# Equivariant local index

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# Purpose

Joint work with Hajime Fujita and Mikio Furuta:

1. H. Fujita, M. Furuta, and T. Y, *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1-26.
2. ______, *Torus fibrations and localization of index II*, arXiv:0910.0358.
3. ______, *Torus fibrations and localization of index III*, arXiv:1008.5007.

In these joint works we are developing an index theory for a Dirac-type operator on a possibly noncompact Riemannian manifold.

## Purpose of this talk

1. *To explain our theory in a simple symplectic case.*

# $L$-acyclic orbit

$(M, \omega)$ symplectic manifold with Hamiltonian $S^1$-action
$(L, \nabla^L)$ $S^1$-equivariant prequantization line bundle

- Each orbit $\mathcal{O}$ is isotropic, namely, $\omega|_{\mathcal{O}} \equiv 0$.
  $\Rightarrow (L, \nabla^L)|_{\mathcal{O}}$ is a flat line bundle. $(\because \frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega)$

**Definition ($L$-acyclic orbit)**

$$\mathcal{O} : L\text{-}acyclic \overset{\text{def}}{\Leftrightarrow} H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0$$

- An orbit consisting of a fixed point is not $L$-acyclic.
- $\mathcal{O}$: $L$-acyclic $\Leftrightarrow H^\bullet(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) = 0$ $(\because \mathcal{O}$ is a circle$)$
  $\Leftrightarrow$ The de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$ has zero kernel. ($\because$ the Hodge theory)

# Example: (Non) $L$-acyclic orbits in $\mathbb{C}P^1$

Let $k$ be a positive integer.

$$(M,\omega) := (\mathbb{C}P^1, k\omega_{FS}) = \left( S^3_k, \frac{\sqrt{-1}}{2\pi}\sum_{j=0}^{1} dz_j \wedge d\bar{z}_j \right) /_{(z_0,z_1)\sim(hz_0,hz_1)\ \ (h\in S^1)}$$

$$(L,\nabla^L) := (H^{\otimes k}, \nabla^L) = \left( S^3_k \times \mathbb{C}, d + \frac{1}{2}\sum_{j=0}^{1} z_j d\bar{z}_j - \bar{z}_j dz_j \right) /_{(z_0,z_1,v)\sim(hz_0,hz_1,h^k v)}$$

where $S^3_k := \{z = (z_0, z_1) \in \mathbb{C}^2 \colon \|z\|^2 = k\}$.

Take and fix an integer $m$ with $0 < m < k$. Define a $S^1$-action on $L$ and $M$ by

$$S^1 \curvearrowright L \colon t[z_0 : z_1, v] := [z_0 : tz_1, t^m v].$$

**Non $L$-acyclic orbits**

$$\mathcal{O}_i := \{[z_0 : z_1] \in M \colon |z_1|^2 = i\}\ (i = 0, 1, \ldots, k)$$

# Equivariant local index

**Theorem 1 (Fujita-Furuta-Y.)**

*Let $(L, \nabla^L) \to (M, \omega)$ be as above. Let $V \subset M$ be an $S^1$-invariant open set which contains only L-acyclic orbits and whose complement $M \setminus V$ is compact. For these data, there exists an element $\mathrm{ind}_{S^1}(M, V; L) \in R(S^1)$ satisfying the following properties:*

1. $\mathrm{ind}_{S^1}(M, V; L)$ *is deformation invariant.*
2. *For a closed $M$,* $\mathrm{ind}_{S^1}(M, V; L) = \mathrm{ind}_{S^1} D$.
3. *If $M'$ is an $S^1$-invariant open neighborhood of $M \setminus V$, then*

$$\mathrm{ind}_{S^1}(M, V; L) = \mathrm{ind}_{S^1}(M', V \cap M'; L|_{M'}). \quad \textit{(excision property)}$$

4. *Gluing formula*
5. *Product formula*

*We call* $\mathrm{ind}_{S^1}(M, V; L)$ *an equivariant local index.*

# Definition of $\mathrm{ind}_{S^1}(M, V; L)$

First consider the case where $M$ has a cylindrical end $V = N \times (0, \infty)$ and all the data are translation invariant on the end.

1. For $t \geq 0$ consider the following perturbation of the Dirac operator $D$

$$D_t := D + t\rho D_{\text{fiber}},$$

   where $D_{\text{fiber}}$ is the de Rham operator of $V$ along orbits, i.e.

   - $D_{\text{fiber}}$ contains only derivatives along orbits
   - $D_{\text{fiber}}|_{\mathcal{O}}$ is the de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$

$$\Rightarrow \ \ker D_{\text{fiber}}|_{\mathcal{O}} = 0 \text{ for } \forall \mathcal{O} \subset V \ (\because \ V \text{ consists of } L\text{-acyclic orbits})$$

2. By using "$\ker D_{\text{fiber}}|_{\mathcal{O}} = 0$ for $\forall \mathcal{O} \subset V$", one can show
   1. $\dim \ker D_t \cap L^2 < +\infty$ for a sufficiently large $t \gg 0$.
   2. $\ker D_t|_{\wedge^{0,even}T^*M \otimes L} \cap L^2 - \ker D_t|_{\wedge^{0,odd}T^*M \otimes L} \cap L^2$ is independent of a sufficiently large $t \gg 0$.

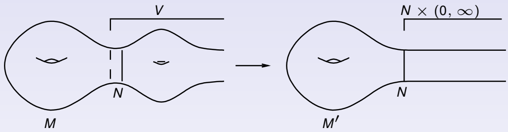
**Definition (Equivariant local index)**

$$\mathrm{ind}_{S^1}(M, V; L) := \ker D_t|_{\wedge^{0,even}} \cap L^2 - \ker D_t|_{\wedge^{0,odd}} \cap L^2 \text{ for } \forall t \gg 0.$$

# Definition of $\mathrm{ind}_{S^1}(M, V; L)$

In general case,

1. Deform $V$ cylindrically so that all the data are translation invariant, and come down to the cylindrical case.




2. Check the localization is independent of a choice of a cut locus $N$.

1. Equivariant local index
2. Another version
3. A special case

## $(L, \sigma)$-acyclic orbit

Let $\mathfrak{t}^*_{\mathbb{Z}}$ be the weight lattice of $S^1$. For $\sigma \in \mathfrak{t}^*_{\mathbb{Z}}$ and $U \in R(S^1)$ we define by

$$U^\sigma := \dim \mathrm{Hom}_{S^1}(\mathbb{C}_\sigma, U).$$

In Theorem 1, by taking the multiplicity of the irreducible representation with weight $\sigma$, we obtain a corresponding theorem for $\mathrm{ind}_{S^1}(M, V; L)^\sigma$. Here we introduce the following notion of "$(L, \sigma)$-acyclicity" which is milder condition than that of $L$-acyclicity, and we obtain a version of a local index, which is denoted by $\mathrm{ind}^\sigma_{S^1}(M, O; L)$. In particular, the theorem for $\mathrm{ind}_{S^1}(M, V; L)^\sigma$ is obtained as a special case of that for $\mathrm{ind}^\sigma_{S^1}(M, O; L)$.

**Definition ($(L, \sigma)$-acyclic orbit)**

$$\mathcal{O} : (L, \sigma)\text{-}acyclic \overset{\text{def}}{\Leftrightarrow} \begin{cases} \mathcal{O} : \text{not a fixed point,} \\ H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}})^\sigma = 0 \end{cases}$$

- $(L, \sigma)$-acyclic orbits $\supset$ $L$-acyclic orbits
- $\mathcal{O}$: $(L, \sigma)$-acyclic $\Leftrightarrow$ The kernel of the de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$ does not contain $\mathbb{C}_\sigma$.

## Example: (Non) $(L, 0)$-acyclic orbits in $\mathbb{C}P^1$

Let $k$ be a positive integer.

$$(M, \omega) := (\mathbb{C}P^1, k\omega_{FS}) = \left( S_k^3, \frac{\sqrt{-1}}{2\pi} \sum_{j=0}^{1} dz_j \wedge d\bar{z}_j \right) \Big/_{(z_0, z_1) \sim (hz_0, hz_1) \quad (h \in S^1)}$$

$$(L, \nabla^L) := (H^{\otimes k}, \nabla^L) = \left( S_k^3 \times \mathbb{C}, d + \frac{1}{2} \sum_{j=0}^{1} z_j d\bar{z}_j - \bar{z}_j dz_j \right) \Big/_{(z_0, z_1, v) \sim (hz_0, hz_1, h^k v)}$$

where $S_k^3 := \{z = (z_0, z_1) \in \mathbb{C}^2 \colon \|z\|^2 = k\}$.

Take and fix an integer $m$ with $0 < m < k$. Define a $S^1$-action on $L$ and $M$ by

$$S^1 \curvearrowright L \colon t[z_0 : z_1, v] := [z_0 : tz_1, t^m v].$$

### Non $(L, 0)$-acyclic orbits

$$\mathcal{O}_0 = \{[z_0 : 0]\}, \ \mathcal{O}_k = \{[0 : z_1]\}, \mathcal{O}_m = \{[z_0 : z_1] \in M \colon |z_1|^2 = m\}$$

# Localization of $\operatorname{ind}^{\sigma}_{S^1}(M, O; L)$

**Theorem 2 (Fujita-Furuta-Y.)**

*Let $(L, \nabla^L) \to (M, \omega)$ be as above. Let $O \subset M$ be an $S^1$-invariant open set which contains only $(L, \sigma)$-acyclic orbits and whose complement $M \setminus O$ is compact. For these data, there exists an integer $\operatorname{ind}^{\sigma}_{S^1}(M, O; L) \in \mathbb{Z}$ satisfying the same properties as in Theorem 1.*

- By replacing the $L$-acyclic condition by $(L, \sigma)$-acyclic condition in the construction of $\operatorname{ind}_{S^1}(M, V; L)$ and define

$$\text{“}\operatorname{ind}^{\sigma}_{S^1}(M, O; L) := (\ker D|_{\wedge^{0,even}} \cap L^2)^{\sigma} - (\ker D|_{\wedge^{0,odd}} \cap L^2)^{\sigma}\text{”} \quad \forall t \gg 0.$$

- Since $L$-acyclic orbits are $(L, \sigma)$-acyclic we can take V in Theorem 1 as O, and we have

$$\operatorname{ind}_{S^1}(M, V; L)^{\sigma} = \operatorname{ind}^{\sigma}_{S^1}(M, V; L).$$

# $L$-acyclic orbit, $(L, \sigma)$-acyclic orbit, and moment map

$(M, \omega)$ symplectic manifold with an effective $S^1$-action

$(L, \nabla^L)$ $S^1$-equivariant prequantization line bundle

- The moment map $\mu \colon M \to \mathfrak{t}^*$ ($\mathfrak{t} := \mathrm{Lie}(S^1)$) is defined by the following Kostant formula
$$\mathcal{L}_{X_\xi} s = \nabla_{X_\xi} s + 2\pi\sqrt{-1}\,\langle \mu, \xi \rangle\, s$$
for $\forall \xi \in \mathfrak{t}$ and $\forall s \in \Gamma(L)$, where $X_\xi$ is the infinitesimal action of $\xi$.

**Lemma**

*Let $\mathcal{O}$ is an orbit.*
*(1) If $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}) \neq 0$, then, $\mathcal{O} \subset \mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$. In particular, fixed points are contained in $\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$.*
*(2) If $H^0(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}})^\sigma \neq 0$, then, $\mathcal{O} \subset \mu^{-1}(\sigma)$.*

## Localization formula for $\mathrm{ind}_{S^1}(M, V; L)$

Suppose $\mu$ is proper and the cardinality of $\mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$ is finite. Then, we have the following corollary of Theorem 1.

### Corollary

*Let $\{V_\gamma\}_{\gamma \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}}$ be mutually disjoint sufficiently small $S^1$-invariant neighborhoods of $\mu^{-1}(\gamma)$'s and put $V := M \setminus \mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$. Then,*

$$\mathrm{ind}_{S^1}(M, V; L) = \bigoplus_{\gamma \in \mu(M) \cap \mathfrak{t}_{\mathbb{Z}^*}} \mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) \in R(S^1).$$

## $\mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})$

We can determine $\mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})$ for some $\gamma$. To do so we remember the Riemann-Roch numbers for symplectc quotients.

Suppose $\gamma \in \mathfrak{t}^*$ is a regular value of $\mu$. Then, a new symplectic manifold $(M_\gamma, \omega_\gamma)$ with prequantization line bundle $(L_\gamma, \nabla^{L_\gamma})$ is obtained by

$$(L_\gamma, \nabla^{L_\gamma}) := \left((L, \nabla^L) \otimes \mathbb{C}_\gamma|_{\mu_\gamma^{-1}(0)}\right)/S^1$$
$$\downarrow$$
$$(M_\gamma, \omega_\gamma) := \left(\mu_\gamma^{-1}(0), \omega|_{\mu_\gamma^{-1}(0)}\right)/S^1,$$

where $\mu_\gamma := \mu - \gamma$. The Riemann-Roch number is defined to be

$$RR(M_\gamma, \omega_\gamma) := \int_{M_\gamma} e^{\omega_\gamma} Td(M_\gamma).$$

**Theorem**

*Let $\gamma \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$ be a regular value of $\mu$. Then,*

$$\mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})^\sigma = \begin{cases} 0 & \text{if } \sigma \neq \gamma \\ RR(M_\gamma, \omega_\gamma) & \text{if } \sigma = \gamma. \end{cases}$$

## Sketch of proof

- Since an $L$-acyclic orbit is $(L,\sigma)$-acyclic,

$$\mathrm{ind}_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})^\sigma = \mathrm{ind}^\sigma_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}).$$

  The right hand side is determined by NON $(L,\sigma)$-acyclic orbits, i.e. fixed points and orbits $\mathcal{O}$ with $H^0\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^\sigma \neq 0$.

- If $\sigma \neq \gamma$, $V_\gamma$ does not contain orbits with $H^0\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^\sigma \neq 0$. A complicated computation shows there is no contribution from fixed points.

- Suppose $\sigma = \gamma$. Since $\gamma$ is a regular value of $\mu$ $V_\gamma$ contains NO fixed points. A direct computation using a local model and the product formula shows

$$\mathrm{ind}^\gamma_{S^1}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) = RR(M_\gamma, \omega_\gamma).$$

Thank you for your attention!