PROBLEMS ON TORIC TOPOLOGY

(Ivan Arzhantsev) In the talk of Hirokaki Ishida it was proved that a torus manifold with an $(S^1)^n$-invariant complex structure is equivariantly biholomorphic to a toric manifold (a joint work with Yael Karshon). After the talk I asked whether every action of a compact torus $T = (S^1)^k$ on a compact complex manifold M of complex dimension n by holomorphic automorphisms may be extended to an action of the complexification $T_{\mathbb{C}}$ of the torus T provided M contains a T-fixed point. The result of Ishida and Karshon shows that this is the case if $n = k$.

Remarks. It was observed by Taras Panov that the restriction that M contains a T-fixed point is essential: otherwise we may consider the torus T^2 equipped with a complex structure as an elliptic curve and the transitive T^2-action on T^2 by left translations. This action is holomorphic (and even algebraic), but it can not be extended to an action of the complex 2-torus by dimension reason.

(Victor Buchstaber) Let $s(P)$ be the maximal dimension of a subtorus acting freely on $\mathcal{Z}_P$ (the Buchstaber number). Express $s(P \times Q)$ in terms of $s(P)$ and $s(Q)$.

Remarks. A guess is $s(P) + s(Q)$, but this might be not true in general. This is known to be wrong for simplicial complexes, as observed by Ayzenberg.

(Shintaro Kuroki) Two questions about topological classification of aspherical small covers (or aspherical real moment angle manifolds):

Q1. Are two aspherical small covers homeomorphic if they have the same homotopy type?

Remark for Q1: An aspherical small cover is a small cover with contractible universal covering space. This question is originally asked by Masuda for all small covers ("Cohomological non-rigidity of generalized real Bott manifolds of height 2", Proceedings of the Steklov Institute of Mathematics, 2010, Vol. 268, 242-247). In the case of all aspherical manifolds, this question is known as the Borel conjecture (see e.g. W. Lück "K- and L-theory of group rings" Proceedings of the 26-th ICM in Hyderabad 2010, Volume II, Section 6 Topology, 1071–1098). So, this question may be regarded as the Borel conjecture for small covers.

One of the merits to restrict our attention to aspherical small covers is the following Davis-Januszkiewicz-Scott's theorem ("Nonpositive curvature of blow-ups",

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Selecta Math. 4 (1998), no. 4, 491-547): the orbit polytopes of aspherical small covers are flag. Here, a simple convex polytope P is flag if the dual of its boundary K (simplicial complex) satisfies that a subset J of the vertex set of K spans a simplex in K whenever any two vertices in J are joined by a 1-simplex in K. For example, the n-dimensional cube ($n \geq 1$) is flag but the m-dimensional simplex ($m \geq 2$) is not flag. Some partial affirmative answers are also known for Q1.

Using the well known Mostow's rigidity, if an aspherical small cover has a hyperbolic structure, then the answer to Q1 is affirmative. Davis-Januszkiewicz proved that the 3-dimensional small cover admits a hyperbolic structure if and only if there are no 3-gon and 4-gon facets in its orbit polytope P ("Convex polytopes, Coxeter orbifolds and torus actions", Duke Math. J. 62 (1991), no.2, 417-451). Therefore, for 3-dimensional small covers with such orbit polytopes, the answer to Q1 is also affirmative. According to Kuroki-Masuda-Yu ("Small Cover, infra-solvmanifold and curvature" arXiv1111.2174), if the fundamental group of an n-dimensional aspherical small cover is virtually solvable, then its orbit polytope is the n-dimensional cube. Namely, this manifold is a real Bott manifold. By Masuda-Kamishima's Z_2-cohomological rigidity for real Bott manifolds, their homeomorphism (diffeomorphism) types are determined by their Z_2 cohomology. Hence, for aspherical small covers with virtually solvable fundamental groups, the answer to Q1 is affirmative.

Q2. Is the Z_2 cohomological rigidity true for aspherical small covers?

Remark for Q2: For the Z_2 cohomological rigidity of all small covers, there exist Masuda's counter examples ("Cohomological non-rigidity of generalized real Bott manifolds of height 2", Proceedings of the Steklov Institute of Mathematics, 2010, Vol. 268, 242-247). His counter examples are the 2-stage generalized Bott manifolds, i.e., their orbit polytopes are the product of two simplices (not cube). Therefore, their orbit polytopes are not flag, i.e., they are not aspherical small covers. So his counter examples are not the case.

Using the same argument mentioned in the above Remark, the answer to Q2 is affirmative in the case when the fundamental group is virtually solvable.

For the real moment angle manifolds, we may ask the same questions. For example, if an aspherical real moment angle manifold has the virtually solvable fundamental group, then its orbit polytope is the cube as well as the case of small covers; therefore, the manifold is torus (see Kuroki-Masuda-Yu "Small Cover, infra-solvmanifold and curvature" arXiv1111.2174). So the Z_2 cohomological rigidity (and homotopical rigidity) also holds in this case.

Of course, if the answer to Q2 is affirmative, then the answer to Q1 is also affirmative.

(Zhi Lü) Q1: Let $\pi : M^n \longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Is there a quasi-toric manifold $\widetilde{\pi} : M^{2n} \longrightarrow P^n$ over P^n such that M^n is the fixed point set of the *conjugation involution* on M^{2n}?

Remarks. The question is equivalent to asking the existence of the lifting $\widetilde{\lambda}$ of $\lambda : \mathcal{F}(P) \longrightarrow (\mathbb{Z}_2)^n$

$$\begin{array}{ccc} & & (\mathbb{Z})^n \\ & \overset{\widetilde{\lambda}}{\nearrow} & \big\downarrow \text{mod } 2 \\ \mathcal{F}(P) & \xrightarrow[\lambda]{} & (\mathbb{Z}_2)^n \end{array}$$

where $\lambda : \mathcal{F}(P) \longrightarrow (\mathbb{Z}_2)^n$ is the characteristic function determined by $\pi : M^n \longrightarrow P^n$, and $\mathcal{F}(P)$ is the set of all facets of P^n. When $n \leq 3$, the answer is affirmative.

Q2: Let P be a simple convex polytope, and let $s(P)$ (resp. $s_{\mathbb{R}}(P)$) be the Buchstaber invariant (resp. real Buchstaber invariant) of P. Is it true that $s(P) = s_{\mathbb{R}}(P)$?

Remarks. This is true if $\dim P \leq 3$. In addition, it has been known that $s(P) \leq s_{\mathbb{R}}(P)$. However, in the more general case to simplicial complexes, as shown by Anton Ayzenberg recently, there exists an example of the simplicial complex K of dimension 3 such that $s(K) \neq s_{\mathbb{R}}(K)$.

(Mikiya Masuda) Here a toric manifold is a compact smooth toric variety.

(1) Find a toric manifold which is not a quasitoric manifold.

Remarks. Any toric manifold of complex dimension ≤ 3 is a quasitoric manifold. Y. Civan claimed that there is a toric manifold of complex dimension 4 which is not a quasitoric manifold but his argument is unclear unfortunately (see Example 3.2 in arXiv:math.AT/0306029). However, there is a topological toric manifold (introduced by Ishida-Fukukawa-Masuda) which is not a quasitoric manifold (see Section 9 in arXiv:1012.1786).

(2) (Cancellation property) Let X, Y and Z be toric manifolds. It is true that if $X \times Z$ and $Y \times Z$ are diffeomorphic, then X and Y are diffeomorphic?

More generally we may ask

(Unique decomposition property) We say that a toric manifold is *indecomposable* if it is not diffeomorphic to a product of two toric manifolds of positive dimension. Let X_i $(i = 1, 2, \ldots, k)$ and Y_j $(j = 1, 2, \ldots, \ell)$ be indecomposable toric manifolds. Is it true that if $\prod_{i=1}^{k} X_i$ and $\prod_{j=1}^{\ell} Y_j$ are diffeomorphic, then $k = \ell$ and there is a permutation σ on $\{1, 2, \ldots, k\}$ such that $X_i = Y_{\sigma(i)}$ for $i = 1, 2, \ldots, k$.

Remaks. One may replace 'diffeomorohism' above by 'homemomorphism'. One can also ask the problems above for quasitoric manifolds, topological toric manifolds, real toric manifolds, small covers and real topological toric manifolds. It is known that the cancellation property and the unique decomposition property hold for real Bott manifolds (Choi-Masuda-Oum, arXiv:1006.4658).

(Taras Panov) Let P and Q be two simple polytopes, let $\mathcal{Z}_P$ and $\mathcal{Z}_Q$ be the corresponding moment-angle manifolds, and let $\beta^{-i,2j}$ denote the bigraded Betti numbers. Is it true that

$$\mathcal{Z}_P \cong \mathcal{Z}_Q \quad \Longleftrightarrow \quad \beta^{-i,2j}(P) = \beta^{-i,2j}(Q) \quad \text{for all } i,j?$$

Remarks. Here $\cong$ may mean 'homotopy equivalent', 'homeomorphic', or 'diffeomorphic'. Betti numbers may be taken with coefficients in any field. Of course, there is a generalisation to simplicial complexes. One may also replace the equality of Betti numbers in the right hand side by the isomorphism of bigraded cohomology rings (Tor-algebras). I am inclined to believe that the '$\Longleftarrow$' statement is false, while the '$\Longrightarrow$' is true. The latter may be referred to as the 'topological invariance' of bigraded Betti numbers. In all known examples of combinatorially different polytopes with same bigraded Betti numbers (such as vertex truncations of simplices), the moment-angle manifolds are also diffeomorphic.

(Jérôme Tambour) Structure of moment-angle complexes associated to simplicial posets. We know that every moment-angle complexes constructed from a simplicial complex which is the underlying complex of a simplicial complete fan can be endowed with a complex structure. Lu and Panov enlarged the family of moment-angle complexes to simplicial posets. The simplest new example we obtain from this construction is the sphere S^4 which is known not to have a complex structure. One can imagine the following assertion: *Let S be a simplicial poset. If the moment-angle complex $\mathcal{Z}_S$ can be endowed with a complex structure, then S is a simplicial complex.* This assertion seems to be false or very hard to prove, because one of its corollary would be the non-existence of a complex structure on the sphere S^6. So, a more reasonnable assertion would be: *Let S be a simplicial poset. If the moment-angle complex $\mathcal{Z}_S$ can be endowed with a complex structure invariant by the natural torus action, then S is a simplicial complex.*

(Li Yu) Let P and Q be two simple convex polytopes. If the moment-angle manifolds of P and Q are homeomorphic, can we derive that the real moment-angle manifolds of P and Q are also homeomorphic? Similarly, we can reverse the question by assuming that the real moment-angle manifolds of P and Q are homeomorphic and ask whether their moment-angle manifolds are homeomorphic.

Remarks. Counterexamples to this question (if exist) will imply that the study of the topology of moment-angle manifolds and real moment-angle manifolds of simple convex polytopes are independent. This is the reason why I want to ask this question.