On the topological decomposition of the hypersurfaces in projective toric manifolds

Wei WANG

School of mathematical sciences, Fudan University

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Introdunction

Let F^n be a nonsingular algebraic hypersurface in $\mathbb{C}P^{n+1}$, for example:

$$\text{Fermat hypersurface } F_d^n := \{[z_0, \cdots, z_{n+1}] \in \mathbb{C}P^{n+1} | \sum z_i^d = 0\}$$

Kulkarni and Wood proved a decomposition theorem of F^n :

Theorem (Kulkarni, Wood)

For $n \neq 2$, there is a differentiable connected sum decomposition:

$$F^n = M \sharp k(S^n \times S^n)$$

When n is odd, $b_n(M) = 0$ or 2. When n is even, $b_n(M) - |sign(M)| = 0$ or 2.

Projective toric manifolds

It is known that $\mathbb{C}P^{n+1}$ is a classical example of toric manifolds and I wonder if I can get some similar results on the hypersurface in (projective) toric manifolds.
Recall that

Definition

A **toric variety** *is a normal algebraic variety X containing the algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open subset in such a way that the normal action $(\mathbb{C}^*)^n$ on itself extends to an action on X.*
In this talk, we call X a **projective toric manifold** *if X is a compact, smooth toric variety that admits a holomorphic embedding into a certain $\mathbb{C}P^N$.*

Examples: $\mathbb{C}P^n$, $\mathbb{C}P^n \times \mathbb{C}P^m$, $BL_p\mathbb{C}P^n$, $\cdots$

Hypersurface Y in X

Let X be a projective toric manifold. For any holomorphic embedding $X \hookrightarrow \mathbb{C}P^N$, let F_1 be a hyperplane of $\mathbb{C}P^N$, we get a subvariety of X :

$$i : Y = F_1 \cap X \hookrightarrow X$$

Y is called a **hypersurface** of X. By Bertini's theorem, for a generic choice of F_1, Y is smooth. From now on, when we say a hypersurface Y, it always means Y is a **smooth** one.

Y and Y_d

For a hypersurface $i : Y^n \hookrightarrow X^{n+1}$, we can construct the smooth hypersurface $i_d : Y_d \hookrightarrow X$, $d \in \mathbb{Z}^+$ with

$$(i_d)_*[Y_d] = d(i_*[Y]) \in H_n(X, \mathbb{Z})$$

Indeed, we can let $Y_d := F_d \cap X$, where F_d is a generic hypersurface in $\mathbb{C}P^N$ with degree d.

Similar to the degree of a hypersurface in $\mathbb{C}P^N$, we can also define the degree of Y in X. Let α_Y be the element of $H^2(X, \mathbb{Z})$ with $\alpha_Y \cap [X] = i_*[Y]$, we denote the **degree** of Y in X by

$$degY =< \alpha_Y^{n+1}, [X] >$$

We see $degY_d = d^{n+1} degY$

Signature of $H^n(X, \mathbb{Z})$

When n is even, the cohomolgy class $\alpha_{Y_d} \in H^2(X^{n+1})$ induces a bilinear form on $H^n(X^{n+1})$:

$$H^n(X) \otimes H^n(X) \longrightarrow \mathbb{Z}$$

$$(x, y) \mapsto < x \cup y \cup \alpha_{Y_d}, [X] >$$

We denote the signature of this bilinear form by $sign(H^n(X))$.
Example: For the hypersurface $F^n \hookrightarrow \mathbb{C}P^{n+1}$, n even, we know $b_n(\mathbb{C}P^{n+1}) = sign(H^n(\mathbb{C}P^{n+1})) = 1$.

Main result

Let $i : Y^n \hookrightarrow X^{n+1}$ be a hypersurface and Y_d be the hypersurface of X^{n+1} with $(i_d)_*[Y_d] = d(i_*[Y]) \in H_n(X, \mathbb{Z})$. Our main result is:

Theorem

When n is odd, for **any** $d > 0$*:*

$$Y_d = M_d \sharp s_d (S^n \times S^n)$$

$b_n(M_d) = 0$ *or* 2*. When n is even, for* **sufficiently big** d*, $(d > N_X)$,*

$$Y_d = M_d \sharp s_d (S^n \times S^n)$$

$b_n(M_d) - |sign(M_d)| = b_n(X) \pm sign(H^n(X))$.

Remark

Compare to Kulkarni and Wood's theorem:

Theorem (Kulkarni, Wood)

For $F^n \hookrightarrow \mathbb{C}P^{n+1}$, $n \neq 2$, there is a differentiable connected sum decomposition:

$$F^n = M \sharp k(S^n \times S^n)$$

When n is odd, $b_n(M) = 0$ or 2. When n is even, $b_n(M) - |sign(M)| = b_n(\mathbb{C}P^{n+1}) \pm sign(H^n(\mathbb{C}P^{n+1})) = 0$ or 2.

When n is odd, our result is a generalization to the case of **all** projective toric hypersurface.

When n is even, our result is a generalization to the case of toric hypersurface with **big** degree.

Limits of s_d

For the decomposition $Y_d = M_d \sharp s_d(S^n \times S^n)$, we also want to estimate s_d when d is getting bigger and bigger.
When n is odd, we have limit estimate:

$$\lim_{d\to+\infty} \frac{2s_d}{degY_d} = \lim_{d\to+\infty} \frac{b_n(Y_d)}{d^{n+1}degY} = 1$$

When n is even, we have limit estimate:

$$0 < \lim_{d\to+\infty} \frac{2s_d}{degY_d} = \lim_{d\to+\infty} \frac{b_n(Y_d) - |sign(Y_d)|}{d^{n+1}degY}$$

$$= 1 - 2^{n+1}(2^{n+1} - 1)\frac{B_{\frac{n+2}{2}}}{(n+1)!} < 1$$

here $B_{\frac{n+2}{2}}$ is the $\frac{n+2}{2}$-th Bernouli number.

Removing handles: geometric point of view

Choose a point $(x, y) \in S^n \times S^n$ and there are two embedded spheres: $S_1 := S^n \times \{y\}$, $S_2 := \{x\} \times S^n \hookrightarrow S^n \times S^n$ with properties:
(1). S_1 intersects S_2 transversally at one point (x, y).
(2). The normal bundles η_1, η_2 of S_1, S_2 in $S^n \times S^n$ are trivial.
We see $\overline{\eta_1} \cup \overline{\eta_2}$ is a manifold with boundary S^{2n-1} and

$$S^n \times S^n = (\overline{\eta_1} \cup \overline{\eta_2}) \cup_{S^{2n-1}} D^{2n}$$

Conversely, let M^{2n} be a smooth manifold and S_1, S_2 be two embedded n-spheres of M^{2n} with:
(1). S_1 intersects S_2 transversally at one point.
(2). The normal bundles η_1, η_2 of S_1 and S_2 are trivial.
we get:

$$M = M' \sharp\, S^n \times S^n$$

where $M' = (M - \eta_1^\circ \cup \eta_2^\circ) \cup_{S^{2n-1}} D^{2n}$.

Removing handles: homological point of view

From the point of view of homology, let M^{2n} be a simply connected smooth closed manifold of dimension $2n$, $n > 2$ and $h : \pi_n(M) \longrightarrow H_n(M, \mathbb{Z})$ be the Hurewicz map.

For every $\alpha,\ \beta \in h(\pi_n(M)) \subset H_n(M, \mathbb{Z})$ with intersection number $\alpha \cdot \beta = 1$, by Whitney's embedding theory and Whitney's trick, there are two embedding n-spheres $f_\alpha,\ f_\beta : S^n \hookrightarrow M^{2n}$ with:
(1). The homology elements α and β are represented by $f_\alpha,\ f_\beta$, i.e. $(f_\alpha)_*[S^n] = \alpha,\ (f_\beta)_*[S^n] = \beta$
(2). The spheres $f_\alpha(S^n)$ and $f_\beta(S^n)$ intersect transversally at only one point.

The next question is how to determine the normal bundles. In general, the normal bundles of $f_\alpha(S^n),\ f_\beta(S^n)$ are not easy to determine. But in our case, the situation become relatively simpler.

Odd case:

Let Y^n be a hypersurface in X^{n+1}, from this diagram

$$\begin{array}{ccccc} H_{n+1}(X,Y) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(X)=0 \\ \uparrow \cong & & \uparrow h_Y & & \uparrow \\ \pi_{n+1}(X,Y) & \longrightarrow & \pi_n(Y) & \xrightarrow{\pi_n(i)} & \pi_n(X) \end{array}$$

we can prove:

Proposition

Every element $\alpha \in H_n(Y, \mathbb{Z})$ can be represented by an embedding $f_\alpha : S^n \hookrightarrow Y$ such that the normal bundle η_{f_α} of f_α is ***stable trivial****.*

The obstruction: ω

Furthermore, we find an obstruction (a certain Wu class) $\omega \in H^{n+1}(X, \mathbb{Z})$ such that:

When $\omega = 0$, there exists a quadratic function $\psi : H_n(Y, \mathbb{Z}) \longrightarrow \mathbb{Z}_2$ with:

(1). $\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + (\alpha \cdot \beta)_2$, where $(\alpha \cdot \beta)_2 \in \mathbb{Z}_2$ is the mod 2 class of the intersection number $\alpha \cdot \beta \in \mathbb{Z}$.

(2). $\psi(\alpha) = 0$ if and only if α can be represented by an embedded n-sphere f_α with **trivial normal bundle**.

Then $H_n(Y, \mathbb{Z}) \cong \oplus_{i=0}^{s}(\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i)$ with intersection matrix $\oplus_{i=0}^{s} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\psi(\alpha_i) = \psi(\beta_i) = 0, i \neq 0$.

Then

$$Y \cong M \sharp s(S^n \times S^n)$$

with $b_n(M) = 2$. (Although we can not determine $\psi(\alpha_0) = \psi(\beta_0)$).

When $\omega \neq 0$

When $\omega \neq 0$, we can prove there exists an n-sphere $g : S^n \hookrightarrow Y$ with:
(1). $g_*[S^n] = 0 \in H_n(Y, \mathbb{Z})$.
(2). The normal bundle η_g of $g : S^n \hookrightarrow Y$ is isomorphic to the tangent bundle TS^n of S^n.

For every element $\alpha \in H_n(Y, \mathbb{Z})$ which is represented by an embedding f_α, we can use this nullhomological sphere to obtain a new embedding f'_α (modify the normal bundle) with:
(1). $f'_\alpha = f_\alpha + g \in \pi_n(Y)$
(2). The normal bundle of f'_α is trivial.

Since $H_n(Y, \mathbb{Z})$ admits a basis $\{\alpha_i, \beta_i\}$ with intersection matrix $\oplus_{i=0}^{s} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we get:

$$Y \cong M\sharp(s+1)(S^n \times S^n)$$

with $b_n(M) = 0$

Even case

When the complex dimension n of Y_d^n is even, we have diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n+1}(X, Y_d) & \longrightarrow & H_n(Y_d) & \xrightarrow{(i_d)_*} & H_n(X) & \longrightarrow & 0 \\ & & \uparrow \cong & & \uparrow h_{Y_d} & & \uparrow & & \\ & & \pi_{n+1}(X, Y_d) & \longrightarrow & \pi_n(Y_d) & \xrightarrow{\pi_n(i_d)} & \pi_n(X) & & \end{array}$$

The **vanishing cycles** $Ker(i_d)_* \subset h(\pi_n(Y_d))$ are what we mainly concerned, we have:

Proposition

Each element $\alpha \in Ker(i_d)_*$ can be represented by an embedding $f_\alpha : S^n \hookrightarrow Y_d$ such that $f_\alpha[S^n] = \alpha$ and the normal bundle η_{f_α} of f_α is ***stable trivial****.*

Structure of $Ker(i_d)_*$

For the embedding f_α representing $\alpha \in Ker(i_d)_*$, the stable trivial normal bundle of f_α is just determined by the self-intersection number $\alpha \cdot \alpha$ of α. Indeed, $\alpha \cdot \alpha = 0$ if and only if the normal bundle of f_α is trivial.

So for $Ker(i_d)_* \subset H_n(Y_d, \mathbb{Z})$, we can prove: for d is big enough

$$Ker(i_d)_* \cong A \oplus (\oplus_{i=1}^{s_d} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i)$$

where A is definite and the intersection matrix of $(\oplus_{i=1}^{s_d} \mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i)$ is $\oplus_{i=1}^{s_d} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

From this algebraic decomposition, we get our topological decomposition:

$$Y_d = M_d \sharp s_d (S^n \times S^n)$$

Identity of s_d

Since A is definite and $sign(\mathbb{Z}\alpha_i \oplus \mathbb{Z}\beta_i) = 0$, we can prove:

$$s_d = \frac{b_n(Y_d) - b_n(X) - |sign(Y_d) - sign(H^n(X))|}{2}$$

Also it is known that $sign(Y_d) = sign(M_d)$ and $\lim_{d \to +\infty} |sign(M_d)| = +\infty$, we have formula:

$$b_n(M_d) = b_n(Y_d) - 2s_d = b_n(X) - |sign(M_d)| \pm sign(H^n(X))$$

Thanks !