Schubert calculus and cohomology of Lie groups

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Toric Topology 2011 in Osaka, November 28, 2011

Outline of the talk

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- 1929-1949: Results of Brauer, Pontryagin, Hopf, Samleson and Yan for the case of $\mathbb{F} = \mathbb{R}$;
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- Recent works of Duan and Zhao for the case of $\mathbb{F}=\mathbb{Z}$

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1. Preliminaries
2. Earlier results
3. Schubert calculus
4. New results (Duan, Zhao)

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Therefore, we can assume in this talk that
"G is one of the 1-connected simple Lie groups listed above."

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   where $f_r \in \mathbb{F}[x_1, \cdots, x_n]$, and where $\langle f_1, \cdots, f_k \rangle$ is the ideal generated by the polynomials $f_1, \cdots, f_k$.

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We will refer this structure as the

"near Hopf ring structure on $H^*(G;\mathbb{Z})$"

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The idea of Hopf and Samleson (1941) : Given a graded algebra $A = \oplus_{k \geq 0} A^k$ over $\mathbb{R}$ with a co-product $\beta : A \to A \bigotimes A$ (i.e. an Hopf algebra over reals),

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for $P(A)$ with $deg(x_i) =$ even and $deg(y_i) =$ odd.

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【Classification Theorem I, Hopf, Samleson, 1941】

$$A = \mathbb{R}[x_1, \cdots, x_n] \otimes \wedge_{\mathbb{R}}(y_1, \cdots, y_m).$$

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【Corollary 2, Yan, 1949】 Let G be an exceptional Lie group. Then

1. $H^*(G_2;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11})$
2. $H^*(F_4;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11}, y_{15}, y_{23})$
3. $H^*(E_6;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_9, y_{11}, y_{15}, y_{17}, y_{23})$
4. $H^*(E_7;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{11}, y_{15}, y_{19}, y_{23}, y_{27}, y_{35})$
5. $H^*(E_8;\mathbb{R}) = \wedge_{\mathbb{R}}(y_3, y_{15}, y_{23}, y_{27}, y_{35}, y_{39}, y_{47}, y_{59})$

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where each $B(x_i)$ is one of the "monogenic Hopf algebra over F_p":

$B(x_i)$	$\deg(x_i)$ odd	$\deg(x_i)$ even
$p \neq 2$	$\Lambda_{\mathbb{F}_p}(x_i)$	$\mathbb{F}_p(x_i)/(x_i^{p^r})$
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- if $p = 3$:

  $\mathbb{F}_3[x_8, x_{20}] / \langle x_8^3, x_{20}^3 \rangle \otimes \Lambda_{\mathbb{F}_3}(\zeta_3, \zeta_7, \zeta_{15}, \zeta_{19}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47});$

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- If $p = 5$:

  $\mathbb{F}_5[x_{12}] / \langle x_{12}^5 \rangle \otimes \Lambda_{\mathbb{F}_5}(\zeta_3, \zeta_{11}, \zeta_{15}, \zeta_{23}, \zeta_{27}, \zeta_{35}, \zeta_{39}, \zeta_{47})$

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【**Remark**】 For the classical Lie groups $G=U(n),Sp(n),Spin(n)$, the near Hopf rings $H^*(G;\mathbb{Z})$ have been determined by Borel (1952) and Pitties (1991).

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1. $E_2^{*,*}(G; \mathbb{F}) = H^*(G/T) \otimes \Lambda_{\mathbb{F}}(t_1, \cdots, t_n)$
2. the differential $d_2 : E_2^{*,*}(G; \mathbb{F}) \to E_2^{*,*}(G; \mathbb{F})$ is given by

$$d_2(x \otimes t_k) = x\omega_k \otimes 1,\ x \in H^*(G/T; \mathbb{F}),\ 1 \leq k \leq n$$

3. Schubert Calculus

Take a maximal torus T in G and let $\omega_1, \cdots, \omega_n \subset H^2(G/T; \mathbb{F})$ be a set of fundamental dominant weights of G.

In the Leray–Serre spectral sequence $\{E_r^{*,*}(G; \mathbb{F}), d_r\}$ of the fibration $T \to G \xrightarrow{\pi} G/T$ one has

1. $E_2^{*,*}(G; \mathbb{F}) = H^*(G/T) \otimes \Lambda_{\mathbb{F}}(t_1, \cdots, t_n)$
2. the differential $d_2 : E_2^{*,*}(G; \mathbb{F}) \to E_2^{*,*}(G; \mathbb{F})$ is given by
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Leray (1951) showed that $E_3^{*,*}(G; \mathbb{R}) = H^*(G; \mathbb{R})$.

Serre (1964) proved that $E_3^{*,*}(G; \mathbb{Z}_p) = H^*(G; \mathbb{Z}_p)$.

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Kac (1984) and Marlin (1991) conjectured that $E_3^{*,*}(G; \mathbb{Z}) = H^*(G; \mathbb{Z})$. (This conjecture has been confirmed by Duan and Zhao in this work).

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$$\mathbb{Z}[\omega_1, \cdots, \omega_n; y_1, \cdots, y_m] / \langle e_i, f_j, g_j \rangle_{1 \leq i \leq k; 1 \leq j \leq m}$$

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where

1. for each $1 \le i \le k$, $e_i \in \langle \omega_1, \cdots, \omega_n \rangle$
2. for each $1 \le j \le m$, the pair (f_j, g_j) of polynomials is related to the Schubert class y_j in the fashion
$$f_j = p_j y_j + \alpha_j; \quad g_j = y_j^{k_j} + \beta_j$$
with $p_j \in \{2, 3, 5\}$ and $\alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$

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In view of the fibration $\pi : G \to G/T$, the set $\{y_1, \cdots, y_m\}$ of Schubert classes on G/T specified in the Lemma gives rise to the integral classes

$$x_{\deg y_i} := \pi^*(y_i) \in H^*(G;\mathbb{Z}), 1 \leq i \leq m.$$

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the polynomials $e_i, \alpha_j, \beta_j \in \langle \omega_1, \cdots, \omega_n \rangle$ yield the integral classes

$$\varrho_k := \kappa \circ \iota(e_i) \in H^*(G;\mathbb{Z}),\ k = \deg e_i - 1$$

$$\varrho_k := \kappa \circ \iota(p_j \beta_j - y_j^{k_j - 1} \alpha_j) \in H^*(G;\mathbb{Z}),\ k = \deg \beta_j - 1$$

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Theorem 1. With respect to the ring presentation

$$H^*(G_2) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}) \oplus \tau_2(G_2),$$

where

$$\tau_2(G_2) = \mathbb{F}_2[x_6]^+ / \left\langle x_6^2 \right\rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3)$$

and where

$$\varrho_3^2 = x_6,\ x_6 \varrho_{11} = 0$$

the reduced co–product ψ is given by

$$\{\varrho_3, x_6\} \subset \mathcal{P}(G_2),$$
$$\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5).$$

Theorem 2. With respect to the ring presentation

$$H^*(F_4) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{23}) \oplus \tau_2(F_4) \oplus \tau_3(F_4)$$

where

$$\tau_2(F_4) = \mathbb{F}_2[x_6]^+ / \left\langle x_6^2 \right\rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23})$$

$$\tau_3(F_4) = \mathbb{F}_3[x_8]^+ / \left\langle x_8^3 \right\rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15})$$

where

$$\varrho_3^2 = x_6,\ x_6\varrho_{11} = 0,\ x_8\varrho_{23} = 0,$$

the reduced co–product ψ is given by

$$\{\varrho_3, x_6, x_8\} \subset \mathcal{P}(F_4)$$

$$\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3$$

$$\psi(\varrho_{15}) = -\delta_3(\zeta_7 \otimes \zeta_7),$$

$$\psi(\varrho_{23}) = \delta_3(\zeta_7 \otimes \zeta_7 x_8 - \zeta_7 x_8 \otimes \zeta_7).$$

Theorem 3. With respect to the ring presentation

$$H^*(E_6) = \Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17}, \varrho_{23}) \oplus \tau_2(E_6) \oplus \tau_3(E_6)$$

where

$$\tau_2(E_6) = \mathbb{F}_2[x_6]^+ / \left\langle x_6^2 \right\rangle \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_9, \varrho_{15}, \varrho_{17}, \varrho_{23}),$$

$$\tau_3(E_6) = \mathbb{F}_3[x_8]^+ / \left\langle x_8^3 \right\rangle \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_9, \varrho_{11}, \varrho_{15}, \varrho_{17})$$

and where

$$\varrho_3^2 = x_6,\ x_6 \varrho_{11} = 0,\ x_8 \varrho_{23} = 0,$$

the reduced co–product ψ is given by

$$\{\varrho_3, \varrho_9, \varrho_{17}, x_6, x_8\} \subset \mathcal{P}(E_6);$$

$$\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3;$$

$$\psi(\varrho_{15}) = x_6 \otimes \varrho_9 - \delta_3(\zeta_7 \otimes \zeta_7);$$

$$\psi(\varrho_{23}) = x_6 \otimes \varrho_{17} + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8).$$

Theorem 4. The ring $H^*(E_7)$ has the presentation

$$\Delta_{\mathbb{Z}}(\varrho_3) \otimes \Lambda_{\mathbb{Z}}(\varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{23}, \varrho_{27}, \varrho_{35}) \oplus \tau_2(E_7) \oplus \tau_3(E_7)$$

where

$$\tau_2(E_7) = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, \mathcal{C}_I]^+}{\langle x_6^2, x_{10}^2, x_{18}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{15}, \varrho_{23}, \varrho_{27})$$

with $t \in e(E_7, 2) = \{3, 5, 9\}$, $I, J, L \subseteq e(E_7, 2)$, $|I|, |J| \geq 2$,

$$\tau_3(E_7) = \frac{\mathbb{F}_3[x_8]^+}{\langle x_8^3 \rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{11}, \varrho_{15}, \varrho_{19}, \varrho_{27}, \varrho_{35})$$

and where

the reduced co–product ψ is given by

$$\{\varrho_3, x_6, x_8, x_{10}, x_{18}\} \subset \mathcal{P}(E_7);$$
$$\psi(\varrho_{11}) = \delta_2(\zeta_5 \otimes \zeta_5) + x_8 \otimes \varrho_3;$$
$$\psi(\varrho_{15}) = \delta_2(\zeta_9 \otimes \zeta_5) + \delta_3(\zeta_7 \otimes \zeta_7);$$
$$\psi(\varrho_{19}) = \delta_2(\zeta_9 \otimes \zeta_9);$$
$$\psi(\varrho_{23}) = \delta_2(\zeta_{17} \otimes \zeta_5) + \delta_3(\zeta_7 x_8 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8);$$
$$\psi(\varrho_{27}) = \delta_2(\zeta_{17} \otimes \zeta_9) - \delta_3(\zeta_7 \otimes \zeta_{19});$$
$$\psi(\varrho_{35}) = \delta_2(\zeta_{17} \otimes \zeta_{17}) + x_8 \otimes \varrho_{27} - \varrho_{27} \otimes x_8 + x_8 \otimes x_8\varrho_{19};$$
$$\psi_2(\zeta_{2i-1}) = 0,\ i \in e(E_7, 2).$$

Theorem 5. The ring $H^*(E_8)$ has the presentation

$$\Delta_{\mathbb{Z}}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{Z}}(\varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}, \varrho_{59}) \bigoplus_{p=2,3,5} \tau_p(E_8)$$

where

$$\tau_2 = \frac{\mathbb{F}_2[x_6, x_{10}, x_{18}, x_{30}, \mathcal{C}_I]^+}{\left\langle x_6^8, x_{10}^4, x_{18}^2, x_{30}^2, \mathcal{D}_J, \mathcal{R}_K, \mathcal{S}_{I,J}, \mathcal{H}_{t,L} \right\rangle} \otimes \Delta_{\mathbb{F}_2}(\varrho_3, \varrho_{15}, \varrho_{23}) \otimes \Lambda_{\mathbb{F}_2}(\varrho_{27})$$

with $t \in e(E_8, 2) = \{3, 5, 9, 15\}$, $K, I, J, L \subseteq e(E_8, 2)$, $|I|, |J| \geq 2$, $|K| \geq 3$;

$$\tau_3 = \frac{\mathbb{F}_3[x_8, x_{20}, \mathcal{C}_{\{4,10\}}]^+}{\left\langle x_8^3, x_{20}^3, x_8^2 x_{20}^2 \mathcal{C}_{\{4,10\}}, \mathcal{C}_{\{4,10\}}^2 \right\rangle} \otimes \Lambda_{\mathbb{F}_3}(\varrho_3, \varrho_{15}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47});$$

$$\tau_5 = \frac{\mathbb{F}_5[x_{12}]^+}{\left\langle x_{12}^5 \right\rangle} \otimes \Lambda_{\mathbb{F}_5}(\varrho_3, \varrho_{15}, \varrho_{23}, \varrho_{27}, \varrho_{35}, \varrho_{39}, \varrho_{47}),$$

and where

$$\varrho_3^2 = x_6,\ \varrho_{15}^2 = x_{30},\ \varrho_{23}^2 = x_6^6 x_{10},$$

$$x_{2s}\varrho_{3s-1} = 0, \text{ for } s = 4, 5$$

$$x_8\varrho_{59} = x_{20}^2\mathcal{C}_{\{4,10\}},\ x_{20}\varrho_{23} = x_8^2\mathcal{C}_{\{4,10\}},$$

$$x_{12}\varrho_{59} = 0,$$

the reduced co–product ψ is given by

$$\{\varrho_3, x_6, x_8, x_{10}, x_{12}, x_{18}, x_{20}\} \subset \mathcal{P}(E_8);$$

$$\psi(\varrho_{15}) = \delta_2(\zeta_9 \otimes \zeta_5) + x_6^2 \otimes \varrho_3 - \delta_3(\zeta_7 \otimes \zeta_7) + x_{12} \otimes \varrho_3;$$

$$\psi(\varrho_{23}) = \delta_2(\zeta_{17} \otimes \zeta_5 + \sum_{s+t=2} x_6^s\zeta_5 \otimes x_6^t\zeta_5) + x_{10}^2 \otimes \varrho_3$$
$$+\delta_3(x_8\zeta_7 \otimes \zeta_7 - \zeta_7 \otimes \zeta_7 x_8) - \delta_5(\zeta_{11} \otimes \zeta_{11});$$

4. New results

$$\psi(\varrho_{27}) = \delta_2(\zeta_{17} \otimes \zeta_9) + \delta_3(\zeta_{19} \otimes \zeta_7) - x_{12} \otimes \varrho_{15} + (x_6^4 + 2x_{12}^2) \otimes \varrho_3;$$

$$\psi(\varrho_{35}) = \delta_2(\zeta_{17} \otimes \zeta_{17}) - \varrho_{27} \otimes x_8 + x_8 \otimes \varrho_{27} + x_{20} \otimes \varrho_{15}$$
$$+\delta_3(x_8\zeta_{19} \otimes \zeta_7) + 2x_{12} \otimes \rho_{23} + \delta_5(x_{12}\zeta_{11} \otimes \zeta_{11} + 3\zeta_{11} \otimes \zeta_{11}x_{12});$$

$$\psi(\varrho_{39}) = \delta_2(\sum_{s+t=2} x_{10}^s\zeta_9 \otimes x_{10}^r\zeta_9) - \delta_3(\zeta_{19} \otimes \zeta_{19}) + x_{12} \otimes \varrho_{27}$$
$$+2x_{12}^2 \otimes \varrho_{15} - x_{12}^3 \otimes \varrho_3;$$

$$\psi(\varrho_{47}) = \delta_2(\sum_{s+t=6} x_6^s\zeta_5 \otimes x_6^r\zeta_5) - x_{20} \otimes \varrho_{27} + \varrho_{39} \otimes x_8$$
$$+\delta_3(x_{20}\zeta_{19} \otimes \zeta_7) + 2x_{12} \otimes \varrho_{35} + x_{12}^2 \otimes \varrho_{23}$$
$$+\delta_5(\zeta_{11} \otimes x_{12}^2\zeta_{11} + \sum_{s+t=2} x_{12}^s\zeta_{11} \otimes x_{12}^r\zeta_{11});$$

$$\psi(\varrho_{59}) = \delta_2(x_{10}^2\zeta_{29}\otimes\zeta_9 + x_{30}\zeta_{17}\otimes\zeta_5 x_6 + x_{18}\zeta_{29}\otimes\zeta_5 x_6 + x_6^4\zeta_{29}\otimes\zeta_5$$

$$\zeta_{29}\otimes\zeta_{29} + x_{10}^2\zeta_{17}\otimes\zeta_9 x_6^2 + \zeta_{17}\otimes x_6^2\zeta_{29} + x_6^4\zeta_{17}\otimes\zeta_5 x_6^2$$

$$+x_{18}\zeta_{17}\otimes\zeta_5 x_6^4 + x_6^4 x_{10}^2\otimes\zeta_5\zeta_9 + x_{10}^2\otimes\zeta_9\zeta_{29} + x_6^4\otimes\zeta_5\zeta_{29})$$

$$\delta_3(\sum_{s+t=1}(-x_{20})^s\zeta_{19}\otimes x_{20}^r\zeta_{19}) + 2\delta_5(\sum_{s+t=4}(-x_{12})^s\zeta_{11}\otimes x_{12}^r\zeta_{11});$$

and for $(p,i) = (2,3),(2,5),(2,9),(3,4),(3,10),(5,6)$

$$\psi_p(\zeta_{2i-1}) = 0;$$

$$\psi_2(\zeta_{29}) = x_{10}^2\otimes\zeta_9 + \zeta_{17}\otimes x_6^2 + x_6^4\otimes\zeta_5.$$

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Main idea in the computation:

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Since the cohomology classes x_i, ϱ_k and $\mathcal{C}_I$ are constructed from the polynomials e_i, α_j, β_j in the Schubert classes, one can boil down the calculation in the cohomology ring $H^*(G;\mathbb{Z})$ to the computation with those polynomials.

Thanks!