NilBott Tower of Aspherical Manifolds and Torus Actions

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Introduction

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We shall explain a notion of **nilBott tower** to study fiber space with nil-geometry on smooth aspherical manifolds. A nilBott manifold is the top space of an iterated fiber space with fiber a nilmanifold.

As an application, we shall prove the *smooth rigidity* of nilBott manifolds of *finite type*. Our construction applies to study *homological injective toral actions* on closed aspherical manifolds. Then, the *Halperin-Carlson conjecture* is true for the homological injective T^k-actions. Especially, Kähler Bott manifolds are supporting examples of the Halperin-Carlson conjecture including complex Riemannian flat manifold.

Definition of nilBott tower

M a closed aspherical mfd associated with a tower of nil-fiber spaces:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}.$$

where each stage is a fiber bundle: $L_i \to M_i \xrightarrow{p_i} M_{i-1}$ with fiber a nilmanifold $L_i = \mathrm{N}_i/\Delta_i$. More precisely, on the universal covering $\tilde{M}_i$ ($\Delta_i = \pi_1(L_i), \pi_i = \pi_1(M_i)$ and so on), the following are satisfied.

- For each i, $^{\exists}$ Group extension $1 \to \Delta_i \to \pi_i \to \pi_{i-1} \to 1$.
- For each i, $^{\exists}$ Equivariant principal bundle

$$(\Delta_i, \mathrm{N}_i) \to (\pi_i, \tilde{M}_i) \to (\pi_{i-1}, \tilde{M}_{i-1}).$$

- In addition, each π_i normalizes N_i

Structure of nilBott manifolds

From the last condition, the nilBott tower is thought of as an iterated Seifert fiber space. Our advantage is to apply a rigid property of such Seifert manifolds under a rigidity condition on the base manifolds.

Proposition A

(1) *The fundamental group π_i of M_i is virtually polycyclic.*
(2) *If each real algebraic closure $\mathrm{A}(\pi_i)$ normalizes the nilpotent Lie group* N_i*, then M_i is diffeomorphic to the infrasolvmanifold U_i/π_i.*

Proof.

Using an embedding of π_i into $\mathrm{A}(\pi_i) = U_i T_i \leq U_i \rtimes \mathrm{Aut}(\mathrm{U_i})$, we can construct an infrasolvmanifold U_i/π_i which is a fiber space over the base space again an infrasolvmanifold U_{i-1}/π_{i-1} with fiber a nilmanifold N_i/Δ_i. Inductively if we suppose $\tilde{M}_{i-1}/\pi_{i-1}$ is diffeomorphic to U_{i-1}/π_{i-1}, then the Seifert rigidity with the same nilfiber shows the total spaces $\tilde{M}_i/\pi_i$ and U_i/π_i are diffeomorphic. □

$$\begin{array}{ccccc}
(\Delta_i, \mathrm{N}_i) & \longrightarrow & (\pi_i, \tilde{M}_i) & \longrightarrow & (\pi_{i-1}, \tilde{M}_{i-1}) \\
|| & & H\big\downarrow & & h\big\downarrow \\
(\Delta_i, \mathrm{N}_i) & \longrightarrow & (\pi_i, U_i) & \longrightarrow & (\pi_{i-1}, U_{i-1}).
\end{array}$$

Examples of nilBott manifolds: New & Old 1

(1) **Real Bott manifolds** $M(A)$
a quotient of a flat torus T^n by a free isometric action of $(\mathbb{Z}_2)^n$ defined by a Bott matrix A. $M(A)$ occurs as a tower of S^1-fiber spaces:

$$M(A) = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \dots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{\mathrm{pt}\}.$$

- $\pi_1(M(A)) = \pi \leq \mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ is a Bieberbach group.
- $M(A) = \mathbb{R}^n/\pi$ is a Riemannian flat manifold.

Examples of nilBott manifolds: New & Old 2

(2) **S^1-fibred nilBott manifolds** M
the top space of a tower of S^1-fiber spaces:

$$M = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \dots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{\mathrm{pt}\}.$$

- $\pi \leq \mathrm{E}(\mathsf{N}) = \mathsf{N} \rtimes K$ (virtually nilpotent).
- M is diffeomorphic to an infranilmanifold N/π.

Remark that Masuda & Lee proved the similar result.

In this direction we introduce the following nilBott complex manifolds.

Examples of nilBott manifolds: New & Old 3

(3) **Holomorphic nilBott manifold** M
a complex manifold which is the top space of a tower of $T^1_{\mathbb{C}}$-holomorphic fiber spaces:

$$M = M_n \xrightarrow{T^1_{\mathbb{C}}} M_{n-1} \xrightarrow{T^1_{\mathbb{C}}} \dots \xrightarrow{T^1_{\mathbb{C}}} M_1 \xrightarrow{T^1_{\mathbb{C}}} \{\mathrm{pt}\}.$$

- $\pi_1(M) = \pi$ is virtually nilpotent.
- M is diffeomorphic to an infranilmanifold U/π.

Finite type and Infinite type

Let M be a holomorphic nilBott manifold.

$1 \to \mathbb{Z}^2 \to \pi_i \to \pi_{i-1} \to 1$ defines a 2-cocycle $[f_i] \in H^2_\phi(\pi_{i-1}; \mathbb{Z}^2)$.

Definition

If each $[f_i]$ is a torsion element, then we call M a holomorphic nilBott mfd of finite type.
Otherwise it is infinite type.

Similarly, it is defined on the S^1-fibred nilBott manifold.

Theorem A

If M is a holomorphic nilBott manifold **of finite type**, *then M is diffeomorphic to a* **complex Riemannian flat manifold**.

Similarly an S^1-fibred nilBott manifold **of finite type** *is diffeomorphic to a* **Riemannian flat manifold.**

Application 1: Homologically injective actions

An effective T^k-action on M.
the orbit map $\mathrm{ev} : T^k \to M$ at $x \in M$, $\mathrm{ev}(t) = tx$,
the induced homomorphism $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$.

Definition

If ev_* is injective, the effective T^k-action is said to be homologically injective.

Proposition B (Carrell(1972))

Any holomorphic (isometric) $T^k_{\mathbb{C}}$-action on a closed aspherical Kähler manifold (M, Ω) is homologically injective.

Halperin-Carlsson conjecture

Theorem B

If T^k is a homologically injective action on a closed aspherical n-manifold M, then

$$ {}_kC_j \leq b_j \ (= \text{the } j\text{-th Betti number of } M). $$

In particular $2^k \leq \sum_{j=0}^{n} \mathrm{Rank}\, H_j(M)$, *i.e. the Halperin-Carlsson conjecture is true.*

Remark

Any effective T^k-action *on Real Bott manifolds, Holomorphic (or S^1-fibred) nilBott manifolds* **of finite type** *are* **homologically injective**.

Sketch of proof$_1$ of Theorem B

- $\mathrm{ev}_* : \mathbb{Z}^k \to \pi_1(M) = \pi$ induces $1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1$ $(Q = \pi/\mathbb{Z}^k)$
- *Homologically injectivity $\Longrightarrow$ Splitting subgroup of finite index*
  This means that
  There is a finite index normal splitting subgroup $\pi' \leq \pi$;
  $$\pi' = \mathbb{Z}^k \times Q'.$$
- $T^k = \mathbb{R}^k/\mathbb{Z}^k$ lifts to a principal action $(\mathbb{R}^k, \tilde{M})$, $\tilde{M} = \mathbb{R}^k \times W$.
  $$(\pi', \tilde{M}) \hookrightarrow (\pi, \tilde{M}).$$
  $\tilde{M}/\pi' = T^k \times W/Q'$.
- $(\pi, \tilde{M})$ induces $(H, T^k \times W/Q')$ (here $H = \pi/\pi'$);
  $$\alpha(t, z) = (t \cdot t_\alpha, \alpha z) \ \ (\forall\, \alpha \in H, \forall\, (t, z) \in T^k \times W/Q').$$
  (Note that translation t_α on the T^k-summand.)

Sketch of proof$_2$ of Theorem B

- As the homology action of H on $H_j(T^k) \otimes H_0(W/Q')$ is trivial

$$H_j(T^k) = H_j(T^k) \otimes H_0(W/Q') \subset H_j(T^k \times W/Q')^H.$$

- With the aid of transfer homomorphism, the projection

$$\nu : T^k \times W/Q' \to T^k \underset{H}{\times} W/Q' = \tilde{M}/\pi' \Big/ \pi/\pi' = M$$

  induces an isomorphism

$$\nu_* : H_j(T^k \times W/Q'; \mathbb{Q})^H \longrightarrow H_j(M; \mathbb{Q}).$$

- Hence, ${}_kC_j = \mathrm{Rank}\, H_j(T^k) \leq \mathrm{Rank}\, H_j(M; \mathbb{Q}) = b_j$.

Application 2: Kähler Bott tower

Let $T^1_{\mathbb{C}} \to M_i \xrightarrow{\mathrm{p}_i} M_{i-1}$ be a fiber space of a holomorphic nilBott tower. Suppose that each M_i is a Kähler manifold with Kähler form Ω_i where

$$p_i : (M_i, \Omega_i) \to (M_{i-1}, \Omega_{i-1})$$

is a Kähler map, that is "preserves the Kähler form on each". Moreover, if $\mathbb{C}$ is the lift of $T^1_{\mathbb{C}}$, then $\mathbb{C}$ leaves invariant a Kähler form $\tilde{\Omega}_i$ on $\tilde{M}_i$ (that is, $\mathbb{C}$ acts as Kähler isometries w.r.t. $\tilde{\Omega}_i$.)

Definition

The top space of the above Kähler Bott tower is called a Kähler Bott manifold.

Theorem C

A Kähler Bott manifold M is diffeomorphic to a complex Riemannian flat manifold $T^n_{\mathbb{C}}/F$.

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- the cocycle $[f_i]$ for the group extension $1\to\mathbb{Z}^2\to\pi_i \xrightarrow{p_i} \pi_{i-1}\to 1$ has finite order.
  $\Leftrightarrow$ *M is finite type*
  $\Rightarrow$ a complex Riemannian flat manifold (by Theorem A).
- There is an equivariant fibration:

$$(\mathbb{Z}^2, \mathbb{C})\to(\pi_i, \tilde{M}_i) \xrightarrow{p_i} (\pi_{i-1}, \tilde{M}_{i-1}).$$

  Taking a finite index subgroup Δ_i in π_i, there induces a central group extension: $1\to\mathbb{Z}^2\to\Delta_i \xrightarrow{p_i} \Delta_{i-1}\to 1$.

Proof.

- $T^1_{\mathbb{C}} = \mathbb{C}/\mathbb{Z}^2$, $Y_i = \tilde{M}_i/\Delta_i$, and $Y_{i-1} = \tilde{M}_{i-1}/\Delta_{i-1}$.
  (Y_i as a finite covering of M_i, is a Kähler manifold.)
- a principal holomorphic fibration: $T^1_{\mathbb{C}} \to Y_i \xrightarrow{q_i} Y_{i-1}$.
- The Kähler action $(T^1_{\mathbb{C}}, Y_i)$ is homologically injective (Proposition B) $\Rightarrow \pi$ has a finite index splitting subgroup $\Rightarrow \Delta_i$ splits;

$$\Delta_i = \mathbb{Z}^2 \times \Delta_{i-1} \text{ that is, } H^2_\phi(\Delta_{i-1}; \mathbb{Z}^2) \ni \iota^*[f_i] = 0.$$

- Using the transfer homomorphism $\tau : H^2_\phi(\Delta_{i-1}; \mathbb{Z}^2) \to H_\phi(\pi_{i-1}; \mathbb{Z}^2)$,
  $\tau \circ \iota^* = [\pi_{i-1} : \Delta_{i-1}] = \ell : H_\phi(\pi_{i-1}; \mathbb{Z}^2) \to H_\phi(\pi_{i-1}; \mathbb{Z}^2)$.
  As $\iota^*[f_i] = 0$, $\ell[f_i] = 0$.

□

Final remark

Although T^k-actions on compact Riemannian flat manifolds are not necessarily homologically injective, we have the following result.

Theorem D

The Halperin-Carlsson conjecture is true for any effective T^k-action on compact Riemannian flat n-manifolds M.

Proof.

An effective T^k-action on M induces a group extension $1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1$. There is a unique maximal abelian subgroup $\mathbb{Z}^n$ by Bieberbach's Theorem. $\mathbb{Z}^k \leq \mathbb{Z}^n$. Choosing a finite index subgroup G from $\mathbb{Z}^n$ such that $\mathrm{G} = \mathbb{Z}^k \oplus \mathbb{Z}^{n-k}$. Thus π has a finite index splitting subgroup G. Apply the proof of Theorem B. □