Young diagrams and intersection numbers on toric manifolds associated with Weyl chambers

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Φ : a root system $\quad \rightsquigarrow \quad X(\Phi)$: a toric manifold

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$\rightsquigarrow$ $X(\Phi)$: the toric manifold associated with $\Delta(\Phi)$

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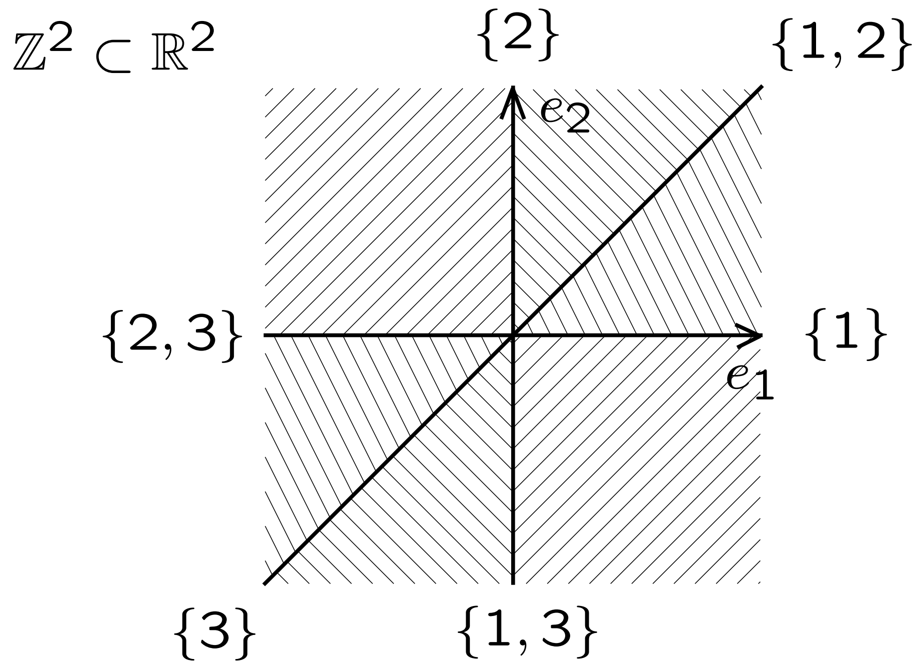

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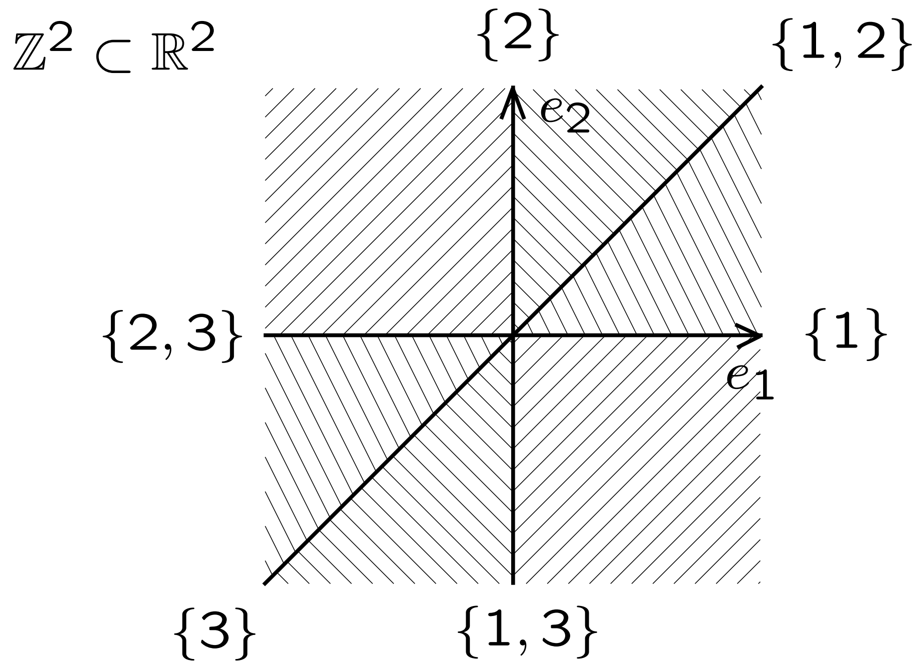

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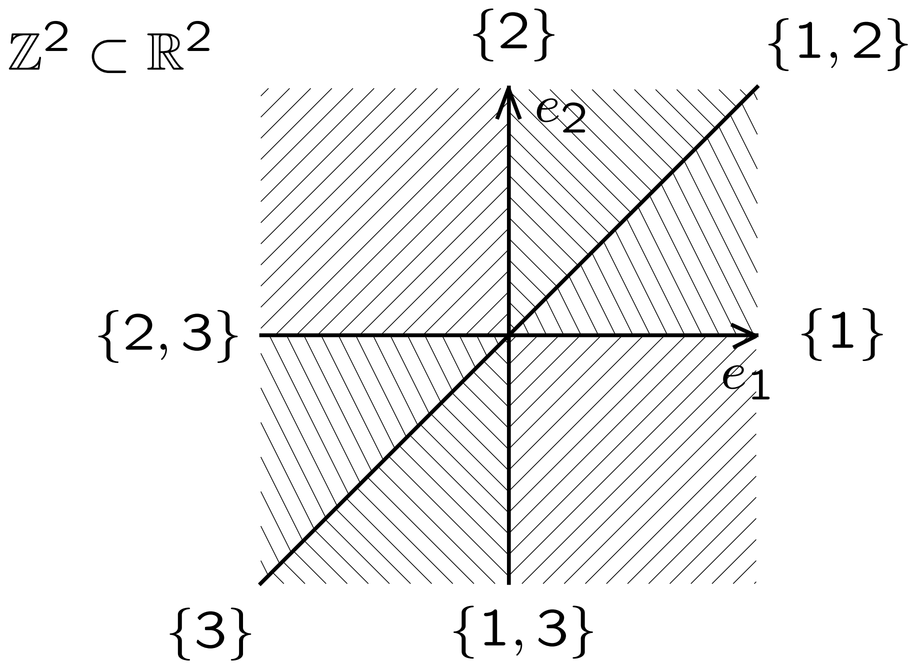

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$\rightsquigarrow$ $X(\Phi)$: the root theoretic generalization!

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$X_S \subset X(A_n)$ invariant divisor for each $\emptyset \subsetneq S \subsetneq [n+1]$

$X_{S_1} \cap \cdots \cap X_{S_n} = \emptyset$ unless $S_1 \subset \cdots \subset S_n$

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For each $u \in \mathfrak{S}_{n+1}$,

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$$[X_u][X_v] = \sum_w c_{u,v}^w [X_w], \quad c_{u,v}^w \in \mathbb{Z}$$

For example, we have

$$[X_{s_i}][X_{s_j}] = \begin{cases} [X_{s_i s_j}](= [X_{s_j s_i}]) & \text{if } |i-j| \geq 2, \\ 0 & \text{if } |i-j| = 1. \end{cases}$$

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In general,

$$c_{uv}^{w} = \sum_{w'} (\mathcal{I}^{-1})_{ww'} \int_X [w_0 X_{w_0 w'}][X_u][X_v] \qquad \left(\mathcal{I}_{uv} = \int_X [w_0 X_{w_0 u}][X_v] \right).$$

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$\rightsquigarrow$ Need a combinatorial formula for intersection numbers.

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Lemma (W-invariance)

Let $\emptyset \subsetneq S_1 \subset \cdots \subset S_n \subsetneq [n+1]$ and $\emptyset \subsetneq S'_1 \subset \cdots \subset S'_n \subsetneq [n+1]$.
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$\rightsquigarrow \quad \int_X \tau_{S_1} \cdots \tau_{S_n}$ depends only on $|S_1| \leq \cdots \leq |S_n|$!

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$\rightsquigarrow \quad \lambda := (|S_n|, \cdots, |S_1|)$: a Young diagram

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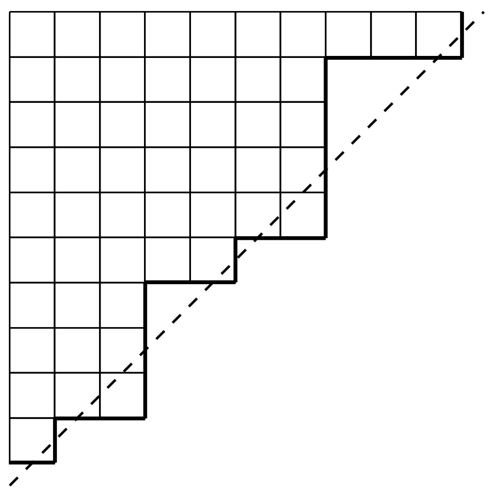

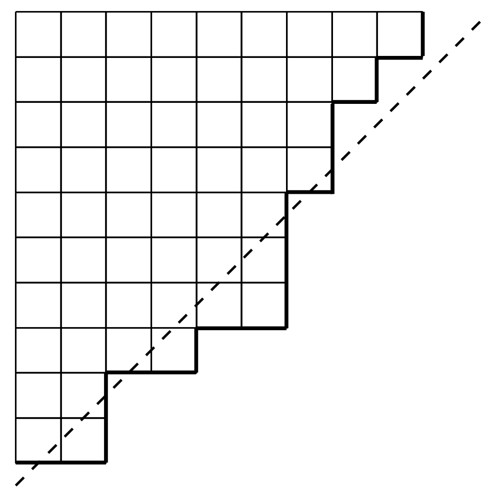

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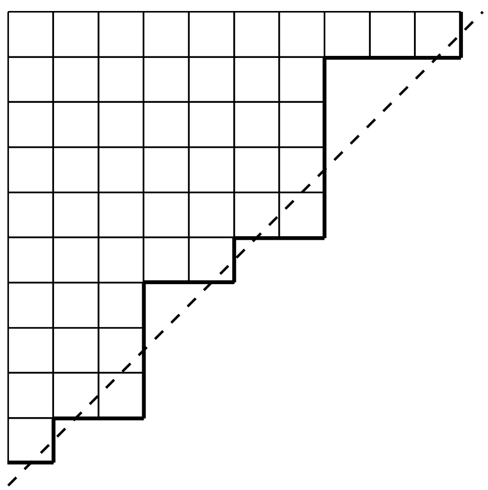

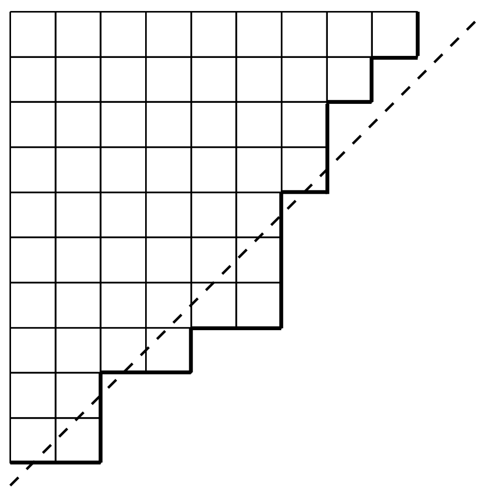

The linear relations for divisor classes :

$$\sum_{\substack{\emptyset \subsetneq S \subsetneq [n+1] \\ k \in S, l \notin S}} \tau_S - \sum_{\substack{\emptyset \subsetneq S \subsetneq [n+1] \\ k \notin S, l \in S}} \tau_S = 0 \qquad \text{for each } k, l \in [n+1].$$

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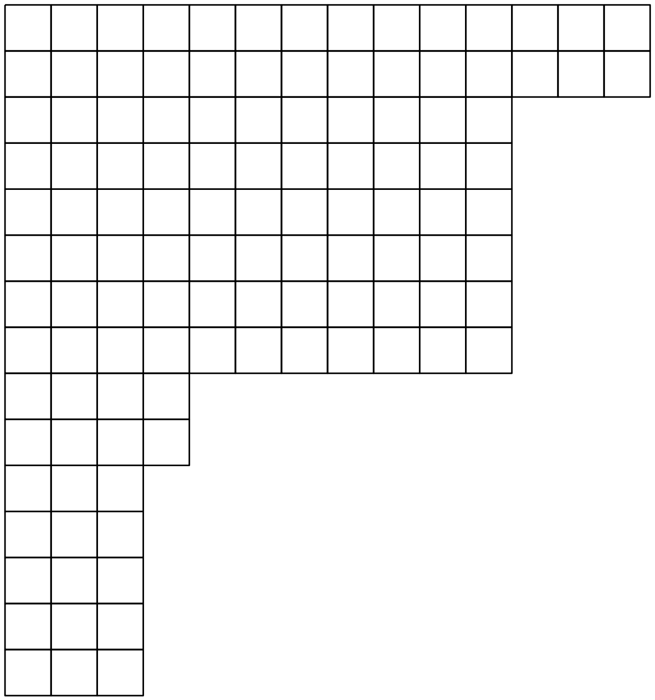

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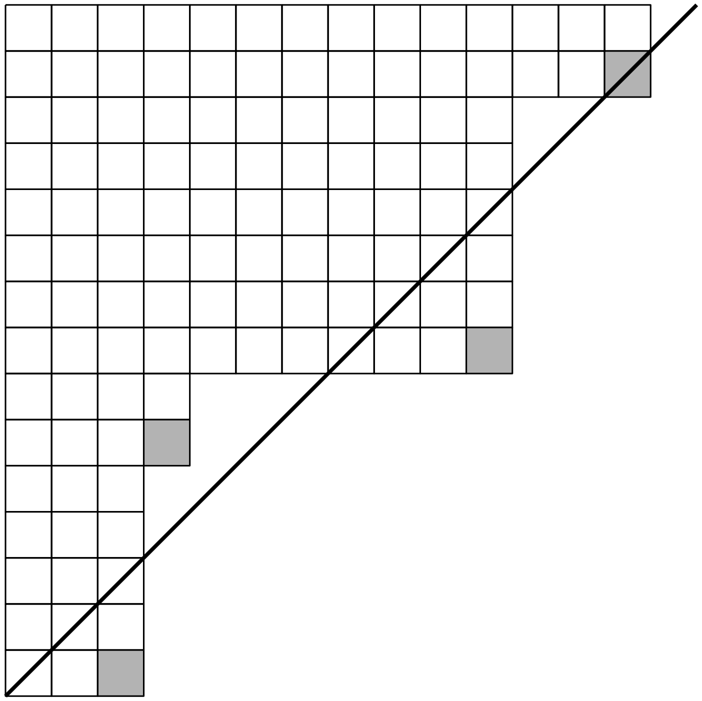

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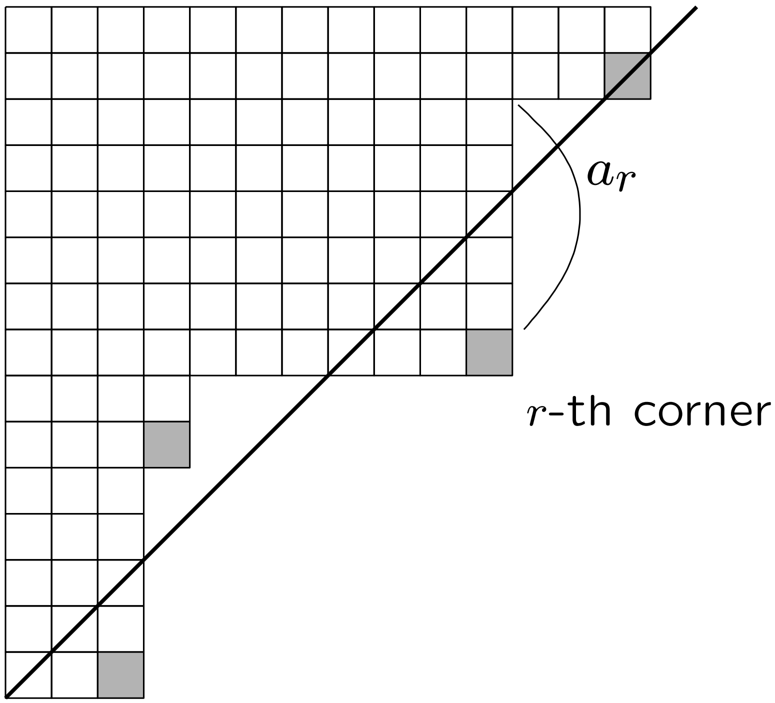

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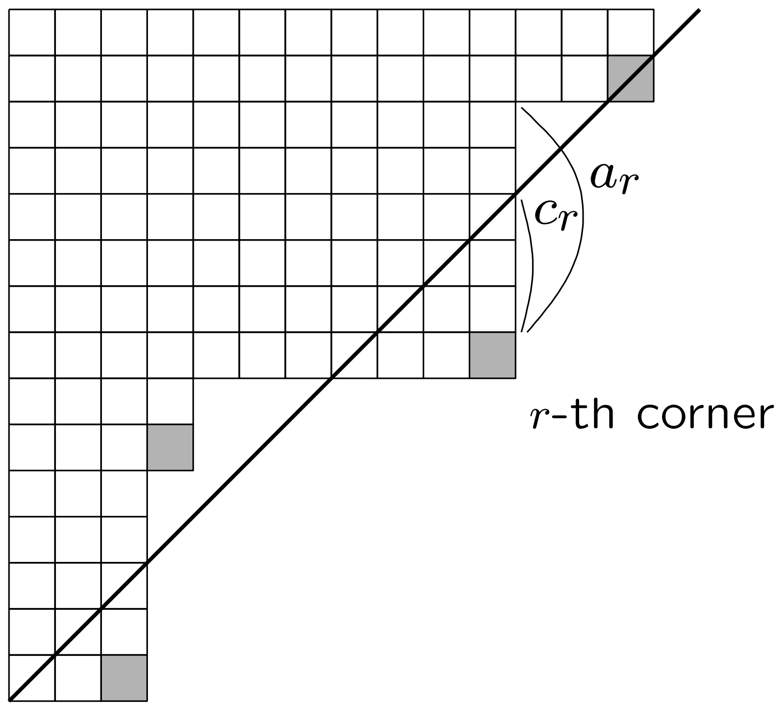

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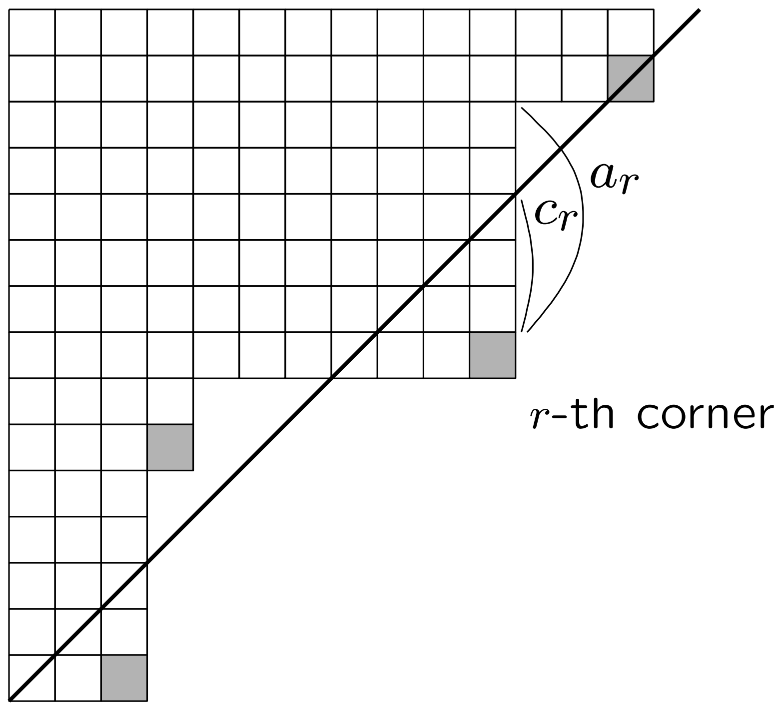

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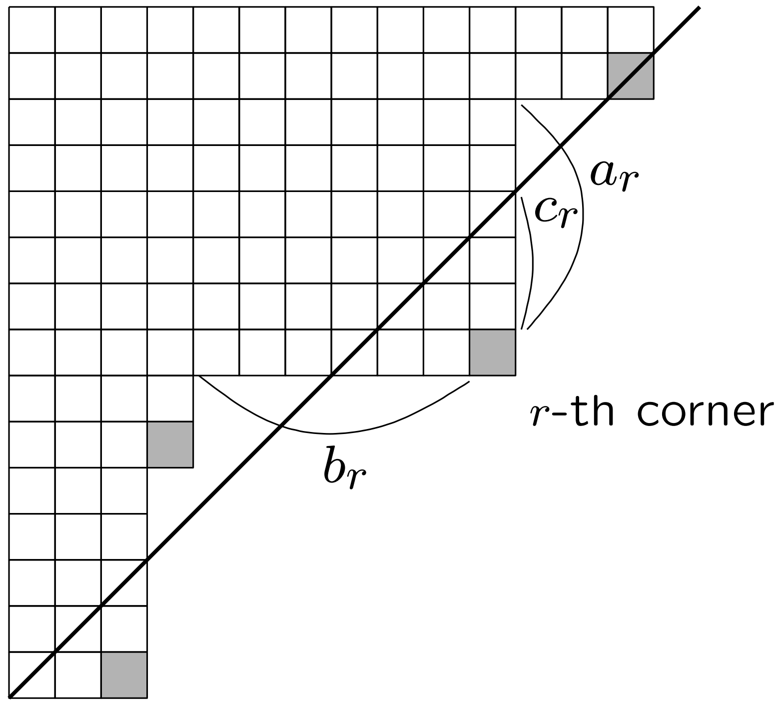

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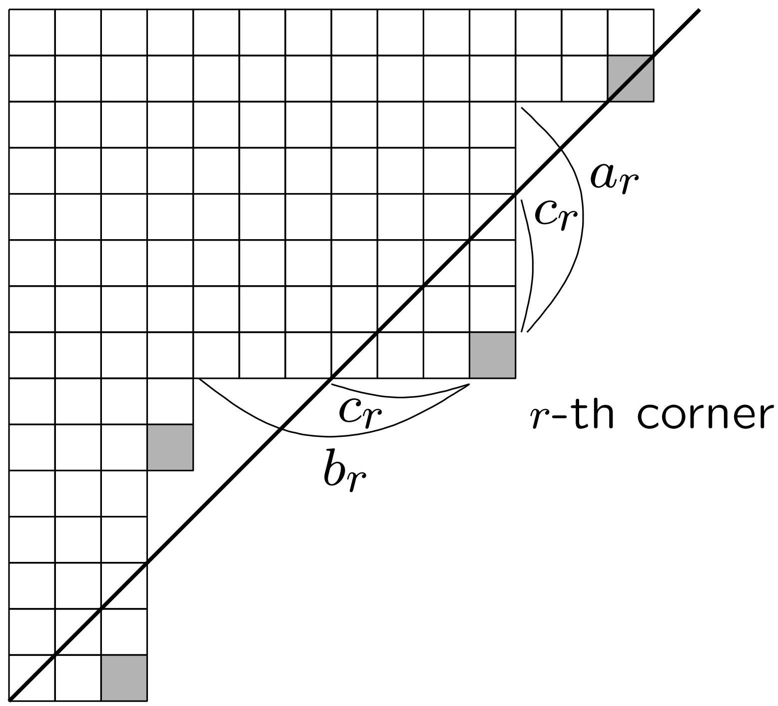

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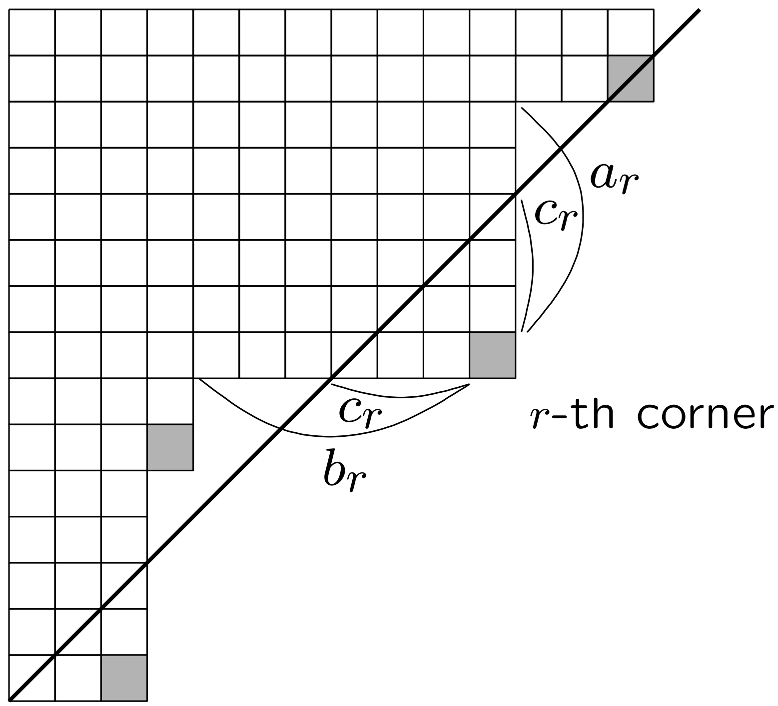

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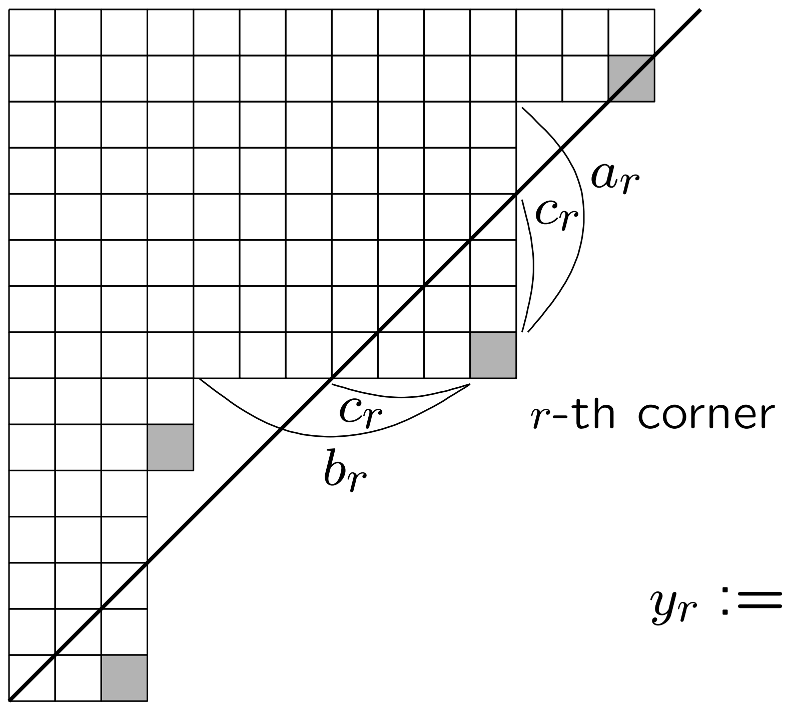

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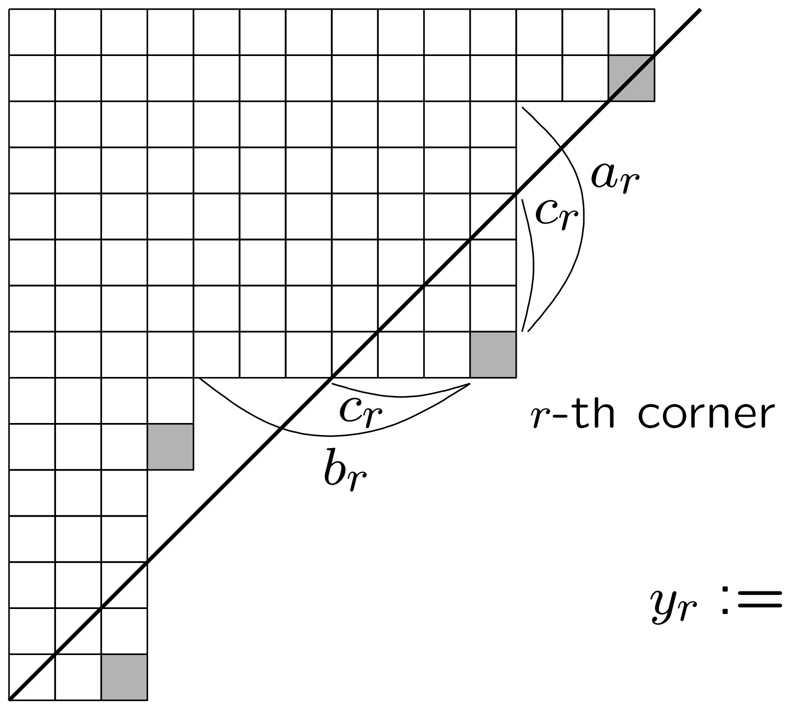

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Theorem

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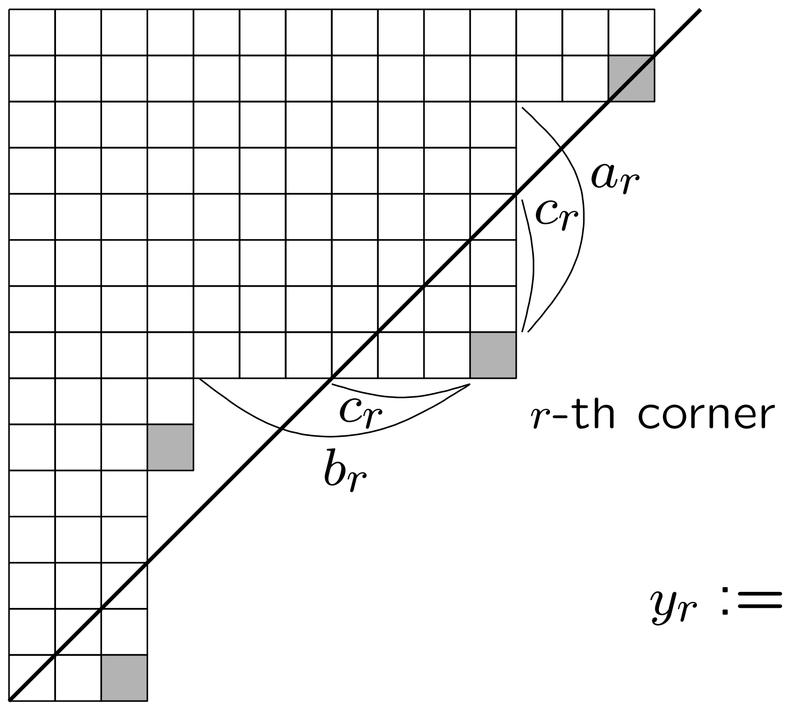
Intersection numbers for type A_n :



$$y_r := \binom{a_r}{c_r}\binom{b_r}{c_r}$$

Theorem

Let $\emptyset \subsetneq S_1 \subset \cdots \subset S_n \subsetneq [n+1]$ and $\lambda = (|S_n|, \cdots, |S_1|)$.
Then,

$$\int_X \tau_{S_1} \cdots \tau_{S_n} = (-1)^{n-s} y_1 \cdots y_s.$$

$$[X_u] = \prod_i \tau_{\{u(1),\cdots,u(i)\}} \quad \text{(running over all } i \text{ s.t. } u(i) > u(i+1))$$

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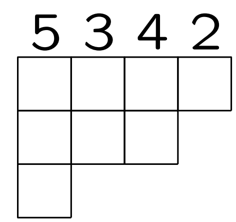
In $n = 5$, for $534216 \in \mathfrak{S}_6$,

$$[X_{534216}] = \tau_{\{5\}}\tau_{\{5,3,4\}}\tau_{\{5,3,4,2\}} =$$



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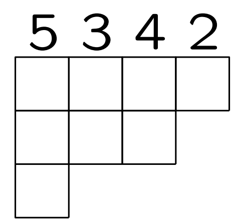

$\rightsquigarrow$ A combinatorial formula for $\int_X [w_0 X_{w_0 w}][X_u][X_v]$!

For example, for $n = 4$,

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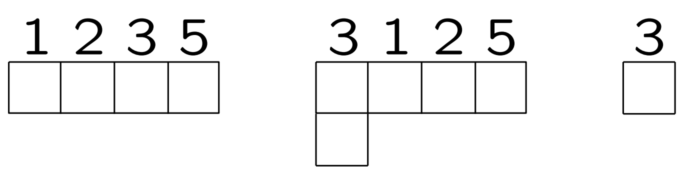

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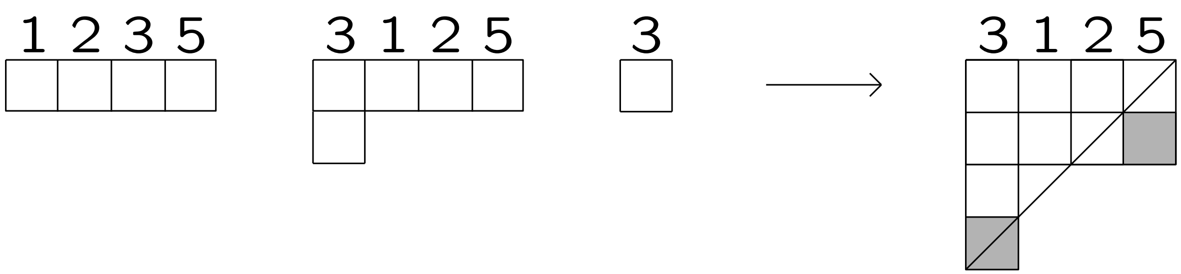
$$= \int_X (\tau_{\{3\}})^2 (\tau_{\{3,1,2,5\}})^2$$



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$$\begin{aligned}
&\textstyle\int_X [X_{12354}][X_{31254}][w_0 X_{31245}] \\
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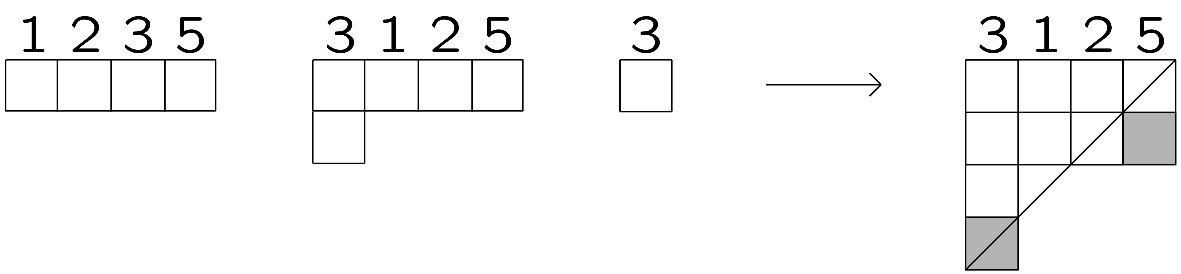
&\quad = \textstyle\int_X (\tau_{\{3\}})^2 (\tau_{\{3,1,2,5\}})^2 \\
&\quad = 2.
\end{aligned}$$



Now we can compute

$$\begin{aligned}[X_{s_1}]^2 &= [X_{2134}][X_{2134}] \\ &= [X_{2431}] - [X_{4213}] - [X_{3421}] - [X_{3241}] - [X_{3214}].\end{aligned}$$

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Closed formula for the expansion of $[X_{s_i}]^2$??

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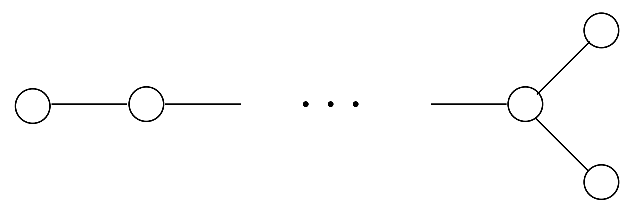
type D_n :

$$[1] \subset [2] \subset \cdots \subset [n-2] \begin{array}{l} \subset \{1, \cdots, n\} \\ \subset \{1, \cdots, -n\} \end{array}$$



Thank you for your attention!