Invariance of Pontrjagin classes of torus manifolds

Suyoung Choi
schoi@ajou.ac.kr

Ajou University

28th November 2011, Osaka City Univ., Osaka, Japan

Invariance of Pontrjagin classes

- The Pontrjagin class is preserved by any diffeomorphism

[a] S. P. Novikov, *Topological invariance of rational classes of Pontrjagin*, Dokl. Akad. Nauk SSSR 163(1965).
[b] D.Montgomery, C.T.Yang, *Differentiable actions on homotopy seven spheres*, Trans. AMS. 122 (1966)
[c] W.-c.Hsiang, *A note on free differentiable actions of* S^1 *and* S^3 *on homotopy spheres*, Ann. of Math.(2) 83(1966)

Invariance of Pontrjagin classes

- The Pontrjagin class is preserved by any diffeomorphism, homeomorphism (if the cohomology ring is torsion-free).[a]

[a] S. P. Novikov, *Topological invariance of rational classes of Pontrjagin*, Dokl. Akad. Nauk SSSR 163(1965).

[b] D.Montgomery, C.T.Yang, *Differentiable actions on homotopy seven spheres*, Trans. AMS. 122 (1966)

[c] W.-c.Hsiang, *A note on free differentiable actions of* S^1 *and* S^3 *on homotopy spheres*, Ann. of Math.(2) 83(1966)

Invariance of Pontrjagin classes

- The Pontrjagin class is preserved by any diffeomorphism, homeomorphism (if the cohomology ring is torsion-free).[a]
- However, the Pontrjagin class is not preserved by homotopy equivalences.

Montgomery-Yang (1966),[b] Hsiang (1966)[c]
There are infinitely many homotopy $\mathbb{C}P^n$'s whose Pontrjagin classes are distinct for $n \geq 3$.

---

[a] S. P. Novikov, *Topological invariance of rational classes of Pontrjagin*, Dokl. Akad. Nauk SSSR 163(1965).

[b] D.Montgomery, C.T.Yang, *Differentiable actions on homotopy seven spheres*, Trans. AMS. 122 (1966)

[c] W.-c.Hsiang, *A note on free differentiable actions of S^1 and S^3 on homotopy spheres*, Ann. of Math.(2) 83(1966)

Invariance of Pontrjagin classes

- The Pontrjagin class is preserved by any diffeomorphism, homeomorphism (if the cohomology ring is torsion-free).[a]
- However, the Pontrjagin class is not preserved by homotopy equivalences.

Montgomery-Yang (1966),[b] Hsiang (1966)[c]
There are infinitely many homotopy $\mathbb{C}P^n$'s whose Pontrjagin classes are distinct for $n \geq 3$.

- There is a smooth effective T^n-action on $\mathbb{C}P^n$.
- Not all homotopy $\mathbb{C}P^n$ admit an action of the compact torus T^k!!

[a] S. P. Novikov, *Topological invariance of rational classes of Pontrjagin*, Dokl. Akad. Nauk SSSR 163(1965).
[b] D.Montgomery, C.T.Yang, *Differentiable actions on homotopy seven spheres*, Trans. AMS. 122 (1966)
[c] W.-c.Hsiang, *A note on free differentiable actions of S^1 and S^3 on homotopy spheres*, Ann. of Math.(2) 83(1966)

Petrie's theorem

Petrie (1973)[d]

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.
- $f: M \to \mathbb{C}P^n$: a homotopy equivalence

Then, $f^*(p(\mathbb{C}P^n)) = p(M)$.

[d] T. Petrie, *Torus actions on homotopy complex projective spaces.* Invent. Math. 20 (1973).

[e] S. P. Novikov, *Homotopically equivalent smooth manifolds. I.*, Izv. Akad. Nauk SSSR Ser. Mat.28 (1964)

Petrie's theorem

Petrie (1973)[d]

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.
- $f: M \to \mathbb{C}P^n$: a homotopy equivalence

Then, $f^*(p(\mathbb{C}P^n)) = p(M)$.

Combining with well-known Novikov's results, [e]

Corollary $\exists$ at most a finite number homotopy $\mathbb{C}P^n$'s having smooth T^n-actions.

[d] T. Petrie, *Torus actions on homotopy complex projective spaces.* Invent. Math. 20 (1973).

[e] S. P. Novikov, *Homotopically equivalent smooth manifolds. I.*, Izv. Akad. Nauk SSSR Ser. Mat.28 (1964)

Petrie's theorem

Petrie (1973)[d]

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.
- $f : M \to \mathbb{C}P^n$: a homotopy equivalence

Then, $f^*(p(\mathbb{C}P^n)) = p(M)$.

Combining with well-known Novikov's results, [e]

Corollary $\exists$ at most a finite number homotopy $\mathbb{C}P^n$'s having smooth T^n-actions.

Petrie's conjecture

The above theorem also holds if M admits smooth T^k-action for any k.

[d] T. Petrie, *Torus actions on homotopy complex projective spaces.* Invent. Math. 20 (1973).

[e] S. P. Novikov, *Homotopically equivalent smooth manifolds. I.*, Izv. Akad. Nauk SSSR Ser. Mat.28 (1964)

Torus manifolds

A torus manifold of dim. $2n$ is a closed smooth manifold having smooth effective T^n-action with the non-empty fixed point set.

Torus manifolds

A torus manifold of dim. $2n$ is a closed smooth manifold having smooth effective T^n-action with the non-empty fixed point set.

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.

Then, M is a torus manifold. ($\because \chi(M^{T^n}) = \chi(M) = \chi(\mathbb{C}P^n) = n + 1 \neq 0$)

Torus manifolds

A torus manifold of dim. $2n$ is a closed smooth manifold having smooth effective T^n-action with the non-empty fixed point set.

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.

Then, M is a torus manifold. ($\because \chi(M^{T^n}) = \chi(M) = \chi(\mathbb{C}P^n) = n + 1 \neq 0$)

Revisited to Petrie's theorem.

- M : a torus manifold.
- $f : M \to \mathbb{C}P^n$: a homotopy equivalence, then

$$f^*(p(\mathbb{C}P^n)) = p(M).$$

Torus manifolds

A torus manifold of dim. $2n$ is a closed smooth manifold having smooth effective T^n-action with the non-empty fixed point set.

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.

Then, M is a torus manifold. ($\because \chi(M^{T^n}) = \chi(M) = \chi(\mathbb{C}P^n) = n+1 \neq 0$)

Revisited to Petrie's theorem.

- M, N : torus manifolds
- $f : M \to N$: a homotopy equivalence, then

$$f^*(p(N)) = p(M).$$

Torus manifolds

A torus manifold of dim. $2n$ is a closed smooth manifold having smooth effective T^n-action with the non-empty fixed point set.

- M : a homotopy $\mathbb{C}P^n$ having an effective T^n-action.

Then, M is a torus manifold. ($\because \chi(M^{T^n}) = \chi(M) = \chi(\mathbb{C}P^n) = n+1 \neq 0$)

Revisited to Petrie's theorem.

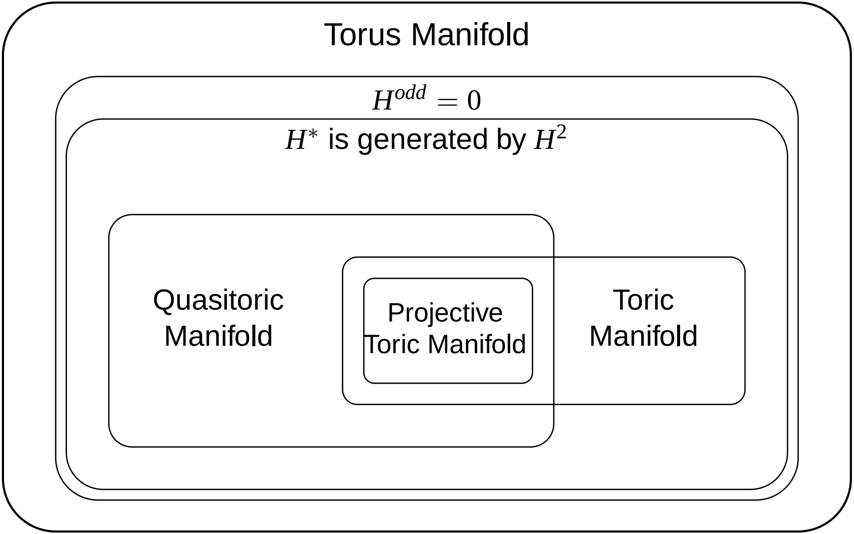
- M, N : torus manifolds
- $f : M \to N$: a homotopy equivalence, then

$$f^*(p(N)) = p(M).$$

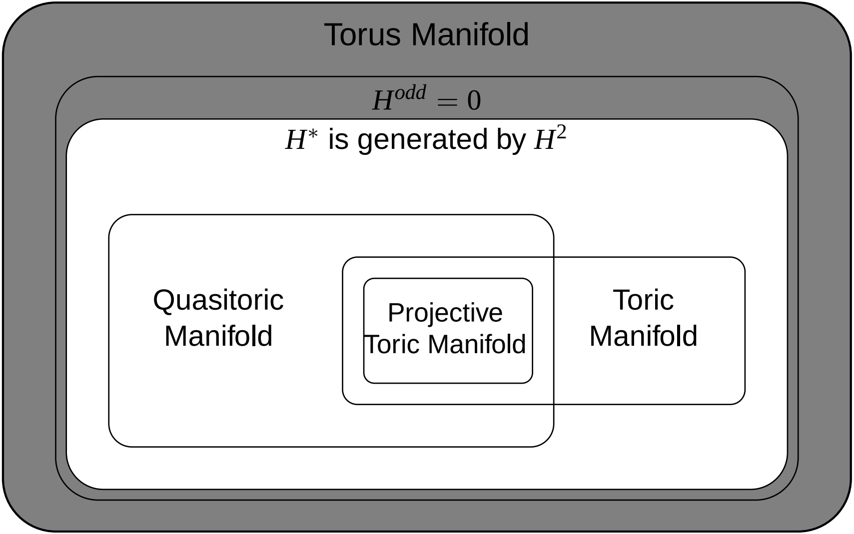
Toric topological analog of Petrie's conjecture

Does any homotopy equivalence between two torus manifolds preserve their Pontrjagin classes?

Categories

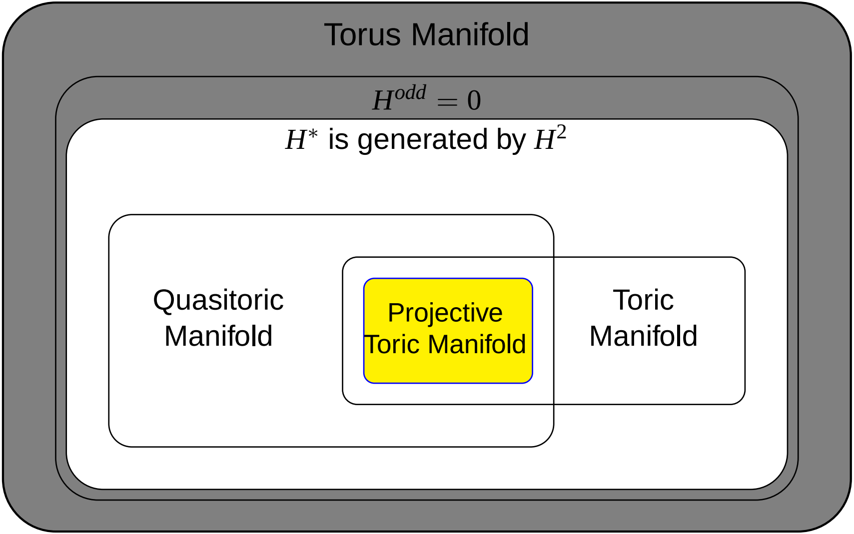

Torus Manifold
$H^{odd} = 0$
H^* is generated by H^2
Quasitoric Manifold
Projective Toric Manifold
Toric Manifold

Categories



$\exists$ counter example (M. Wiemeler (2011))

Categories


Torus Manifold
$H^{odd} = 0$
H^* is generated by H^2
Quasitoric Manifold
Projective Toric Manifold
Toric Manifold

On Masuda-Panov's theories

A torus manifold M is locally standard if M is locally weakly equivariantly diffeomorphic to the standard T^n-action on $\mathbb{C}^n$.

Masuda-Panov (2006)[f] $H^{\mathrm{odd}}(M) = 0 \quad \Leftrightarrow \quad M$ is locally standard

[f] M. Masuda and T. E. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. **43** (2006)

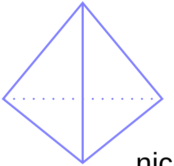
On Masuda-Panov's theories

A torus manifold M is locally standard if M is locally weakly equivariantly diffeomorphic to the standard T^n-action on $\mathbb{C}^n$.

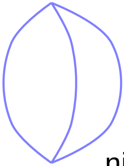
Masuda-Panov (2006)[f] $H^{\text{odd}}(M) = 0 \quad \Leftrightarrow \quad M$ is locally standard

Set $Q := M/T^n$ as a manifold with corners.
Each facet is corresponding to an element of $\mathrm{Hom}(S^1, T^n)$.

M is locally standard $\Rightarrow$ Q has the nice face structure; exactly n facets intersect at each vertex.



nice



nice

[f] M. Masuda and T. E. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. **43** (2006)

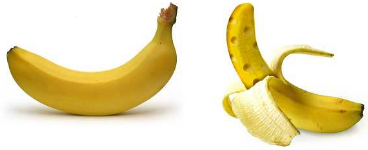
On Masuda-Panov's theories

A torus manifold M is locally standard if M is locally weakly equivariantly diffeomorphic to the standard T^n-action on $\mathbb{C}^n$.

Masuda-Panov (2006)[f] $H^{\text{odd}}(M) = 0 \quad \Leftrightarrow \quad M$ is locally standard

Set $Q := M/T^n$ as a manifold with corners.
Each facet is corresponding to an element of $\mathrm{Hom}(S^1, T^n)$.

M is locally standard $\Rightarrow$ Q has the nice face structure; exactly n facets intersect at each vertex.



[f] M. Masuda and T. E. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. **43** (2006)

Q is called a homology polytope if all faces of Q, including Q itself, and, for any subset $I \subset \{1, \dots, m\}$, $\bigcap_{i \in I} Q_i$ are acyclic. (Eg. simple polytope)

Masuda-Panov (2006)

$$H^*(M) \text{ is gen. by } H^2(M) \quad \Leftrightarrow \quad Q \text{ is a homology polytope.}$$

[g] M. W. Davis, T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991)

Q is called a homology polytope if all faces of Q, including Q itself, and, for any subset $I \subset \{1, \ldots, m\}$, $\bigcap_{i \in I} Q_i$ are acyclic. (Eg. simple polytope)

Masuda-Panov (2006)

$$H^*(M) \text{ is gen. by } H^2(M) \quad \Leftrightarrow \quad Q \text{ is a homology polytope.}$$

Masuda-Panov (2006), Davis-Januskiewicz (1991)[g] If $H^*(M)$ is gen. by $H^2(M)$, then M is determined by a pair of data (λ, Q), i.e.,

$$M = Q \times T^n / \sim .$$

[g] M. W. Davis, T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991)

Topological invariants of torus manifold

If $H^*(M)$ is gen. by $H^2(M)$, [h]

$$H_T^*(M) = \mathbb{Z}[x_1, \dots, x_m]/I$$
$$p_T(M) = (1+x_1^2)(1+x_2^2)\cdots(1+x_m^2),$$

where I is the face ring ideal.

In addition,
We have

$$H^*(M) = \mathbb{Z}[x_1, \dots, x_m]/I+J$$
$$p(M) = (1+x_1^2)(1+x_2^2)\cdots(1+x_m^2) \in H^*(M),$$

for some ideal J gen. by linear elements.

If Q itself is a simple polytope, then M is a *quasitoric manifold*.

[h] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. (2) **51** (1999)

Bott manifold

Bott tower

$$B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \; = \{\text{a point}\},$$

where $B_j = P(\gamma^{\alpha_j} \oplus \underline{\mathbb{C}})$ for $j = 1, \ldots, n$ and γ^{α_j} is the $\mathbb{C}$-line bundle with $c_1(\gamma^{\alpha_j}) = \alpha_j \in H^*(B_{j-1})$ and $\underline{\mathbb{C}}$ is the trivial line bundle over B_{j-1}.

We call B_n an (n-stage) Bott manifold.

Example

- B_1 : $\mathbb{C}P^1 \cong S^2$, $H^*(\mathbb{C}P^1) = \mathbb{Z}[x]/x^2$.
- B_2 : Hirzebruch Surface Σ_a ($a \in \mathbb{Z}$)

$$\Sigma_a = P(\gamma^{ax} \oplus \underline{\mathbb{C}}) \to \mathbb{C}P^1$$

Generalized Bott manifolds

Generalized Bott tower

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 \ = \{\text{a point}\},$$

where $B_j = P(\underline{\mathbb{C}} \oplus \gamma^{\alpha_{j1}} \oplus \cdots \oplus \gamma^{\alpha_{jn_j}})$.

We call each B_h an (h-stage) generalized Bott manifold.

- B_h is a torus manifold; i.e., it admits natural $T^{n_1+\cdots+n_h}$-action.
- B_h/T^n is $\prod_{i=1}^h \Delta^{n_i}$.
- If all fibrations are trivial, then $B_h = \prod_{i=1}^h \mathbb{C}P^{n_i}$.

A generalized Bott manifold is a generalization of $\mathbb{C}P^n$ in toric topology.

Cohomology generalized Bott manifolds

A smooth closed manifold M of even dimension (say, $2n$) is called a cohomology (generalized) Bott manifold if

$$H^*(M) \cong H^*(B_h)$$

for some (generalized) Bott manifold B_h.

Cohomology generalized Bott manifolds

A smooth closed manifold M of even dimension (say, $2n$) is called a cohomology (generalized) Bott manifold if

$$H^*(M) \cong H^*(B_h)$$

for some (generalized) Bott manifold B_h.

- Any homotopy (generalized) Bott manifold is a cohomology (generalized) Bott manifold.
- $H^*(M)$ is generated by $H^2(M)$.
- If M admits an effective smooth T^n-action, then M is a torus manifold.

Rigidity of product of simplicies

$f(Q)$: face ring of Q with $\mathbb{Q}$-coefficients.

C-Panov-Suh (2010)[i]

$$f(Q)/J \cong f(Q')/J' \Longrightarrow H^*(\mathcal{Z}_Q) = H^*(\mathcal{Z}_{Q'})$$

In particular, if $Q' = \prod_{i=1}^{h} \Delta^{n_i}$, then

$$f(Q) \cong f(Q').$$

[i] S. Choi, T. E. Panov and D. Y. Suh, *Toric cohomological rigidity of simple polytopes*, J. London Math. Soc. **82**(2) (2010)

Rigidity of product of simplicies

$f(Q)$: face ring of Q with $\mathbb{Q}$-coefficients.

C-Panov-Suh (2010)[i]

$$f(Q)/J \cong f(Q')/J' \Longrightarrow H^*(\mathcal{Z}_Q) = H^*(\mathcal{Z}_{Q'})$$

In particular, if $Q' = \prod_{i=1}^{h} \Delta^{n_i}$, then

$$f(Q) \cong f(Q').$$

M^{2n} : cohomology generalized Bott manifold having T^n action. Then,

$H^*(M)$ and $p(M)$ can be identified with
those of some quasitoric manifold M' over $\prod_{i=1}^{h} \Delta^{n_i}$.

[i] S. Choi, T. E. Panov and D. Y. Suh, *Toric cohomological rigidity of simple polytopes*, J. London Math. Soc. **82**(2) (2010)

Generalization of Petrie's theorem

M' : a quasitoric manifold over $\prod_{i=1}^{h} \Delta^{n_i}$.

C-Suh [j] Any $\varphi\colon H^*(M') \xrightarrow{iso} H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$ is realizable by homeomorphism.

Theorem

M : torus manifold

$$\varphi\colon H^*(M) \xrightarrow{iso} H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i}) \implies \varphi(p(M)) = p\left(\prod_{i=1}^{h} \mathbb{C}P^{n_i}\right).$$

[j] S. Choi and D. Y. Suh, *Strong cohomological rigidity of a product of projective spaces*, arXiv:0912.4791.

Generalization of Petrie's theorem

M' : a quasitoric manifold over $\prod_{i=1}^{h} \Delta^{n_i}$.

C-Suh [j] Any $\varphi\colon H^*(M') \xrightarrow{iso} H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$ is realizable by homeomorphism.

Theorem

M : torus manifold

$$\varphi\colon H^*(M) \xrightarrow{iso} H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i}) \implies \varphi(p(M)) = p\left(\prod_{i=1}^{h} \mathbb{C}P^{n_i}\right).$$

Corollary

Any homotopy equivalence $f\colon M \to \prod_{i=1}^{h} \mathbb{C}P^{n_i}$ preserves their Pontjragin classes.

---

[j] S. Choi and D. Y. Suh, *Strong cohomological rigidity of a product of projective spaces*, arXiv:0912.4791.

Generalization of Petrie's theorem

M' : a quasitoric manifold over I^n

C-Suh (2011) [k] $H^*(M') \cong H^*(B_n)$ implies that $M = B'_n$ for some Bott manifolds B_n and B'_n.

Theorem

M : torus manifold, B_n : Bott manifold

$$\varphi\colon H^*(M) \xrightarrow{iso} H^*(B_n) \quad \Longrightarrow \quad \varphi(p(M)) = p(B_n).$$

[k] S. Choi and D. Y. Suh, *Properties of Bott manifolds and cohomological rigidity*, Algebr. Geom. Topol. **11**, (2011)

Generalization of Petrie's theorem

M' : a quasitoric manifold over I^n
C-Suh (2011) [k] $H^*(M') \cong H^*(B_n)$ implies that $M = B'_n$ for some Bott manifolds B_n and B'_n.

Theorem

M : torus manifold, B_n : Bott manifold

$$\varphi\colon H^*(M) \xrightarrow{iso} H^*(B_n) \quad \Longrightarrow \quad \varphi(p(M)) = p(B_n).$$

Corollary

Any homotopy equivalence $f: M \to B_n$ preserves their Pontjragin classes.

[k] S. Choi and D. Y. Suh, *Properties of Bott manifolds and cohomological rigidity*, Algebr. Geom. Topol. **11**, (2011)

Cohomological rigidity problem

B_n, B'_n : (generalized) Bott manifolds

Cohomological Rigidity (CR) conjecture

$$B_n \cong B'_n \text{ up to diff.} \quad \Longleftrightarrow \quad H^*(B_n) \cong H^*(B'_n)$$

Cohomological rigidity problem

B_n, B'_n : (generalized) Bott manifolds

Cohomological Rigidity (CR) conjecture

$$B_n \cong B'_n \text{ up to diff.} \iff H^*(B_n) \cong H^*(B'_n)$$

Strong Cohomological Rigidity (SCR) conjecture

$$\varphi \colon H^*(B_n) \xrightarrow{\cong} H^*(B'_n) \implies \exists f : B'_n \xrightarrow{\cong} B_n \quad \text{s.t.} \quad f^* = \varphi$$

Remarkable Results - CR

Masuda-Panov (2008)[l]
$B_n \cong (\mathbb{C}P^1)^n$ up to diff. $\iff$ $H^*(B_n) \cong H^*((\mathbb{C}P^1)^n)$

C-Masuda-Suh (2010)[m]

1. $B_n \cong \prod_{i=1}^{h} \mathbb{C}P^{n_i}$ up to diff. $\iff$ $H^*(B_n) \cong H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$
2. CR holds for 2-stage generalized Bott manifolds.
3. CR holds for 3-stage Bott manifolds.

C CR holds for 4-stage Bott manifolds.

[l] M. Masuda, T. E. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, Mat. Sb. **199** (2008)

[m] S. Choi, M. Masuda, D. Y. Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. **362** (2010)

Remarkable Results - SCR

The following isomorphisms are realizable by diffeomorphism

C-Suh[n] $\varphi\colon H^*(B_n) \to H^*(\prod_{i=1}^{h} \mathbb{C}P^{n_i})$

Ishida (2011)[o] $\varphi\colon H^*(B_n) \to H^*(\text{Bott manifolds})$ preserving the filtration

$$H^*(B_1) \to \cdots \to H^*(B_n).$$

C SCR holds for 3-stage Bott manifolds.

C-Masuda-Suh (2010)[c] Any $\mathbb{Z}/2$-cohomology ring isomorphism preserves their S-W classes.

[n] S. Choi and D. Y. Suh, *Strong cohomological rigidity of a product of projective spaces*, arXiv:0912.4791.

[o] H. Ishida, *(Filtered) cohomological rigidity of Bott towers*, to appear in Osaka J. Math.

Theorem

Theorem

Any cohomology ring isomorphism of two Bott manifolds preserves their Pontrjagin classes.

Theorem

Any cohomology ring isomorphism of two Bott manifolds preserves their Pontrjagin classes.

Corollary

$$H^*(B_n) \cong H^*(B'_n) \quad \Longrightarrow \quad [B_n] = [B'_n] \in \Omega_*^{SO}$$

Corollary

$\exists$ *finitely many diffeomorphism types for each homotopy equivalence type.*

Thank you for your attention!