Equivariant local index

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Joint work with Hajime Fujita and Mikio Furuta:


In these joint works we are developing an index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold.

**Purpose of this talk**

1. *To explain our theory in a simple symplectic case.*
1. Equivariant local index

2. Another version

3. A special case
1. Equivariant local index
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3. A special case
(\(M, \omega\)) symplectic manifold with Hamiltonian \(S^1\)-action

(\(L, \nabla^L\)) \(S^1\)-equivariant prequantization line bundle

- Each orbit \(O\) is isotropic, namely, \(\omega|_O \equiv 0\).
  \(\Rightarrow (L, \nabla^L)|_O\) is a flat line bundle. (\(\because \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega\))

**Definition (L-acyclic orbit)**

\[ O : L\text{-acyclic} \iff H^0(\mathcal{O}; (L, \nabla^L)|_O) = 0 \]

- An orbit consisting of a fixed point is not \(L\)-acyclic.
- \(O: L\text{-acyclic} \iff H^\bullet(\mathcal{O}; (L, \nabla^L)|_O) = 0 \ (\because \mathcal{O}\) is a circle)
  \(\iff\) The de Rham operator of \(\mathcal{O}\) with coefficients in \((L, \nabla^L)|_O\) has zero kernel. (\(\because\) the Hodge theory)
Let $k$ be a positive integer.

$$(M, \omega) := (\mathbb{C}P^1, k\omega_{FS}) = \left( S_k^3, \frac{\sqrt{-1}}{2\pi} \sum_{j=0}^{1} dz_j \wedge d\bar{z}_j \right) / (z_0, z_1) \sim (hz_0, hz_1) \ (h \in S^1)$$

$$(L, \nabla^L) := (H^\otimes k, \nabla^L) = \left( S_k^3 \times \mathbb{C}, d + \frac{1}{2} \sum_{j=0}^{1} z_j d\bar{z}_j - \bar{z}_j dz_j \right) / (z_0, z_1, v) \sim (hz_0, hz_1, h^k v)$$

where $S_k^3 := \{ z = (z_0, z_1) \in \mathbb{C}^2 : \|z\|^2 = k \}$.

Take and fix an integer $m$ with $0 < m < k$. Define a $S^1$-action on $L$ and $M$ by

$$S^1 \curvearrowright L: t[z_0 : z_1, v] := [z_0 : tz_1, t^m v].$$

**Non $L$-acyclic orbits**

$$\mathcal{O}_i := \{ [z_0 : z_1] \in M : |z_1|^2 = i \} \ (i = 0, 1, \ldots, k)$$
Theorem 1 (Fujita-Furuta-Y.)

Let \((L, \nabla^L) \to (M, \omega)\) be as above. Let \(V \subset M\) be an \(S^1\)-invariant open set which contains only \(L\)-acyclic orbits and whose complement \(M \setminus V\) is compact. For these data, there exists an element \(\text{ind}_{S^1}(M, V; L) \in R(S^1)\) satisfying the following properties:

1. \(\text{ind}_{S^1}(M, V; L)\) is deformation invariant.
2. For a closed \(M\), \(\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1} D\).
3. If \(M'\) is an \(S^1\)-invariant open neighborhood of \(M \setminus V\), then
   \[\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1}(M', V \cap M'; L|_{M'})\]. (excision property)

4. Gluing formula
5. Product formula

We call \(\text{ind}_{S^1}(M, V; L)\) an equivariant local index.
Definition of $\text{ind}_{S^1}(M, V; L)$

First consider the case where $M$ has a cylindrical end $V = N \times (0, \infty)$ and all the data are translation invariant on the end.

1. For $t \geq 0$ consider the following perturbation of the Dirac operator $D$

   $$D_t := D + t\rho D_{\text{fiber}},$$

   where $D_{\text{fiber}}$ is the de Rham operator of $V$ along orbits, i.e.

   - $D_{\text{fiber}}$ contains only derivatives along orbits
   - $D_{\text{fiber}}|_{\mathcal{O}}$ is the de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$

   $$ \Rightarrow \ker D_{\text{fiber}}|_{\mathcal{O}} = 0 \text{ for } \forall \mathcal{O} \subset V \ (\because V \text{ consists of } L\text{-acyclic orbits})$$

2. By using “$\ker D_{\text{fiber}}|_{\mathcal{O}} = 0 \text{ for } \forall \mathcal{O} \subset V$”, one can show

   1. $\dim \ker D_t \cap L^2 < +\infty$ for a sufficiently large $t \gg 0$.
   2. $\ker D_t|_{\Lambda^0,\text{even} T^*M \otimes L} \cap L^2 - \ker D_t|_{\Lambda^0,\text{odd} T^*M \otimes L} \cap L^2$ is independent of a sufficiently large $t \gg 0$.

Definition (Equivariant local index)

$$\text{ind}_{S^1}(M, V; L) := \ker D_t|_{\Lambda^0,\text{even}} \cap L^2 - \ker D_t|_{\Lambda^0,\text{odd}} \cap L^2 \text{ for } \forall t \gg 0.$$
In general case,

1. Deform $V$ cylindrically so that all the data are translation invariant, and come down to the cylindrical case.

2. Check the localization is independent of a choice of a cut locus $N$. 
1. Equivariant local index

2. Another version

3. A special case
Let $t^*_Z$ be the weight lattice of $S^1$. For $\sigma \in t^*_Z$ and $U \in R(S^1)$ we define by

$$U^\sigma := \dim \text{Hom}_{S^1}(\mathbb{C}_\sigma, U).$$

In Theorem 1, by taking the multiplicity of the irreducible representation with weight $\sigma$, we obtain a corresponding theorem for $\text{ind}_{S^1}(M, V; L)^\sigma$. Here we introduce the following notion of "$(L, \sigma)$-acyclicity" which is milder condition than that of $L$-acyclicity, and we obtain a version of a local index, which is denoted by $\text{ind}_{S^1}^g(M, O; L)$. In particular, the theorem for $\text{ind}_{S^1}(M, V; L)^\sigma$ is obtained as a special case of that for $\text{ind}_{S^1}^g(M, O; L)$.

**Definition ($(L, \sigma)$-acyclic orbit)**

$O: (L, \sigma)$-acyclic $\iff$

$$O: \text{not a fixed point,}$$

$$H^0(O; (L, \nabla^L)|_O)^\sigma = 0$$

- $(L, \sigma)$-acyclic orbits $\supset L$-acyclic orbits
- $O: (L, \sigma)$-acyclic $\iff$ The kernel of the de Rham operator of $O$ with coefficients in $(L, \nabla^L)|_O$ does not contain $\mathbb{C}_\sigma$. 

Let $k$ be a positive integer.

$$(M, \omega) := (\mathbb{CP}^1, k\omega_{FS}) = \left( S^3_k, \frac{\sqrt{-1}}{2\pi} \sum_{j=0}^{1} dz_j \wedge d\bar{z}_j \right) / (z_0, z_1) \sim (h z_0, h z_1) \ (h \in S^1)$$

$$(L, \nabla^L) := (H^\otimes k, \nabla^L) = \left( S^3_k \times \mathbb{C}, d + \frac{1}{2} \sum_{j=0}^{1} z_j d\bar{z}_j - \bar{z}_j dz_j \right) / (z_0, z_1, v) \sim (h z_0, h z_1, h^k v)$$

where $S^3_k := \{ z = (z_0, z_1) \in \mathbb{C}^2 : \| z \|^2 = k \}$.

Take and fix an integer $m$ with $0 < m < k$. Define a $S^1$-action on $L$ and $M$ by

$$S^1 \curvearrowright L: t[z_0 : z_1, v] := [z_0 : t z_1, t^m v].$$

Non $(L, 0)$-acyclic orbits

$$\mathcal{O}_0 = \{ [z_0 : 0] \}, \quad \mathcal{O}_k = \{ [0 : z_1] \}, \quad \mathcal{O}_m = \{ [z_0 : z_1] \in M : |z_1|^2 = m \}$$
Theorem 2 (Fujita-Furuta-Y.)

Let \((L, \nabla^L) \to (M, \omega)\) be as above. Let \(O \subset M\) be an \(S^1\)-invariant open set which contains only \((L, \sigma)\)-acyclic orbits and whose complement \(M \setminus O\) is compact. For these data, there exists an integer \(\text{ind}_{S^1}^\sigma (M, O; L) \in \mathbb{Z}\) satisfying the same properties as in Theorem 1.

- By replacing the \(L\)-acyclic condition by \((L, \sigma)\)-acyclic condition in the construction of \(\text{ind}_{S^1}^\sigma (M, V; L)\) and define

  \[
  \text{"ind}_{S^1}^\sigma (M, O; L) := (\ker D|_{\wedge^0, \text{even}} \cap L^2)^\sigma - (\ker D|_{\wedge^0, \text{odd}} \cap L^2)^\sigma \text{" } \forall t \gg 0.
  \]

- Since \(L\)-acyclic orbits are \((L, \sigma)\)-acyclic we can take \(V\) in Theorem 1 as \(O\), and we have

  \[
  \text{ind}_{S^1} (M, V; L)^\sigma = \text{ind}_{S^1}^\sigma (M, V; L).
  \]
1. Equivariant local index

2. Another version

3. A special case
L-acyclic orbit, \((L, \sigma)\)-acyclic orbit, and moment map

\((M, \omega)\) symplectic manifold with an effective \(S^1\)-action

\((L, \nabla^L)\) \(S^1\)-equivariant prequantization line bundle

- The moment map \(\mu: M \to \mathfrak{t}^*\) \((\mathfrak{t} := \text{Lie}(S^1))\) is defined by the following Kostant formula

\[
\mathcal{L}_{X_\xi} s = \nabla_{X_\xi} s + 2\pi \sqrt{-1} \langle \mu, \xi \rangle s
\]

for \(\forall \xi \in \mathfrak{t}\) and \(\forall s \in \Gamma(L)\), where \(X_\xi\) is the infinitesimal action of \(\xi\).

Lemma

Let \(\mathcal{O}\) is an orbit.

1. If \(H^0(\mathcal{O}; (L, \nabla^L)|_\mathcal{O}) \neq 0\), then, \(\mathcal{O} \subset \mu^{-1}(t^*_\mathbb{Z})\). In particular, fixed points are contained in \(\mu^{-1}(t^*_\mathbb{Z})\).
2. If \(H^0(\mathcal{O}; (L, \nabla^L)|_\mathcal{O})^\sigma \neq 0\), then, \(\mathcal{O} \subset \mu^{-1}(\sigma)\).
Suppose $\mu$ is proper and the cardinality of $\mu(M) \cap t^*_Z$ is finite. Then, we have the following corollary of Theorem 1.

**Corollary**

Let $\{V_\gamma\}_{\gamma \in \mu(M) \cap t^*_Z}$ be mutually disjoint sufficiently small $S^1$-invariant neighborhoods of $\mu^{-1}(\gamma)$'s and put $V := M \setminus \mu^{-1}(t^*_Z)$. Then,

$$\text{ind}_{S^1}(M, V; L) = \bigoplus_{\gamma \in \mu(M) \cap t^*_Z} \text{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) \in R(S^1).$$
We can determine $\text{ind}_{S^1}(V_{\gamma}, V_{\gamma} \cap V; L|_{V_{\gamma}})$ for some $\gamma$. To do so we remember the Riemann-Roch numbers for symplectic quotients.

Suppose $\gamma \in t^*$ is a regular value of $\mu$. Then, a new symplectic manifold $(M_{\gamma}, \omega_{\gamma})$ with prequantization line bundle $(L_{\gamma}, \nabla^{L_{\gamma}})$ is obtained by

$$(L_{\gamma}, \nabla^{L_{\gamma}}) := \left( (L, \nabla^{L}) \otimes \mathbb{C}_{\gamma}|_{\mu_{\gamma}^{-1}(0)} \right) / S^1$$

$$\downarrow$$

$$(M_{\gamma}, \omega_{\gamma}) := \left( \mu_{\gamma}^{-1}(0), \omega|_{\mu_{\gamma}^{-1}(0)} \right) / S^1,$$

where $\mu_{\gamma} := \mu - \gamma$. The Riemann-Roch number is defined to be

$$RR(M_{\gamma}, \omega_{\gamma}) := \int_{M_{\gamma}} e^{\omega_{\gamma}} Td(M_{\gamma}).$$

**Theorem**

*Let $\gamma \in \mu(M) \cap t^*_\mathbb{Z}$ be a regular value of $\mu$. Then,*

$$\text{ind}_{S^1}(V_{\gamma}, V_{\gamma} \cap V; L|_{V_{\gamma}})^\sigma = \begin{cases} 0 & \text{if } \sigma \neq \gamma \\ RR(M_{\gamma}, \omega_{\gamma}) & \text{if } \sigma = \gamma. \end{cases}$$
Since an $L$-acyclic orbit is $(L, \sigma)$-acyclic,

$$\text{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})^\sigma = \text{ind}_{S^1}^\sigma(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}).$$

The right hand side is determined by NON $(L, \sigma)$-acyclic orbits, i.e. fixed points and orbits $O$ with $H^0(O; (L, \nabla^L)|_{O})^\sigma \neq 0$.

If $\sigma \neq \gamma$, $V_\gamma$ does not contain orbits with $H^0(O; (L, \nabla^L)|_{O})^\sigma \neq 0$. A complicated computation shows there is no contribution from fixed points.

Suppose $\sigma = \gamma$. Since $\gamma$ is a regular value of $\mu$ $V_\gamma$ contains NO fixed points. A direct computation using a local model and the product formula shows

$$\text{ind}_{S^1}^\gamma(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) = RR(M_\gamma, \omega_\gamma).$$
Thank you for your attention!