Geometry of moment-angle complexes

Jérôme Tambour

KAIST (Daejeon, South Korea)

Osaka City University, Osaka
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Moment-angle complexes

Let K be a (finite and non empty) simplicial complex on a set $\{1, \ldots, n\}$. If $\sigma \in K$, we put

$$B_\sigma = \{\ z \in \mathbb{D}^n \ / \ \forall j \notin \sigma,\ |z_j| = 1\ \} \simeq \mathbb{D}^{|\sigma|} \times (S^1)^{n-|\sigma|}$$

The moment-angle complex parametrized by K is the set

$$\mathcal{Z}_{K,n} = \bigcup_{\sigma \in K} B_\sigma$$

Problem: the characterization of moment-angle complexes which can be endowed with a smooth or complex structure.

Examples:

If $K = \{\emptyset\}$, then $\mathcal{Z}_{K,n} = (S^1)^n$.

If $K = \{\ \emptyset, \{1\}, \{2\}\ \}$, then $\mathcal{Z}_{K,2} = (S^1 \times \mathbb{D}) \bigcup_{S^1 \times S^1} (\mathbb{D} \times S^1) \simeq S^3$

If K is the set of all the subsets of $\{1, \ldots, n\}$, then $\mathcal{Z}_{K,n} = \mathbb{D}^n$.

Si K is the set of all the subsets of $\{1, \ldots, n\}$ which are different of $\{1, \ldots, n\}$, then $\mathcal{Z}_{K,n} \simeq S^{2n-1}$.

If K is a shifted complex, then $\mathcal{Z}_{K,n}$ is a wedge of spheres.

Group action and complex manifolds

Theorem:
The orbit space X/G of a free, proper and holomorphic action of a complex Lie group G on a complex manifold X can be endowed with a structure of complex compact manifold such that the natural surjection $\pi : X \to X/G$ is a holomorphic map.

Some hypothesis

Parameters:

- m, n two positive integers (avec $2m + 1 \leq n$)
- $\lambda_1, \cdots, \lambda_n$ vectors of $\mathbb{C}^m$
- $\mathcal{P}$ a pure simplicial complex on $\{1, ..., n\}$ with dimension $n - 2m - 2$.

Hypothesis: We assume that $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mathcal{P}$ fulfill the following condition:
For every P in $\mathcal{P}$, $(\lambda_j, j \notin P)$ spans $\mathbb{C}^m$ as a real affine space.

We note:

$$\mathcal{S} = \{ z \in \mathbb{C}^n \ / \ I_z \notin \mathcal{P} \} \text{ where } I_z = \{ k \ / \ z_k \neq 0 \}$$

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$$\mathcal{S} = \mathbb{C}^n \setminus \bigcup_{\sigma^c \in \mathcal{P}} L_\sigma$$

$$L_\sigma = \{ \ z \in \mathbb{C}^n \ / \ z_{j_1} = z_{j_2} = \cdots = z_{j_p} = 0 \ \}$$

$$\sigma = \{j_1, \ldots, j_p\} \subset \{1, \ldots, n\}$$

$\mathbb{C}^* \times \mathbb{C}^m$ acts holomorphically and freely on $\mathcal{S}$ by

$$(\alpha, T) \cdot z = (\alpha e^{<\lambda_1, T>} z_1, \cdots, \alpha e^{<\lambda_n, T>} z_n)$$

We put $\mathcal{E} = \{P^c | P \text{ is a facet of } \mathcal{P}\}$

Theorem (Bosio, 2001):

The orbit space $\mathcal{N}$ of $\mathcal{S}$ for the previous action can be endowed with a structure of complex manifold if and only if the two following conditions are fulfilled:

- (SE) For every P in $\mathcal{E}$ and every k in $\{1, ..., n\}$, there exists an élément k' in P such that $(P \backslash \{k'\}) \cup \{k\}$ is in $\mathcal{E}$.
- (IC) For every P and Q in $\mathcal{E}$, the (relative) interiors of $Conv(\lambda_p, p \in P)$ and $Conv(\lambda_q, q \in Q)$ are not disjoint.

In this case, $\mathcal{N}$ is called *LVMB manifold* (parametrized by the good system $(\mathcal{P}, \lambda)$).

History

1. LVMB = Lopez de Medrano - Verjovsky - Meersseman - Bosio
2. S.López de Medrano, *Topology of the intersection of quadrics in* $\mathbb{R}^n$.
3. S.López de Medrano and A.Verjovsky, *A new family of complex, compact, non-symplectic manifolds.*, 1997
4. L.Meersseman, *Un procédé géométrique de construction de variétés compactes, complexes, non algébriques, en dimension quelconque.*, 1998
5. F.Bosio, *Variétés complexes compactes: une généralisation de la construction de Meersseman et López de Medrano-Verjovsky.*, 2001

Bosio conditions in terms of the complex $\mathcal{P}$

Theorem (T.,2010)

1. *If $(\mathcal{P}, \lambda)$ is a good system, then $\mathcal{P}$ is a starshaped sphere.*
2. *Conversely, if $\mathcal{P}$ is a starshaped sphere, then there exists a (non unique) family λ of vectors of $\mathbb{C}^m$ such that $(\mathcal{P}, \lambda)$ is a good system.*

Remark: The same results where obtained independantly by Panov and Ustinovsky.

LVMB manifolds and moment-angle complexes

Proposition:

Let $(\mathcal{P}, \lambda)$ be a good system. Then the moment-angle complex $\mathcal{Z}_{\mathcal{P},n}$ intersects every orbit of the action of $\mathbb{C}^* \times \mathbb{C}^m$ on $\mathcal{S}$.

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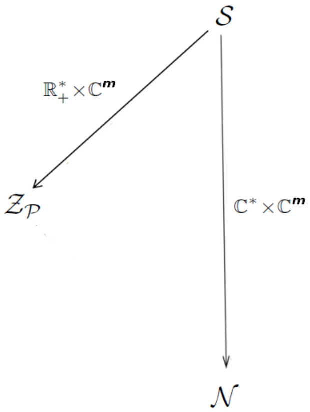
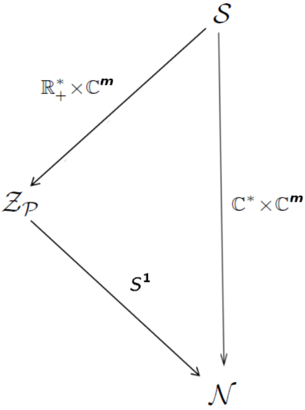
Moreover, if we consider the restriction of this action to $\mathbb{R}^*_+ \times \mathbb{C}^m$, then $\mathcal{Z}_{\mathcal{P},n}$ meets every orbit of $\mathcal{S}$ in exactly one point.

Consequence: the orbit space $\mathcal{S}/(\mathbb{R}^*_+ \times \mathbb{C}^m)$ is homeomorphic to $\mathcal{Z}_{\mathcal{P},n}$.


$\mathcal{S}$
$\mathbb{R}_+^* \times \mathbb{C}^{\boldsymbol{m}}$
$\mathcal{Z}_{\mathcal{P}}$
$\mathbb{C}^* \times \mathbb{C}^{\boldsymbol{m}}$
$\mathcal{N}$


$\mathcal{S}$
$\mathbb{R}_+^* \times \mathbb{C}^m$
$\mathcal{Z}_\mathcal{P}$
$\mathbb{C}^* \times \mathbb{C}^m$
S^1
$\mathcal{N}$

$\mathbb{C}^*$ is diffeomorphic to the product $S^1 \times \mathbb{R}^*_+$.

Consequence: the LVMB manifold $\mathcal{N}$ is the orbit space of an action of the circle S^1 on $\mathcal{Z}_{\mathcal{P},n}$.

$$\mathcal{N} = \mathcal{Z}_{\mathcal{P},n}/S^1$$

In many frequent cases, $\mathcal{Z}_{\mathcal{P},n} = \mathcal{Z}_{\mathcal{P},n-1} \times S^1$ and we can show that $\mathcal{N}$ is homeomorphic to $\mathcal{Z}_{\mathcal{P},n-1}$.

Complex structure on moment-angle complexes

Corollary:

Every moment-angle associated to a starshaped sphere and with even dimension can with a complex structure.

First examples

Example:

If $m = 0$, then $\mathcal{N}$ is the projective space $\mathbb{CP}^{n-1}$

Example:

If $n = 2m + 1$, then $\mathcal{N}$ is a complex torus. Conversely, every complex structure on tori (of every dimension) can be obtained as a LVMB manifold.

An important theorem

Theorem (Meersseman, Bosio):

If $n > 2m + 1 > 1$, then $\mathcal{N}$ is a non projective complex compact manifold.

Two classical non projective examples

Theorem (Hopf, Calabi-Eckmann):

There exists a complex structure on $S^{2q+1} \times S^{2p+1}$ for every nonnegative integer p and q. This structure is non projective (except if $p = q = 0$).

Example:

Hopf manifolds and Calabi-Eckmann manifolds can be obtained as LVMB manifolds.

For instance, if $\lambda = (\underbrace{-1-i, \dots, -1-i}_{p+1}, 1-i, \underbrace{0, \dots, 0}_{q+1})$, then $\mathcal{N}$ is diffeomorphic to $S^{2p+1} \times S^{2q+1}$.

Simplicial sphere with few vertices

Theorem:
Lopez de Medrano (1989) If $\mathcal{P}$ is a d-sphere with $n = d + 4$ vertices, then $\mathcal{Z}_{\mathcal{P},n}$ is

- either a product of 3 spheres of odd dimension;
- or a connected sum of products of spheres.

Polytopal case vs non polytopal case

If $\mathcal{P}$ is polytopal, then there exists a family $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\mathcal{Z}_{\mathcal{P},n}$ can be embedded as a smooth (but not complex) submanifold of $\mathbb{CP}^{n-1}$:

$$\mathcal{N} = \left\{ [w_1 : \cdots : w_n] \in \mathbb{CP}^{n-1} \;\Big/\; \sum_{j=1}^{n} |w_j|^2 \lambda_j = 0 \right\}$$

Polytopal case vs non polytopal case (II)

Question: Does there exists a moment-angle complex $\mathcal{Z}_{\mathcal{P},n}$ such that

1. $\mathcal{P}$ is a non polytopal starshaped sphere;
2. $\mathcal{Z}_{\mathcal{P},n}$ is not diffeomorphic to $\mathcal{Z}_{K,n}$ for every polytopal complex K.

The question is still open.

Bosio,T.(2011)

The answers is no if $\mathcal{P}$ has dimension 3 and 8 vertices.

Acknowledgements

Thank you for your attention.