Complex torus manifolds

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Let us consider

- M : connected smooth manifold,
- $(S^1)^n \curvearrowright M$: effective,
- $M^{(S^1)^n} \neq \emptyset$.

$\Rightarrow n \leq \frac{1}{2} \dim M$
(see the tangential rep. of $(S^1)^n$ at a fixed point).

We consider the extreme case ($n = \frac{1}{2} \dim M$).

Definition (Hattori-Masuda)

M : torus manifold of dim $2n$

$\underset{def.}{\Longleftrightarrow}$

- M : closed connected oriented manifold of dim $2n$,
- $(S^1)^n \curvearrowright M$: effective,
- $M^{(S^1)^n} \neq \emptyset$.

We consider the case when M is a complex mfd.

Definition

A complex torus manifold M of $\dim_{\mathbb{C}} n$ is

- M : closed conn. complex manifold of $\dim_{\mathbb{C}} n$,
- $(S^1)^n \curvearrowright M$: effective, as biholomorphisms,
- $M^{(S^1)^n} \neq \emptyset$.

Example

S^{2n} : $2n$-dimensional sphere

$$S^{2n} = \{(y, z_1, \dots, z_n) \in \mathbb{R} \times \mathbb{C}^n \mid y^2 + \sum |z_i|^2 = 1\}$$

$(S^1)^n \curvearrowright S^{2n}$ given by

$$(g_1, \dots, g_n) \cdot (y, z_1, \dots, z_n) := (y, g_1, \dots, g_n z_n)$$

Example (Complex)

$(S^1)^n \curvearrowright \mathbb{C}P^n$ given by

$$(g_1, \dots, g_n) \cdot [z_0, \dots, z_n] := [z_0, g_1 z_1, \dots, g_n z_n]$$

Combining BIG RESULTS $+\varepsilon$,

- Donaldson
- Kodaira
- Masuda-Panov
- Orlik-Raymond

Theorem (I.-Masuda)

M : closed connected complex mfd of $\dim_{\mathbb{C}} n$,
$(S^1)^n \curvearrowright M$: effectively, as biholomorphisms,
$H^{odd}(M) = 0$.
$\Longrightarrow$
$T(M) = 1$.

Main Theorem (I.-Karshon)

A complex torus manifold is equivariantly biholomorphic to a toric manifold.

In other words,

Corollary

M *: closed connected complex mfd of* $\dim_{\mathbb{C}} n$,
$(S^1)^n \curvearrowright M$ *: effectively, as biholomorphisms.*

$M^{(S^1)^n} \neq \emptyset.$

$\Longleftrightarrow$

M *is equivariantly biholomorphic to a toric manifold.*

Definition

Y : toric variety of $\dim_{\mathbb{C}} n$

$\underset{def.}{\Longleftrightarrow}$

- Y is a normal algebraic variety over $\mathbb{C}$,
- there is an embedding of $(\mathbb{C}^*)^n$ as a Zariski open subset such that.....

Y is called a toric manifold if Y is complete and non-singular.

$$\begin{array}{ccccccc} (\mathbb{C}^*)^n & \times & Y & \longrightarrow & Y & & \text{action} \\ & & \cup & & \cup & & \\ (\mathbb{C}^*)^n & \times & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n & & \text{binary operation} \end{array}$$

Main Theorem (I.-Karshon)

A complex torus manifold M is equivariantly biholomorphic to a toric manifold.

For our M,

- local structure near a fixed point
- how to find $(\mathbb{C}^*)^n$-action
- how to find the toric structure on M

Proposition

M : smooth manifold

G : compact Lie group, $G \curvearrowright M$ *smoothly*

Then, ${}^\forall p \in M^G$,

- ${}^\exists U_p \ni p$ *: G-inv. open subset of M,*
- ${}^\exists D_p \ni 0$ *: G-inv. open subset of* T_pM

such that $U_p \underset{\text{eq. diff}}{\cong} D_p$.

Proposition

M : complex manifold

G : compact Lie group, $G \curvearrowright M$ as biholo.

Then, $^{\forall} p \in M^G$,

- $^{\exists} U_p \ni p$: *G-inv. open subset of M,*
- $^{\exists} D_p \ni 0$: *G-inv. open subset of T_pM*

such that $U_p \underset{eq.\ biholo}{\cong} D_p$.

From now, we assume

- M : closed connected complex mfd of $\dim_{\mathbb{C}} n$
- $(S^1)^n \curvearrowright M$: effectively and as biholo.
- $M^{(S^1)^n} \neq \emptyset$.

We know that

$$^{\forall}p \in M^{(S^1)^n}, (S^1)^n \curvearrowright M \text{ is standard near } p.$$

$\xi_1, \ldots, \xi_n$: fundamental vector fields of $(S^1)^n$

J : complex structure on M

Then, $-J\xi_1, \ldots, -J\xi_n, \xi_1, \ldots, \xi_n$ satisfy

- holomorphic,
- commutative,
- $\mathbb{R}$-linearly independent,
- complete.

$-J\xi_1, \ldots, -J\xi_n, \xi_1, \ldots, \xi_n$ allow us to define

$$\mathbb{C}^n \curvearrowright M \quad \text{holomorphic}$$

whose global stabilizer is discrete subgroup of rank n.

$\Longrightarrow \mathbb{C}^n \curvearrowright M$ induces

$$(\mathbb{C}^*)^n \curvearrowright M \quad \text{effective, holomorphic.}$$

For $p \in M^{(S^1)^n}$,

$$^{\exists}\varphi_p : U_p \to D_p \subset T_pM \quad (S^1)^n\text{-eq. biholo.}$$

For $g \in (\mathbb{C}^*)^n$,

$$\varphi_p^{(g)} := g \circ \varphi_p \circ g^{-1} : gU_p \to gD_p$$

is a biholomorphism.

Claim

$\varphi_p = \varphi_p^{(g)}$ *on* $gU_p \cap U_p$.

Identity theorem tells us that

$$V_p := \bigcup_{g \in (\mathbb{C}^*)^n} gU_p$$

is $(\mathbb{C}^*)^n$-eq. biholo. to T_pM.

M : closed connected complex mfd of $\dim_{\mathbb{C}} n$
$(S^1)^n \curvearrowright M$: effectively and as biholo.
$M^{(S^1)^n} \neq \emptyset$.

What we have

- $(S^1)^n \curvearrowright M$ extends to $(\mathbb{C}^*)^n \curvearrowright M$.
- ${}^\forall p \in M^{(S^1)^n} = M^{(\mathbb{C}^*)^n}$, ${}^\exists V_p \subset M$ s.t.
  - $p \in V_p \subset M$
  - $V_p \cong T_pM \quad (\mathbb{C}^*)^n$-eq. biholo.

We set

- $X' := \bigcup_{p \in M^{(\mathbb{C}^*)^n}} V_p \quad (\mathbb{C}^*)^n$-inv. submfd. of M
- X : a connected component of X'.

X : $(\mathbb{C}^*)^n$-inv. connected open submfd of M,
$(\mathbb{C}^*)^n \curvearrowright X$ locally looks like a rep. of $(\mathbb{C}^*)^n$.

We can see the followings :

Lemma

X is a non-singular toric variety.

Lemma

X is compact. Therefore, $X = M$.

Theorem (I.-Karshon)

M : closed connected complex mfd of $\dim_{\mathbb{C}} n$,
$(S^1)^n \curvearrowright M$: *effectively, as biholomorphisms.*
$M^{(S^1)^n} \neq \emptyset$.

$\Longleftrightarrow$

M is equivariantly biholomorphic to a toric manifold.