Moment Angle Complexes and Integral Cohomology of Toric Orbifolds

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Remark: $\mathrm{Tor}_0^{\mathbb{Z}[\mathrm{R}^*]}(\mathbb{Z}[K], \mathbb{Z}) = \mathbb{Z}[K]/\langle u_1, \cdots, u_n\rangle$.

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It holds over $\mathbb{Z}$ if $\mathcal{X}$ is smooth.

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Question Can we compute $H^*_\mathrm{G}(\mathcal{Z}_K)$ in terms of $H^*_\mathrm{G}(\mathcal{Z}_{K_\pm})$ and $H_\mathrm{G}(\mathcal{Z}_W)$??

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