The cohomology ring of the GKM graph of a flag manifold

Yukiko Fukukawa

Osaka City University

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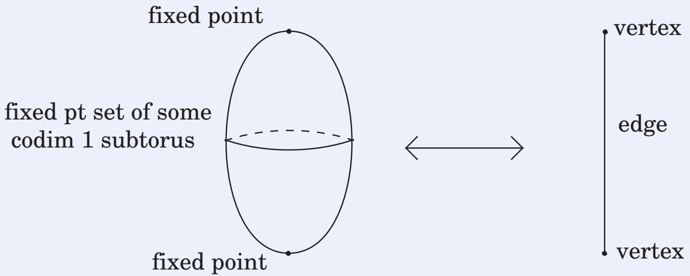
GKM graph

For some nice manifold with a torus action, we can define the graph named "GKM graph".



Then we can describe the equivariant cohomology ring of that manifold by using the combinatorial way.

Let
T be an n- dimensional torus and,
M be a closed smooth T manifold
satisfying following two conditions ; $H^{odd}(M) = 0$ and M^T: finite set.
$H_T^*(M) := H^*(ET \times_T M ; \mathbb{Z})$

- $\iota \colon M^T \to M$
  - $\iota^* : H_T^*(M) \longrightarrow H_T^*(M^T) \quad \cdots$ injective
- $$H_T^*(M^T) \Big(= \bigoplus_{p \in M^T} H_T^*(p) = \bigoplus_{p \in M^T} H^*(BT) \Big) = \bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n].$$

Namely, we shall regard $H_T^*(M)$ as the subring of $\bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n]$ through the map ι^*.

① Goresky-Kottwitz-MacPherson

$\iota^*(H_T^*(M)) \otimes \mathbb{Q}$ is determined by fixed point sets of some codimension 1 subtori of T

② Guillemin-Zara

- fixed point sets of some codimension 1 subtori of T
  $\longrightarrow \mathcal{G}_M$: the GKM graph associated with M
- the graph cohomology ring of $\mathcal{G}_M$:
  $H_T^*(\mathcal{G}_M) \subset H_T^*(M^T) = \bigoplus_{p \in M^T} \mathbb{Z}[t_1, \cdots, t_n]$ (We will define later.)

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$\longrightarrow \quad H_T^*(\mathcal{G}_M) \otimes \mathbb{Q} \cong H_T^*(M) \otimes \mathbb{Q}$
M : flag manifold $\Rightarrow H_T^*(\mathcal{G}_M) \cong H_T^*(M)$. (Arabia)

Def (flag manifold)

$G := U(n)$ (resp. G_2)
$M \cong G/T$: flag manifold of type A_{n-1} (resp. G_2)

Def (labeled graph of a flag manifold)

$\mathcal{G}_\Phi$: the labeled graph associated with the root system Φ

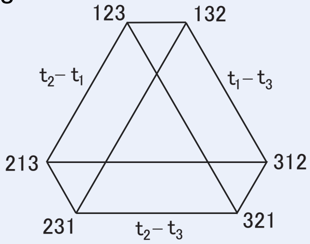
- the vertex set $V(\mathcal{G}_\Phi)$: Weyl group W_Φ
- $w_1, w_2 \in V(\mathcal{G}_\Phi)$ are connected by an edge $e_{w_1 w_2}$
  $\Leftrightarrow \exists \alpha \in \Phi$ s.t. the reflection σ_α satisfies $w_1 = w_2 \sigma_\alpha$
- the label of $e_{w_1 w_2}$: $\ell(e_{w_1 w_2}) = w_2 \alpha \ \in \Phi$

type	root system ($\cdots$ label)	Weyl group ($\cdots$ vertex)
A_2	$\{\pm(t_i - t_j) \mid 1 \le i < j \le 3\}$	the permutation group S_3
G_2	$\{\pm(s_i - s_j),\ \pm s_i \mid 1 \le i < j \le 3\}$	$W(G_2)$

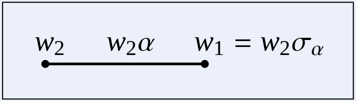
(where $s_1 := t_1 - t_2,\ s_2 := t_2 - t_3,\ s_3 := t_3 - t_1$)

Example

The labeled graph associated with the flag manifold of type A_2 is the following.



$\mathcal{A}_2$



$\Phi(G_2) = \{\pm(s_i - s_j),\ \pm s_i \mid 1 \leq i < j \leq 3\}$
$W(G_2) =< \sigma_1, \sigma_2 \mid (\sigma_1)^2 = (\sigma_2)^2 = (\sigma_1\sigma_2)^6 = 1 >$
where $\sigma_1 := \sigma_{s_1}$ and $\sigma_2 = \sigma_{s_3 - s_1}$.

$\Phi := \{\pm(s_i - s_j) \mid \leq i < j \leq 3\} \ \subset \Phi(G_2)$.
$W(\Phi) =< \sigma_2, \sigma_1\sigma_2\sigma_1 \mid (\sigma_2)^2 = (\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2\sigma_1 \cdot \sigma_2)^3 = 1 >$
$W(\Phi)$ is isomorphic to S_3 the Weyl group of type A_2.

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$$\begin{array}{ccccc} \psi & : & W(\Phi) & \cong & S_3 \\ & & \sigma_{s_1 - s_2} = \sigma_1\sigma_2\sigma_1 & \longmapsto & (1,2) \\ & & \sigma_{s_3 - s_1} = \sigma_2 & \longmapsto & (1,3) \end{array}$$

$W(G_2) = W(\Phi) \sqcup \rho W(\Phi)$ as a set. ($\rho := (\sigma_1\sigma_2)^3$)
$\Psi : W(G_2) \to \{(v, \varepsilon) \mid v \in S_3, \varepsilon = \pm\} = S_3 \times \{\pm\}$

$$\Psi(w) := (\psi(w), +) \quad \text{and} \quad \Psi(\rho w) := (\psi(w), -) \quad \text{for} \quad w \in W(\Phi).$$

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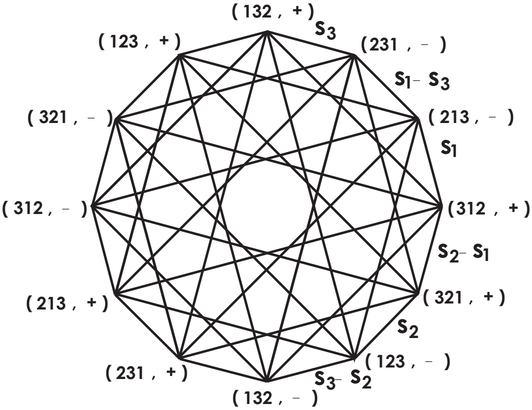

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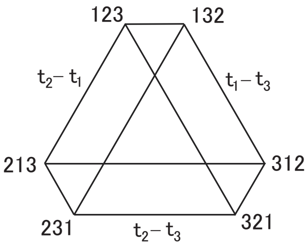

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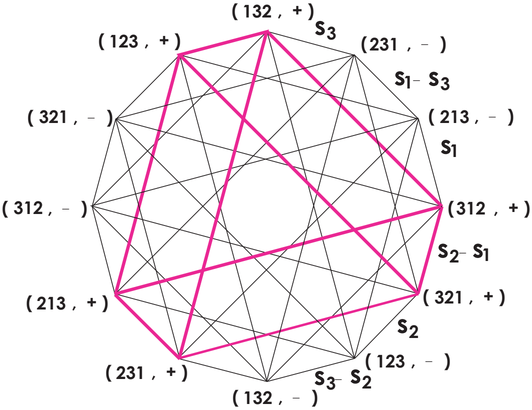

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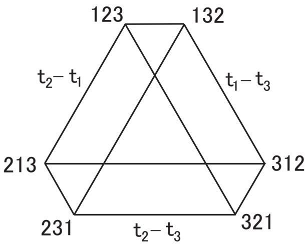

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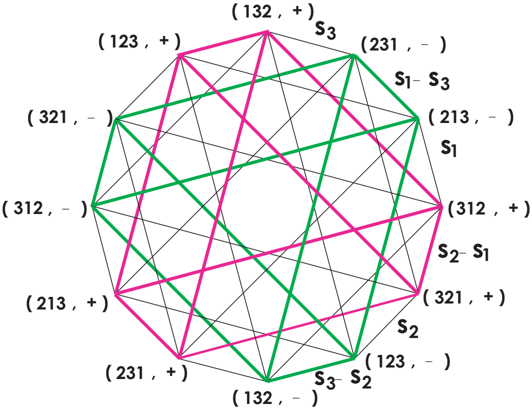

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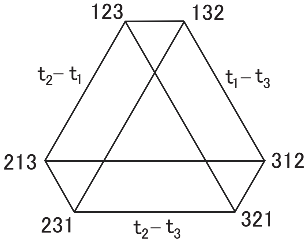


$\mathcal{A}_2$

GKM fiber bundle [Guillemin, Sabatini and Zara]

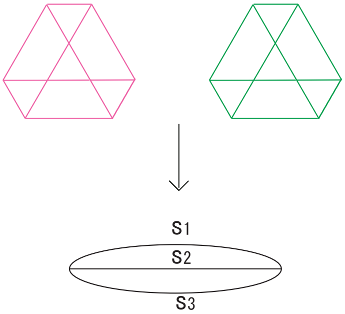
$$SU(n)/T \to G_2/T \to G_2/SU(n)$$

$$\mathcal{A}_2 \to \mathcal{G}_2 \to \mathcal{G}_2/\mathcal{A}_2$$

$$\mathcal{G}_2 \ (\cdots S_3 \times \{\pm\})$$

$$\downarrow$$

$$\mathcal{G}_2/\mathcal{A}_2 \ (\cdots \{\pm\})$$

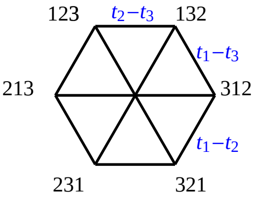


Def (graph cohomology ring of graph $\mathcal{A}_2$)

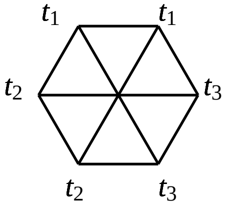
$$H_T^*(\mathcal{A}_2) := \left\{ h \;\middle|\; \begin{array}{l} \text{for } \forall \text{ edge } e = (w, w') \text{ of } \mathcal{A}_2 , \\ \ell(e) \mid h(w) - h(w') \end{array} \right\} \subset \bigoplus_{w \in V(\mathcal{A}_2)} \mathbb{Z}[t_1, t_2, t_3]$$

$\tau_i, t_i \in H_T^*(\mathcal{A}_2) \subset \bigoplus_{w \in S_3} \mathbb{Z}[t_1, t_2, t_3]$ $(i = 1, 2, 3)$

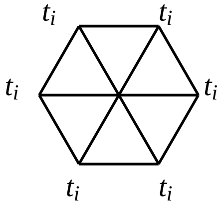
$$\tau_i(w) := t_{w(i)}, \quad t_i(w) := t_i \quad \text{for } w \in S_n.$$



τ_1



t_i



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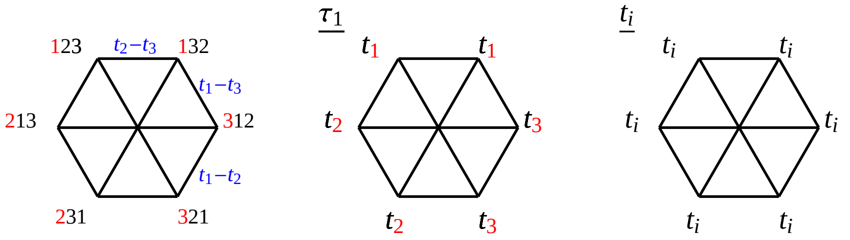

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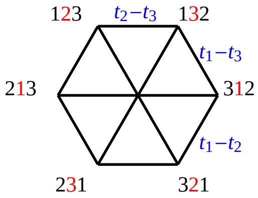

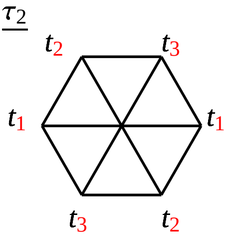

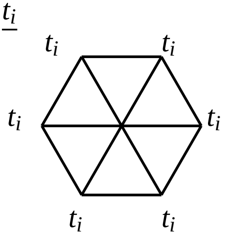

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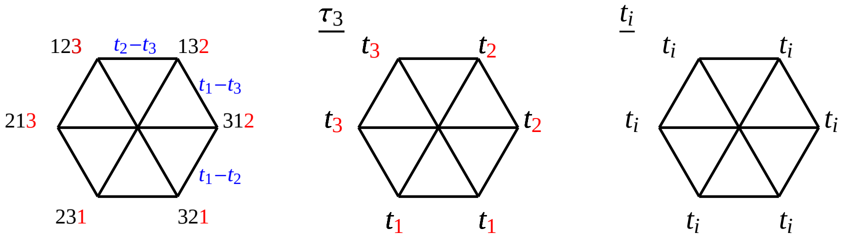


Theorem [F-Ishida-Masuda]

$$H_T^*(\mathcal{A}_n) \cong \mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]/(e_i(\tau) - e_i(t) \mid i = 1, \cdots, n)$$

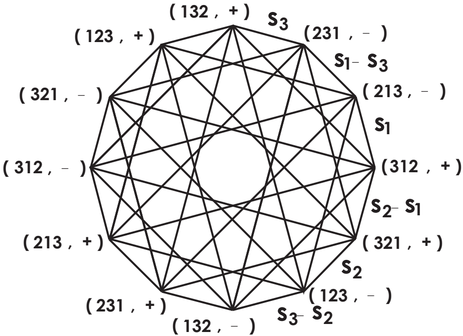
$e_i(\tau)$ (resp. $e_i(t)$) is the i^{th} elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$).

Reference

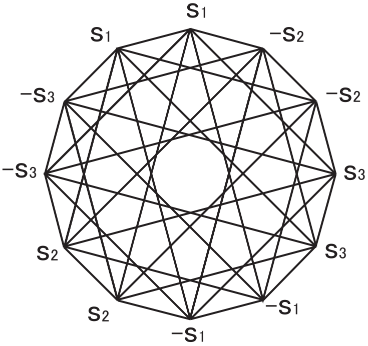
Y. Fukukawa, H. Ishida, and M. Masuda,
The cohomology ring of the GKM graph of a flag manifold of classical type
, arXiv:1104.1832

Def (graph cohomology ring of graph $\mathcal{G}_2$)

$$H_T^*(\mathcal{G}_2) := \left\{ h \;\middle|\; \begin{array}{l} \text{for } \forall \text{ edge } e = (w, w') \text{ of } \mathcal{G}_2 \text{ ,} \\ l(e) \mid h(w) - h(w') \end{array} \right\} \subset \bigoplus_{w \in V(\mathcal{G}_2)} \mathbb{Z}[t_1, t_2, t_3]$$



$\mathcal{G}_2$



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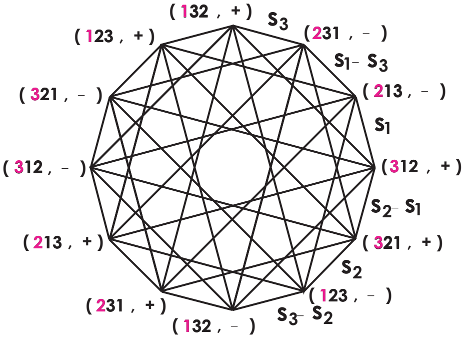
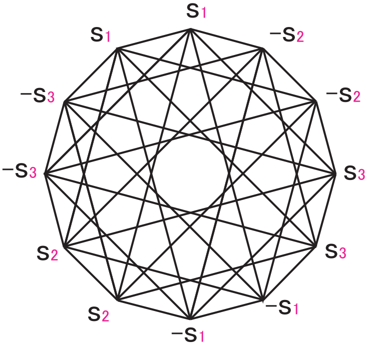


$\mathcal{G}_2$



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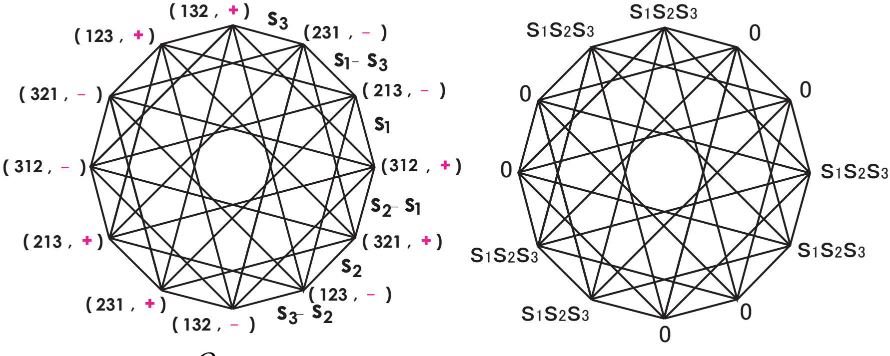

$\mathcal{G}_2$ f

$(s_1 := t_1 - t_2,\ s_2 := t_2 - t_3,\ s_3 := t_3 - t_1)$

Def (elements of $H_T^*(\mathcal{G}_2)$)

$\tau_1, \tau_2, \tau_3, f \in H_T^*(\mathcal{G}_2) \subset \bigoplus_{w \in V(\mathcal{G}_2)} \mathbb{Z}[t_1, t_2, t_3]$

$$\tau_i(w) := \begin{cases} s_{\sigma(i)} & (w = (\sigma, +)) \\ -s_{\sigma(i)} & (w = (\sigma, -)) \end{cases} \quad , \quad f(w) := \begin{cases} s_1 s_2 s_3 & (w = (\sigma, +)) \\ 0 & (w = (\sigma, -)) \end{cases}$$

$t_i(w) := t_i \quad (\forall w)$

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Thm

$$H_T^*(\mathcal{G}_2) \cong \mathbb{Z}[t_1, t_2, t_3, \tau_1, \tau_2, \tau_3, f]/I$$

$$I = (e_1(\tau), e_2(\tau) - e_2(s), 2f - e_3(\tau) - e_3(s), f^2 - f e_3(s))$$

where $e_i(\tau)$ (resp. $e_i(s)$) is the i^{th} elementary symmetric polynomial in τ_1, τ_2, τ_3 (resp. s_1, s_2, s_3).

Thank you for your attention!