Characteristic numbers of toric varieties from combinatorial point of view

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Plan

- Localization formulas
  - in equivariant cohomology of smooth manifolds
  - in equivariant complex cobordism ring
- Applications in combinatorcis and toric topology
  - Cobordisms of toric varieties and combinatorial operations on simplicial fans.
  - Volume polynomial of simple polytope.

Localization formula in equivariant cohomolgy

Theorem (Atiyah, Bott '84)

Let G be a compact Lie group acting on a closed manifold M. Consider an element $\omega \in H^*_G(M)$. Then in $H^*_G(pt)$ one has:

$$\int_M \omega = \sum_{F \in Fix} \int_F \frac{\omega|_F}{e^G(\nu_F)},$$

where F is a connected component of M^G, and $e^G(\nu_F)$ denotes an equivariant Euler class of normal bundle $\nu_F = TM/TF$.

Localization formula and characteristic numbers

Much earlier in the case of holomorphic vector field on a complex manifold Raul Bott has found a residue formula for its characterisitic numbers.

Theorem (Bott's residue)

Let M be a stably complex manifold with holomorphic vector field V. $c(TM) = \sum c_i$. Then

$$\int_M c_1^{i_1} \dots c_n^{i_n} = \sum_{p \in Fix} \frac{g(w_1(p), \dots, w_n(p))}{w_1(p) \dots w_n(p)},$$

where $\{w_i(p)\}_{i=1}^n$ are weights of representation of V at TM_p and $g(t_1, \dots, t_m)$ is a symmetric polynomial

$$g(t_1, \dots, t_n) = \sigma_1^{i_1}(t_1, \dots, t_m) \dots \sigma_n^{i_n}(t_1, \dots, t^n)$$

Initial approach worked only for holomorphic action on complex manifolds, while analogous formula holds even for stably complex manifolds and vector fields induced by an arbitrary Lie group action.

Localization formula in cobordisms

Theorem (Krichiver)

Let M be a stably complex manifold with T^n action, which acts with isolated fixed points. Let $\pi \colon M \times_T ET \to BT$ be a natural projection. Then in $\Omega_U^*[[u_1, \dots, u_m]]$ one has

$$\pi_*(1) = \sum_p sign(p) \frac{1}{u_1[w_1(p)] \dots u_n[w_n(p)]},$$

where w_i are weights of T^n representation in TM_p, $sign(p)$ is a sign of a fixed point and $u_1[w_1(p)] \dots u_n[w_n(p)]$ denotes power degree and multiplication in the formal group of complex cobordisms.

The so-called *miraculous cancellations* are just the fact that all singularities in right-hand side vanish. Bott's formula on characterisitic numbers is nothing but the statement, that series $\pi_*(1)$ starts with $[M]$.

Combinatorial preliminaries

Let K be a simplicial complex on the set of m vertices.

- We say that $\widehat{K}$ is obtained from K by *stellar subdivision* if there is a simplex $\sigma \in K$ s.t.

$$\widehat{K} = (K \backslash \mathrm{star}\sigma) \cup_{\mathrm{lk}\sigma * \partial\sigma} (\mathrm{lk}\sigma * v * \partial\sigma)$$

- Let K be a PL-manifold then we say that PL-manifold $\widehat{K}$ is obtained from K by *bistellar move of type k* if there is a subcomplex of the form $\partial\sigma * \tau$ in K for σ — a $(k-1)$-dim simplex, s.t.

$$\widehat{K} = (K \backslash \partial\sigma * \tau) \cup_{\partial\sigma * \partial\tau} (\sigma * \partial\tau).$$

These two operations are crucial in the study of *PL*-manifolds due to the following two results:

Theorem (Newman '30)

Two simplicial complexes K_1 and K_2 are PL-equivalent iff there is a sequence of stellar subdivisions and inverse operations, connecting K_1 and K_2.

Theorem (Pachner '86)

Two PL-manifolds K_1 and K_2 are PL-equivalent iff there is a sequence of bistellar moves, connecting K_1 and K_2.

In order to construct a nonsingular toric variety, we need not just a simplicial complex K, but its realization as a *regular* simplicial fan Σ_K. The definitions of stellar subdivisions and bistellar moves for fans are naturally extended.

Characteristic numbers and stellar subdivisions

Theorem

Assume that regular simplicial fan Σ_2 is obtained from simplicial fan Σ_1 by a stellar subdivision at cone (simplex) σ, then in the complex cobordism ring Ω_*^U one has:

$$[X_{\Sigma_2}] - [X_{\Sigma_1}] = [Y],$$

where Y is a quasitoric manifold over multifan

$$star\sigma \cup_{lk\sigma * \partial\sigma} (lk\sigma * v * \partial\sigma).$$

As smooth manifold Y is just $\mathbb{P}(\nu(X_{lk_\sigma}) \oplus \mathbb{C})$, however orientation and stably complex structure are not standard.

Example

Blow-up of a toric surface X at a fixed point is a stellar subdivision at a maximal simplex. Thus $[\widetilde{X}] - [X] = [\overline{\mathbb{C}P^2}]$

On the dual language of simple polytopes we have considered face-truncation operation. In particular the above result gives direct combinatorial formula for the characteristic numbers of toric varieties corresponding to 2-truncated 3-cubes — all polytopes, which could be obtained from standard cube I^3 by consecutive truncation of edges.

Proposition

If P^3 is obtained from Q^3 by truncating an edge E, then $s_3(X_P) = s_3(X_Q) - 2k$, where $k = c_1(\nu(X_E))$.

These k-numbers, which are assigned to each edge, could be computed very easily.

Consequence

Two toric varieties over one combinatorial polytope P could represent different cobordism classes.

Characteristic numbers and stellar subdivisions

Theorem

Assume that a regular simplicial fan Σ_2 is obtained from simplicial fan Σ_1 by a bistellar move of type k, then in the complex cobordism ring Ω_*^U one has:

$$[X_{\Sigma_2}] - [X_{\Sigma_1}] = [\mathbb{C}P^n_k]$$

where $\mathbb{C}P^n_k$ is a complex projective space as a smooth manifold, and orientation with stably complex structure are chosen in such a way that exactly k T^n-fixed points in $\mathbb{C}P^n_k$ have $-$-sign.

Example

Again, blow-up of a toric surface X at a fixed point is a bistellar move of type 1. Thus $[\widetilde{X}] - [X] = [\mathbb{C}P^2_1]$. It is easy to see, that $\overline{\mathbb{C}P^2}$ with $T\mathbb{C}P^2 \oplus \mathbb{R}^2 = \eta \oplus \eta \oplus \overline{\eta}$ has exactly one minus-point.

As a consequence of previous computation we have

Proposition

If Σ_1 is obtained from Σ_2 by a sequence of bistellar moves of different types, then

$$[X_{\Sigma_2}] - [X_{\Sigma_1}] = \sum_{i=1}^{n}(h_i(\Sigma_2) - h_i(\Sigma_1))[\mathbb{C}P_k^n].$$

This implies, that there is no analogue of Pacher's theorem for regular simplicial fans, since otherwise $[X_\Sigma]$ depend only on combinatorics of the underlying simplicial comlex.

Volume Polynomial of Simple Polytope

Let P^n be geometric realization of a simple polytope and assume for a moment, that corresponding toric variety X_P is nonsigular. Let $\lambda_1, \ldots, \lambda_m$ be an *integer distances* from the origin to hyperplanes containing facets $F_1, \cdots F_m$ of P. Then there is a formula due to Kushnirenko

Theorem (Kushnirenko)

$$n! Vol(P; \lambda_1, \ldots, \lambda_m) = \int_{X_P} (\lambda_1 D_1 + \cdots + \lambda_m D_m)^n,$$

where $D_i \in H^2(X_P, \mathbb{Z})$ is divisor corresponding to facet F_i.

In particular $Vol(P; \lambda_1, \ldots, \lambda_m)$ is a polynomial of degree n in $\lambda_1, \ldots, \lambda_m$.

Localization formula gives a direct way for computation of the latter integral. Indeed all $D_i \in H^2(X_P, \mathbb{Z})$ could be lifted to a classes $\omega_i \in H^2_{T^n}(X_P, \mathbb{Z})$.

Proposition

For a regular lattice polytope P one has

$$n! Vol(P; \lambda_1, \dots, \lambda_m) = \sum_{v = F_{i_1} \cap \dots \cap F_{i_n}} \frac{\left(\lambda_{i_1} w_{i_1}(v; x) + \dots + \lambda_{i_n} w_{i_n}(v; x)\right)^m}{w_{i_1}(v; x) \dots w_{i_n}(v; x)},$$

where $w_{i_1}(v;\ ), \dots, w_{i_n}(v;\ )$ *are weights of* T^n *representations in* TX_P *at point p. (here x is an arbitrary vector in the Lie algebra* $\mathfrak{t}$ *of a torus* T^n*.)*

As a consequence we obtain that the meromorphic function on right hand side is a constant.

Application of the orbifold localization theorem to the resolutions of singular toric varieties together with continuity reasons give the following result. Here we do not require integer and regularity assumptions and formulate results purely in geometrical terms.

Theorem

Let P be a simple polytope, $\{e_i\}_{i=1}^m$ —unit normal vectors of its facets F_i and d_i — euclidean distances from the origin to the hyperplanes, containing facets F_i. Then

$$n!Vol(P) = \sum_{v=F_{i_1}\cap\dots\cap F_{i_n}} \frac{1}{det(e_{i_1},\dots,e_{i_n})} \frac{\left(d_{i_1}w_{i_1}(v;x)+\dots+d_{i_n}w_{i_n}(v;x)\right)^m}{w_{i_1}(v;x)\dots w_{i_n}(v;x)},$$

where $w_{i_j}(v;x)$ are coordinates of $x\in\mathbb{R}^n$ in the basis $e_{i_1},\dots,e_{i_n}$.

Thank you!