = (\mathbb{C}^{\times})^n$: the complex torus

$T \curvearrowright \mathbb{C}^n$

§2. A torus action and GKM graph

A torus action on wGr(d, n)

$T = (\mathbb{C}^{\times})^n$: the complex torus

$T \curvearrowright \mathbb{C}^n$

$\rightsquigarrow \; T \curvearrowright \bigwedge^d \mathbb{C}^n$ via $t \cdot (x_1 \wedge \cdots \wedge x_d) = (tx_1) \wedge \cdots \wedge (tx_d)$

§2. A torus action and GKM graph

A torus action on wGr(d, n)

$T = (\mathbb{C}^{\times})^n$: the complex torus

$T \curvearrowright \mathbb{C}^n$

$\rightsquigarrow \quad T \curvearrowright \bigwedge^d \mathbb{C}^n \quad$ via $\quad t \cdot (x_1 \wedge \cdots \wedge x_d) = (tx_1) \wedge \cdots \wedge (tx_d)$

$\rightsquigarrow \quad T \curvearrowright \mathrm{w}\mathbb{P}(\bigwedge^d \mathbb{C}^n)$

§2. A torus action and GKM graph

A torus action on wGr(d,n)

$T = (\mathbb{C}^\times)^n$: the complex torus

$T \curvearrowright \mathbb{C}^n$

$\rightsquigarrow \; T \curvearrowright \bigwedge^d \mathbb{C}^n$ via $t \cdot (x_1 \wedge \cdots \wedge x_d) = (tx_1) \wedge \cdots \wedge (tx_d)$

$\rightsquigarrow \; T \curvearrowright$ w$\mathbb{P}(\bigwedge^d \mathbb{C}^n)$

$\bigcup$

wGr(d,n) : T-invariant subset

By the Serre spectral sequence of $\mathrm{wGr}(d,n) \hookrightarrow \mathrm{wGr}(d,n)_T \to BT$,

$$H_T^*(\mathrm{wGr}(d,n):\mathbb{Q}) \text{ : a free } H_T^*(\mathrm{pt}:\mathbb{Q})\text{-module.}$$

By the Serre spectral sequence of wGr$(d, n) \hookrightarrow$ wGr$(d, n)_T \to BT$,

$$H_T^*(\text{wGr}(d,n) : \mathbb{Q}) \text{ : a free } H_T^*(\text{pt} : \mathbb{Q})\text{-module.}$$

Chang-Skjelbred's lemma & Goresky-Kottwitz-MacPherson's argument show the following :

By the Serre spectral sequence of wGr$(d, n) \hookrightarrow$ wGr$(d, n)_T \to BT$,

$$H_T^*(\text{wGr}(d,n):\mathbb{Q}) \text{ : a free } H_T^*(\text{pt}:\mathbb{Q})\text{-module.}$$

Chang-Skjelbred's lemma & Goresky-Kottwitz-MacPherson's argument show the following :

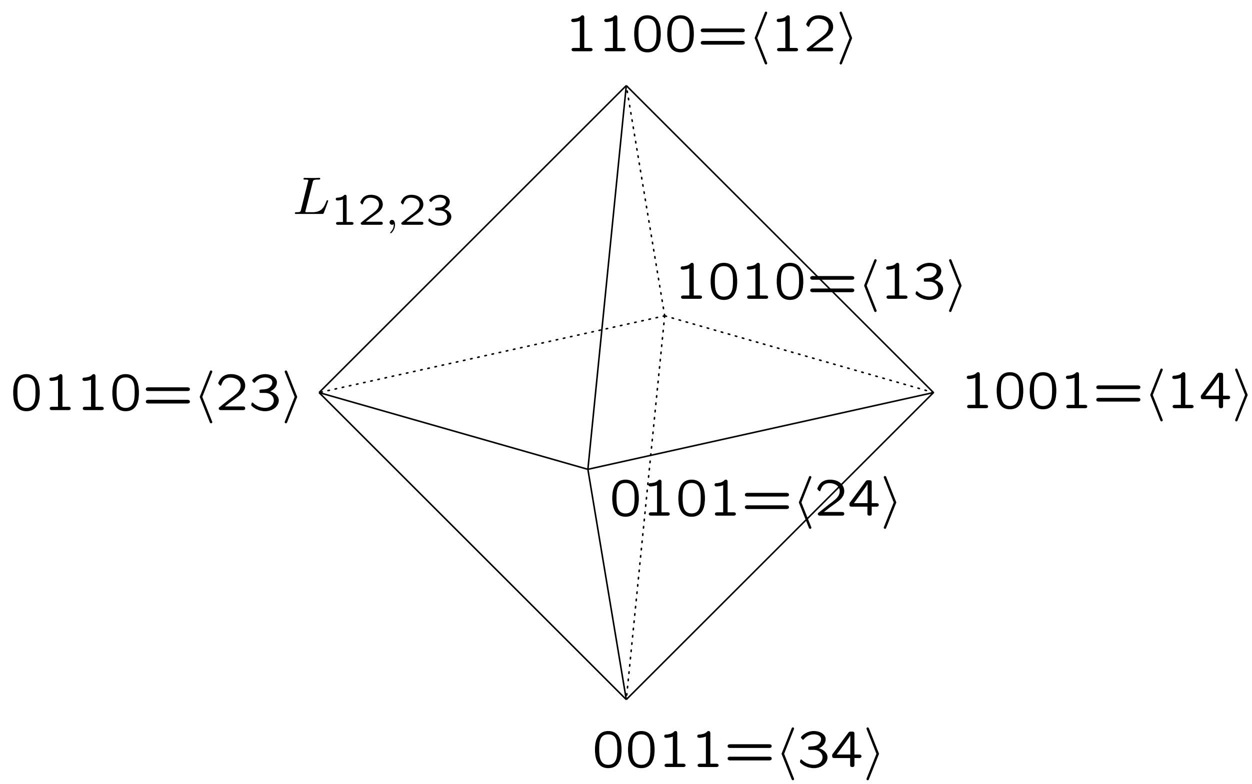
Theorem

$H_T^*(\text{wGr}(d,n) : \mathbb{Q})$ is identified with the GKM subalgebra of $\bigoplus_{\text{wGr}(d,n)^T} \mathbb{Q}[t_1, \cdots, t_n]$

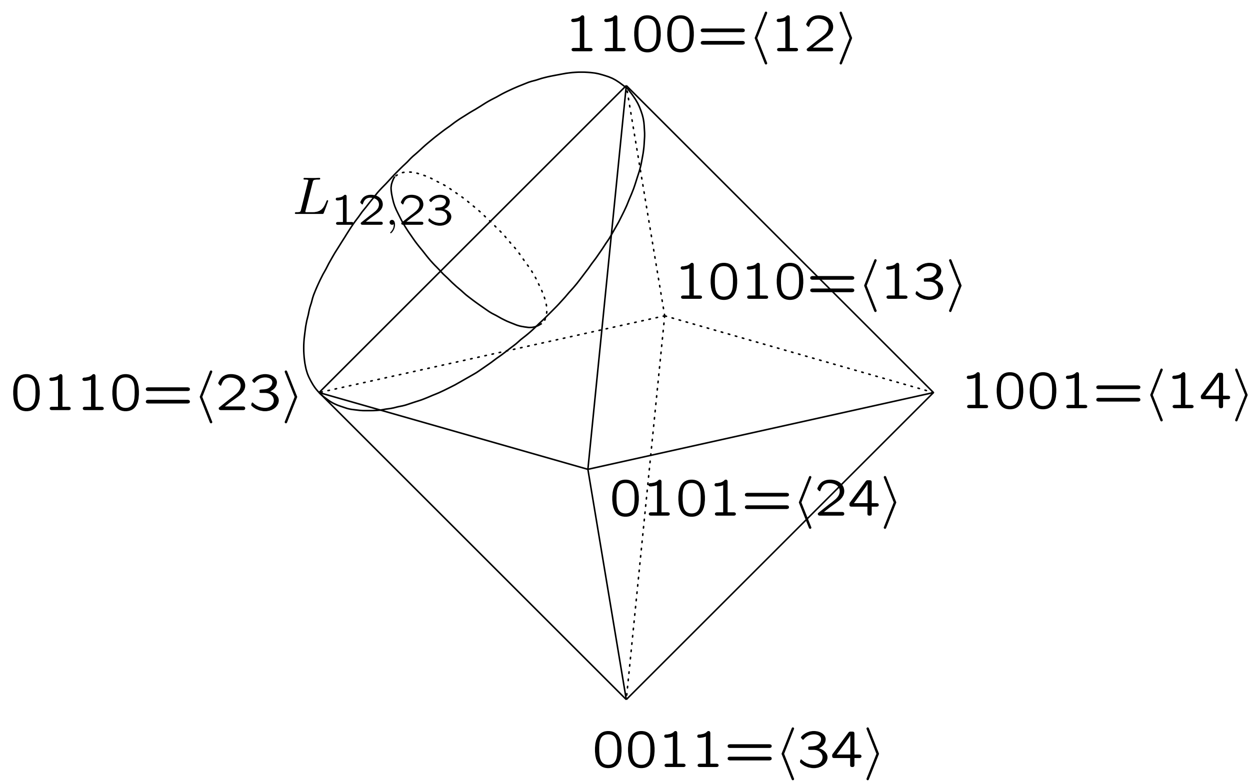
(The GKM subalgebra is described combinatorially!)

GKM graph for Gr$(2, 4)$:

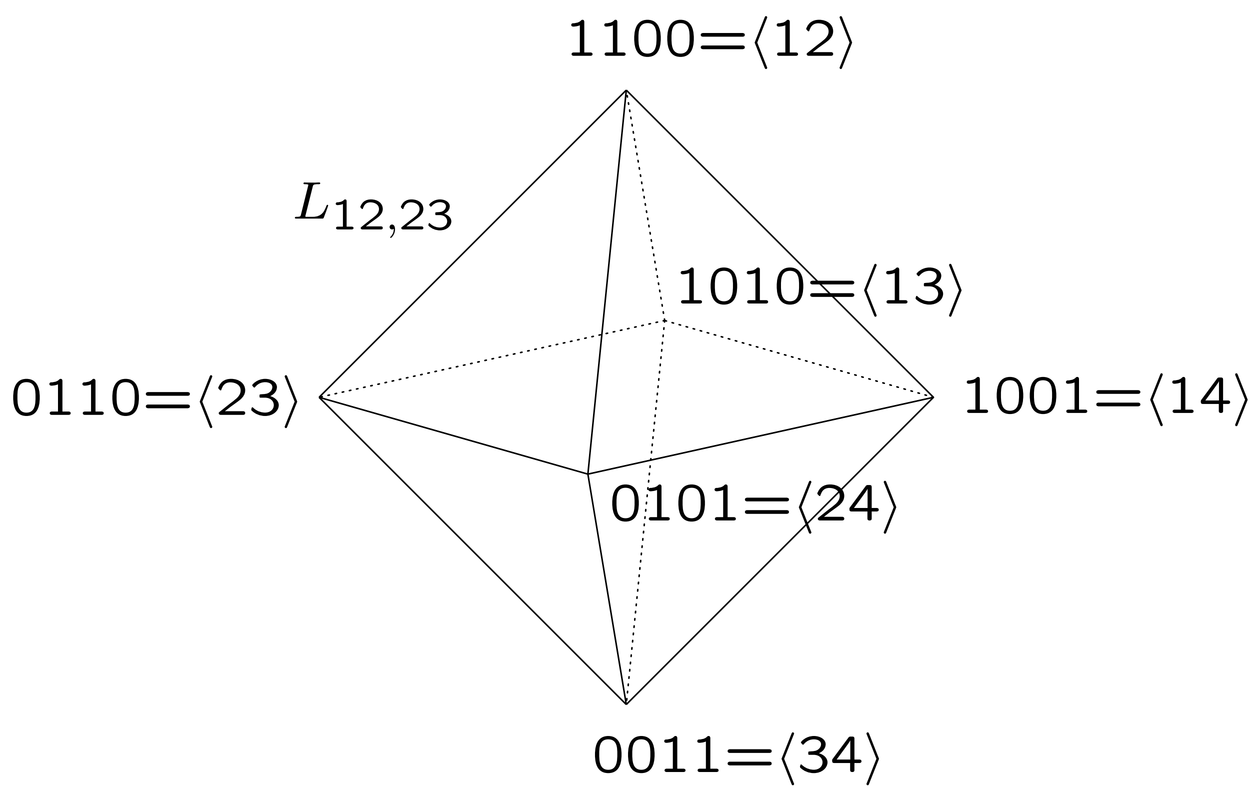
GKM graph for Gr(2, 4):



GKM graph for $\mathrm{Gr}(2,4)$:

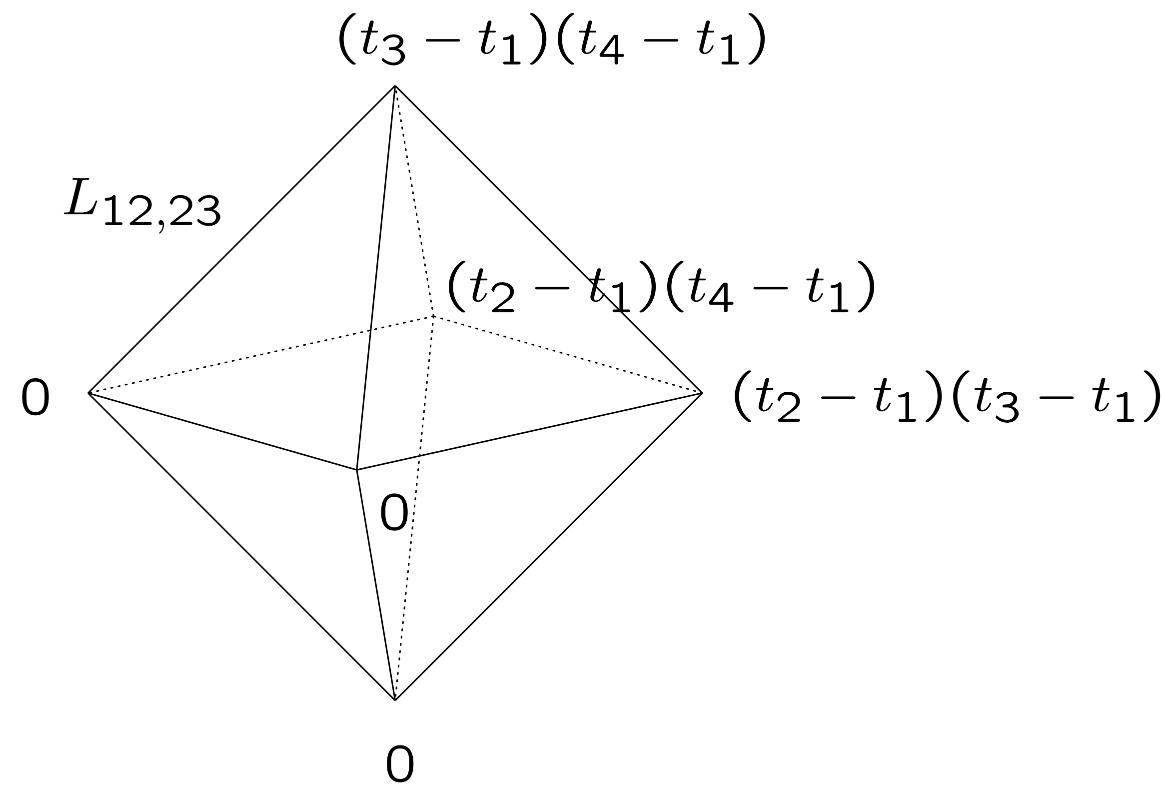


GKM graph for Gr(2, 4):



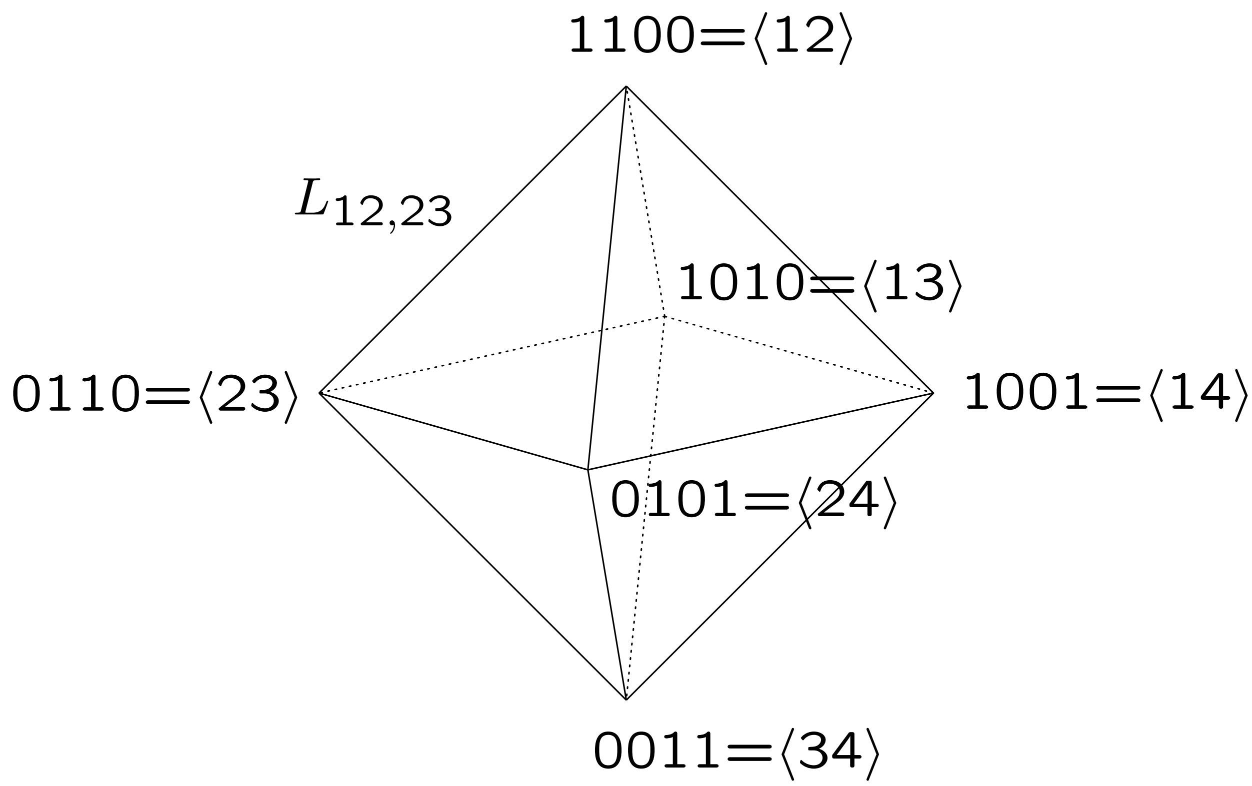
$$L_{12,23} = (t_2 + t_3) - (t_1 + t_2) = t_3 - t_1$$

GKM graph for Gr$(2,4)$:



$$L_{12,23} = (t_2 + t_3) - (t_1 + t_2) = t_3 - t_1$$

GKM graph for wGr(2, 4):



$$L_{12,23} = w_{12}t_{23} - w_{23}t_{12}$$
$$= (w_1 + w_2 + 1)(t_2 + t_3) - (w_2 + w_3 + 1)(t_1 + t_2)$$

§3. Results

Working combinatorially, we obtained the following.

§3. Results

Working combinatorially, we obtained the following.

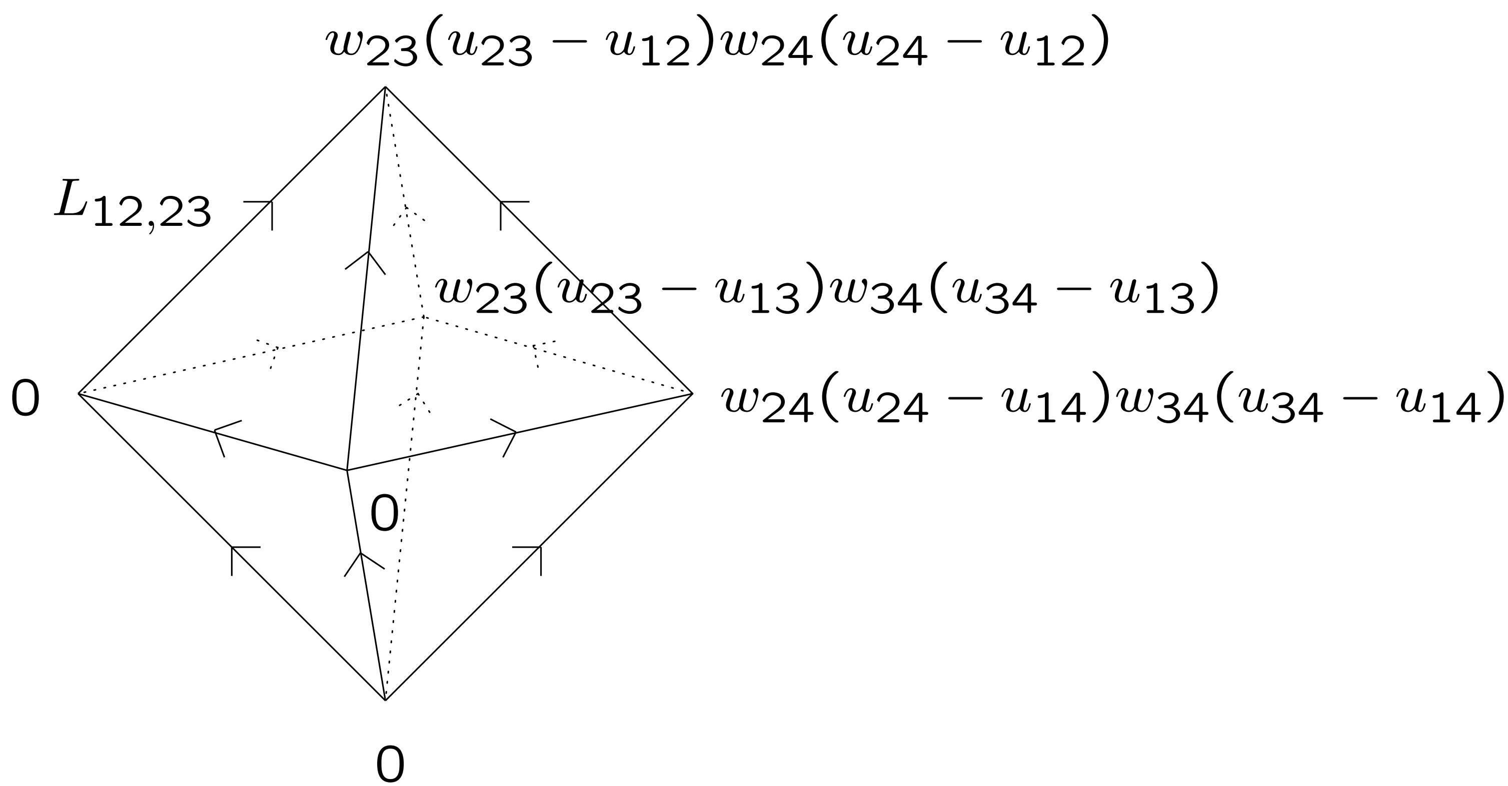
Proposition

There exists a set of canonical classes $\{S_\lambda\}$ for $H_T^*(\mathrm{wGr}(d,n):\mathbb{Q})$, and they form an $H_T^*(\mathrm{pt}:\mathbb{Q})$-module basis.

(In non-weighted case : S_λ is the equivariant Schubert class)

A canonical class for wGr(2, 4):

$$u_{ab} = \frac{1}{w_a + w_b + 1}(t_a + t_b)$$



$$\begin{aligned} L_{12,23} &= (w_1 + w_2 + 1)(t_2 + t_3) - (w_2 + w_3 + 1)(t_1 + t_2) \\ &= w_{23}w_{12}(u_{23} - u_{12}) \end{aligned}$$

§3. Results

Working combinatorially, we obtained the following.

Proposition

There exists a set of Canonical classes $\{S_\lambda\}$ for $H_T^*(\mathrm{wGr}(d,n):\mathbb{Q})$, and they form an $H_T^*(\mathrm{pt}:\mathbb{Q})$-module basis.

(In non-weighted case : S_λ is the equivariant Schubert class)

§3. Results

Working combinatorially, we obtained the following.

Proposition

There exists a set of Canonical classes $\{S_\lambda\}$ for $H_T^*(\text{wGr}(d,n):\mathbb{Q})$, and they form an $H_T^*(\text{pt}:\mathbb{Q})$-module basis.

(In non-weighted case : S_λ is the equivariant Schubert class)

$$S_\lambda S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$

§3. Results

Working combinatorially, we obtained the following.

Proposition

There exists a set of Canonical classes $\{S_\lambda\}$ for $H_T^*(\mathrm{wGr}(d,n):\mathbb{Q})$, and they form an $H_T^*(\mathrm{pt}:\mathbb{Q})$-module basis.

(In non-weighted case : S_λ is the equivariant Schubert class)

$$S_\lambda S_\mu = \sum_\nu c_{\lambda\mu}^\nu S_\nu$$

Proposition

An inductive rule to calculate the structure constants with respect to $\{S_\lambda\}$ can be obtained.

Taking the non-equivariant limit,

$$H^*(\mathsf{wGr}(d,n)\colon\mathbb{Q}) = \bigoplus_{\lambda} \mathbb{Q} S_\lambda$$

$$S_\lambda S_\mu = \sum_{\nu} c_{\lambda\mu}^{\nu} S_\nu \qquad \text{in } H^*(\mathsf{wGr}(d,n)\colon\mathbb{Q})$$

Examples of a product in $H^*(\mathrm{Gr}(2,4):\mathbb{Q})$:

$$S_{24}S_{24} = \qquad S_{23} + \qquad S_{14}$$

$$S_{23}S_{14} = 0$$

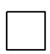
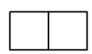
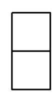
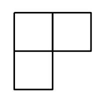
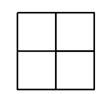
$34 = \emptyset$, $24 = \begin{array}{|c|}\hline \ \\ \hline\end{array}$, $14 = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$, $23 = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, $13 = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $12 = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$

Examples of a product in $H^*(\text{wGr}(2,4):\mathbb{Q})$:

$$w_{ab} = w_a + w_b + 1 \qquad (w_1, w_2, w_3, w_4 \in \mathbb{Z}_{\geq 0})$$

$$S_{24}S_{24} = \frac{w_{34}}{w_{24}} S_{23} + \frac{w_{34}}{w_{24}} S_{14}$$

$$S_{23}S_{14} = \frac{w_4 - w_1}{w_{13}} S_{12}$$

$34 = \emptyset$, $24 =$ , $14 =$ , $23 =$ , $13 =$ , $12 =$ 

Thank you!

Future problems

Relations between canonical classes and (equivariant) quasi-Schubert classes?

$H_T^*(\mathrm{wGr}(d,n)\!:\!\mathbb{Q}) \cong H_T^*(\mathrm{Gr}(d,n)\!:\!\mathbb{Q})$ as $H^*(BT\!:\!\mathbb{Q})$-algebras?

Equivariant Giambelli for canonical classes?

Localization formulas for canonical classes?

$H_T^*(\mathrm{wGr}(d,n)\!:\!\mathbb{Z})$?

$\lambda = (0110100) = < 235 >$

$$w_\lambda = w_2 + w_3 + w_5 + 1$$

$$t_\lambda = t_2 + t_3 + t_5$$

Recurrence relations for the equivariant structure constants $\{c^\nu_{\lambda\mu}\}_{\lambda\mu\nu}$:

$$(1)\quad c^\lambda_{\lambda\lambda} = \prod_{(k,l)\in \mathrm{inv}(\lambda)} w_{(k,l)\lambda} \left(\frac{t_{(k,l)\lambda}}{w_{(k,l)\lambda}} - \frac{t_\lambda}{w_\lambda} \right),$$

$$(2)\quad w_\mu \left(\frac{t_\mu}{w_\mu} - \frac{t_\lambda}{w_\lambda} \right) c^\lambda_{\lambda\mu} = \sum_{\mu' : \mu' \to \mu} c^\lambda_{\lambda\mu'},$$

$$(3)\quad w_\lambda \left(\frac{t_\lambda}{w_\lambda} - \frac{t_\nu}{w_\nu} \right) c^\nu_{\lambda\mu} = \sum_{\lambda' : \lambda' \to \lambda} c^\nu_{\lambda'\mu} - \sum_{\nu' : \nu \to \nu'} \frac{w_\lambda}{w_{\nu'}} c^{\nu'}_{\lambda\mu}.$$