Riemannian metrics with positive Ricci curvature on moment-angle manifolds

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The question of «Geometry in large»:

- Is it possible assuming some local behavior of the space (usually formulated using curvature assumption) to derive some information about the global shape of the space?

The classical «local shape» condition is positivity of sectional curvature, and classical result is sphere theorem:

Theorem (Berger, Klingenberg, 1960 (homeomorphism); Brendle-Schoen, 2007)

Let sectional curvature of simply-connected closed Riemannian manifold M^n satisfies condition $\frac{1}{4} < K \leq 1$.Then M is diffeomorphic to S^n.

- Problem: The lack of examples with $K > 0$.

All known examples were constructed via some Lie group action constructions (homogeneous spaces, biquotients, cohomogeneity one manifolds).

Consider weakening of condition $K > 0$ (or $K \geq 0$): positivity of Ricci curvature, $Ric > 0$.
Examples of positively Ricci-curved manifolds:

- Homogeneous spaces $M = G/H$ and biquotients $K \backslash G/H$, where $Z(G)$ is finite (Milnor).
- Connected sums $\#_{i=1}^N S^2 \times S^2$ (Sha-Yang, 1989).
- Connected sums $\#_{i=1}^N \mathbb{C}P^2$ (Anderson, 1990; improved by Perelman in 1997).
- Connected sums $\#_{i=1}^N S^n \times S^m$ (Sha-Yang, 1991).
- Connected sums of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ (Sha-Yang, 1993).
- Homotopy spheres which bound parallelisable manifolds (Wraith, 1997).
- Connected sums $\#_{i=1}^N S^{n_i} \times S^{m_i}$ with $n_i + m_i$ fixed (Wraith, 2007).

Let remark following:

- All simply connected four-dimensional T^2-manifolds admit Ricci-positive Riemannian metrics.

Moreover, these metrics can be chosen invariant with respect to the T^2-action:

Theorem (B.-Matvienko)

Every simply-connected four-dimensional T^2-manifold admits T^2-invariant Ricci-positive Riemannian metric.

Another simple observation:

- Certain connected sums of products of spheres are moment-angle manifolds.

Our main idea is that (quasi-)toric constructions can give manifolds which admit Riemannian metrics with positive Ricci curvature.

Constructions of positively curved (K>0) or positively Ricci-curved (Ric>0) Riemannan manifolds are based on Riemannian submersion concept.

- Let E, B be Riemannian manifolds and $p : E \to B$. Consider $e \in E, b \in B, p(e) = b$. We call p submersion if

$$d_e p : T_e E \to T_b B$$

be surjection. Put
$V_e = d_e p(0)$ - vertical subspace of submersion,
$H_e = V_e^{\perp}$ - horizontal subspace of submersion.
The map

$$d_e p|_{H_e} : H_e \to T_b B$$

is linear isomorphism. Submersion p is Riemannian, iff $d_e p|_{H_e}$ is an isometry.

Typical example of Riemannian submersion arises from the following construction.

- Let Lie group G acts freely by isometries on Riemannian manifold M. Then there exist canonical Riemannian metric on orbit space — manifold M/G and natural projection $M \to M/G$ is Riemannian submersion.

Theorem (O'Neil)

Let $p : M \to N$ be Riemannian submersion and manifold M has positive (nonnegative) sectional curvature $K > 0$ $(K \geq 0)$. Then N also has $K > 0$ $(K \geq 0)$.

Theorem (Gilkey-Park-Tuschmann, 1998)

Let (Y, g_Y) be a compact connected Riemannian manifold with positive Ricci curvature. Let P be a principal bundle over Y with compact connected structure group G so that $\pi_1(P)$ is finite. Then there exists a G-invariant metric g_P on P so that g_P has positive Ricci curvature and so that $\pi : (P, g_P) \to (Y, g_Y)$ is a Riemannian submersion.

Let M^{2n} be quasi-toric manifold and $P = M/T^n$ — correspondent polyhedra in Euclidean space $\mathbb{R}^n$ with m faces. Then for moment-angle manifold Z_P there exist free action of torus T^{m-n} such that we have principal torus bundle

$$Z_p \to M.$$

If T^{m-n} acts on Z_P not freely such that $M = Z_P/T^{m-n}$ is orbifold we call M quasi-toric orbifold.
In the above situation we have following improvement of previous theorem.

Theorem

If quasi-toric orbifold admits Riemannian metric with positive Ricci curvature then there exist Ricci-positive Riemannian metric on Z_P such that projection above is Riemannian submersion.

Theorem (Model for blowup construction)

For any linear action of $\mathbb{Z}_p$ on $\mathbb{C}P^q$ with $p \geq q$ and with isolated fixed point z there exist Ricci-positive Riemannian metric on some singular blowup P' of $\mathbb{C}P^q$ in the point z. Moreover, there exist arbitrary small neighborhood U of z such that the metric outside of U coincides with Fubini-Study metric on $\mathbb{C}P^q$.

Here under the singular blowup we mean $P' = (\mathbb{C}P^q \setminus U) \cup E$, where E is complex line bundle over some weighted projective space $\tilde{P}^{q-1}$.

The idea of proof of this theorem is to glue orbifold metric of $\mathbb{C}P^q/\mathbb{Z}_p$ with Ricci-flat metric on above E.

The explicit example of such metric can be done in dimension $q = 2$:

$$ds^2 = U(d\tau + \omega)^2 + \frac{1}{U} d\bar{x}^2$$

where $\tau \in S^1$, $\bar{x} \in \mathbb{R}^3$,

$$U = \sum_{i=1}^{p} \frac{1}{|\bar{x} - \bar{x}_i|},$$

and 1-form ω can be found from equation

$$\star d\omega = dU$$

(necessary condition $\Delta U = 0$ holds evidently). In our case we need take set $\bar{x}_i$ to be equal pair of points in $\mathbb{R}^3$, counted with multiplicities.

We need the following result of Gao:

Theorem

In the ball $U(0, \rho_0) = \{x \in \mathbb{R}^n | |x| \leq \rho_0\}$ consider two Riemannian metrics g_0 and g_1 with positive Ricci curvature and with the same 1-jets $J^1(g_0)$ and $J^1(g_1)$ in the point 0. Then there exists Riemannian metric $\bar{g}$ in $U(0, \rho_0)$ with positive Ricci curvature and $0 < \rho_2 < \rho_1 < \rho_0$ such that $\bar{g} = g_1$ for $|x| < \rho_2$ and $\bar{g} = g_0$ for $|x| > \rho_1$.

Combining Gao's result with theorem above we immediately obtain:

- If we have quasi-toric orbifold with invariant Riemannian Ricci-positive metric having isolated singularities which look like $\mathbb{C}^n/\mathbb{Z}_q$, $q \geq n$ then Riemannian quasi-toric manifold (orbifold) obtained by blow-up construction also has Ricci-positive invariant metric.

Moreover, using generalization of the Gao theorem for normal neighborhoods of submanifolds we can assume that singularities locally look like $\mathbb{R}^{n-k} \times \mathbb{C}^k/\mathbb{Z}_q$, $q \geq k$.

Now we can explain our scheme of constructing Ricci-positive Riemannian metrics on certain moment-angle manifolds:

$$\begin{array}{ccc} Z_Q & & Z_P \\ \downarrow & & \downarrow \\ M & \leftarrow & N \end{array}$$

Let Z_Q be moment-angle manifold with nonnegative sectional curvature $K \geq 0$. By O'Neil theorem quasitoric manifold (orbifold) $M^n = Z_Q/T^{m-n}$ also has nonnegative sectional curvature and has positive Ricci curvature. Consider such action of T^{m-n} that M is an orbifold with singularities satisfying theorems above.
Horizontal arrow on the diagram means blow-up construction preserving positive Ricci curvature. So we obtain quasi-toric Ricci-positive orbifold N and principal torus bundle $Z_P \to N$. Improved version of theorem of Gilkey-Park-Tuschmann gives Ricci-positive metric on Z_P.

As application of our construction we can prove following theorem.

Theorem (B.)

Let $Q = \Delta^{n_1} \times \dots \Delta^{n_k}$ be product of simplexes. Consider polyhedra P obtained by truncating some facets of Q. Then there exist Riemannian metric of positive Ricci curvature on moment-angle manifold Z_P.

Thank You!