Alexander duals to boundaries of polytopes

Anton Ayzenberg
ayzenberga@gmail.com

Osaka City University

January 22, 2014
Toric Topology in Osaka 2014

Plan

1. What is understood by “boundary of a polytope”;
2. To recall the definition of Alexander dual complex;
3. To describe Alexander duals to boundaries of polytopes;
4. Applications (neighborly polytopes, flag polytopes, bigraded Betti numbers, Buchstaber invariant).

Common and important construction

Let Q be a simple d-dimensional polytope (i.e. any vertex of Q belongs to exactly d facets).

The polar dual polytope Q^* is simplicial (i.e. any proper facet of Q^* is a simplex). The boundary ∂Q^* can be considered as an abstract simplicial complex, associated to a simple polytope Q. In fact, ∂Q^* is a simplicial sphere (thus Gorenstein*, Cohen–Macaulay, etc.).

Common and important construction

Let Q be a simple d-dimensional polytope (i.e. any vertex of Q belongs to exactly d facets).

The polar dual polytope Q^* is simplicial (i.e. any proper facet of Q^* is a simplex). The boundary ∂Q^* can be considered as an abstract simplicial complex, associated to a simple polytope Q. In fact, ∂Q^* is a simplicial sphere (thus Gorenstein*, Cohen–Macaulay, etc.).

The complex ∂Q^* is very important for toric topology and toric geometry. It appears in many formulas, e.g. describing cohomology and equivariant cohomology of toric spaces over Q.

General polytopes

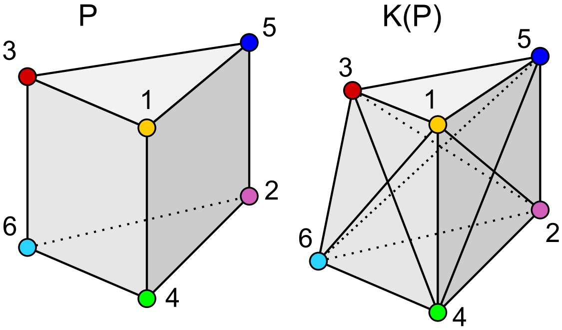
If a polytope P is not simplicial, its boundary is not a simplicial complex anymore. Nevertheless, in this case we can associate a simplicial complex $K(P)$ to such polytope in a very natural way:

Definition of $K(P)$

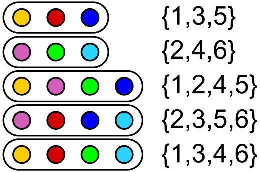
Let P be a d-dimensional polytope with m vertices $v_1, \dots, v_m$. Define a simplicial complex $K(P)$ on the set $[m] = \{1, \dots, m\}$ by the condition $\{i_1, \dots, i_k\} \in K(P) \Leftrightarrow v_{i_1}, \dots, v_{i_k}$ lie in one proper face of P.

This construction "replaces" facets of P by abstract simplices. If P is simplicial, $K(P) = \partial P$.

In the joint work with prof. Buchstaber we called $K(P)$ **nerve-complexes**.



Maximal simplices of K(P) =
= facets of P:



The nerve-complex $K(P)$ is not a simplicial sphere. It is even not pure in this case!

If P is non-simplicial, then $K(P)$ is not a sphere (even not Cohen–Macaulay). Anyway, the complex $K(P)$ has nice properties.

If P is non-simplicial, then $K(P)$ is not a sphere (even not Cohen–Macaulay). Anyway, the complex $K(P)$ has nice properties.

Properties of $K(P)$

For a face $F \subsetneq P$ let $\sigma(F) \subset [m]$ denote the set of its vertices (their labels, strictly speaking). For an "empty face" we also set $\sigma(\varnothing) = \varnothing$. By definition, $\sigma(F)$ is a simplex of $K(P)$. Then

1. $\mathrm{link}_{K(P)}\,\sigma(F) \simeq S^{d-\dim F-2}$;
2. If $I \in K(P)$ and $I \neq \sigma(F)$ for any F, then $\mathrm{link}_{K(P)}\, I$ is contractible.

Two things follow from these properties.

1. The complex $K(P)$ behaves in many aspects like a sphere.
2. The poset of faces of P is contained in $K(P)$. One can retrieve the complete combinatorial information about P from $K(P)$.

Two things follow from these properties.

1. The complex $K(P)$ behaves in many aspects like a sphere.
2. The poset of faces of P is contained in $K(P)$. One can retrieve the complete combinatorial information about P from $K(P)$.

Some aspects of toric topology (e.g. theory of moment-angle complexes) can be developed for non-simple polytopes Q. In this general setting complexes $K(Q^*)$ play the same role as ∂Q^* in a classical case.

Motivation

Minimal non-simplices

Let K be a simplicial complex on a set $[m]$. A subset $I \subset [m]$ is called **minimal non-simplex** (or missing face) of K if $I \notin K$ but $\partial I \subseteq K$. The set of minimal non-simplices will be denoted by $N(K)$.

Motivation

Minimal non-simplices

Let K be a simplicial complex on a set $[m]$. A subset $I \subset [m]$ is called **minimal non-simplex** (or missing face) of K if $I \notin K$ but $\partial I \subseteq K$. The set of minimal non-simplices will be denoted by $N(K)$.

In some tasks we need a convenient description of $N(K)$, when K is the boundary ∂P of a simplicial polytope or, more generally, a nerve-complex $K(P)$.

These tasks include: the study of a Buchstaber invariant, the calculation of bigraded Betti numbers (using Taylor resolution), and others... I will tell about some concrete applications later.

Alexander dual complex

Definition (Alexander dual complex)

If K is a complex on a set $[m]$ and $K \neq \Delta_{[m]}$, then its **Alexander dual** is a simplicial complex on $[m]$, defined by

$$K^{\wedge} = \{I \subset [m] \mid [m] \setminus I \notin K\}$$

Non-simplices of K are exactly complements to simplices of $K^{\wedge}$. So far

To study non-simplices of K $\leftrightsquigarrow$ to study simplices of $K^{\wedge}$.

Well-known facts about Alexander duality

- $K^{\wedge\wedge} = K$;

Well-known facts about Alexander duality

- $K^{\wedge\wedge} = K$;
- $\widetilde{H}_i(K; \Bbbk) \cong \widetilde{H}^{m-3-i}(\widehat{K}; \Bbbk)$;

Well-known facts about Alexander duality

- $K^{\wedge\wedge} = K$;
- $\widetilde{H}_i(K;\Bbbk) \cong \widetilde{H}^{m-3-i}(\widehat{K};\Bbbk)$;
- If $I \in K$, then $(\mathrm{link}_K I)^{\wedge} = \widehat{K}_{[m]\setminus I}$ — a full subcomplex;
- If $J \notin K$, then $(K_J)^{\wedge} = \mathrm{link}_{\widehat{K}}([m] \setminus J)$.

Problem

To describe simplicial complexes, which are Alexander dual to $K(P)$ for convex polytopes P. In particular, to describe complexes, which are Alexander dual to boundaries of simplicial polytopes.

Constellation complexes

Let $\mathbb{R}^{r+1}$ be a Euclidian space with the inner product $\langle \cdot, \cdot \rangle$.

Let $\mathbb{S}^r \subset \mathbb{R}^{r+1}$ be a unit sphere. For a point $x \in \mathbb{S}^r \sqcup \{0\}$ consider the set $H(x) = \{y \in \mathbb{S}^r \mid \langle x, y \rangle > 0\}$.
If $x = 0$, then $H(x)$ is empty. If $x \in \mathbb{S}^r$, the set $H(x)$ is an open hemisphere.

Constellation complexes

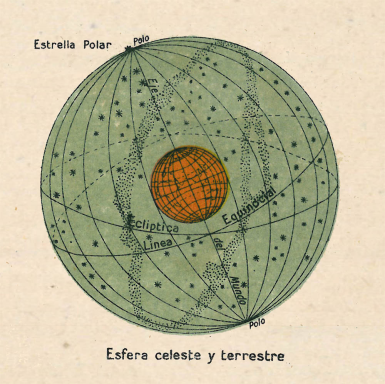
Let $\mathbb{R}^{r+1}$ be a Euclidian space with the inner product $\langle \cdot, \cdot \rangle$.

Let $\mathbb{S}^r \subset \mathbb{R}^{r+1}$ be a unit sphere. For a point $x \in \mathbb{S}^r \sqcup \{0\}$ consider the set $H(x) = \{y \in \mathbb{S}^r \mid \langle x, y \rangle > 0\}$.
If $x = 0$, then $H(x)$ is empty. If $x \in \mathbb{S}^r$, the set $H(x)$ is an open hemisphere.

For a finite sequence of points $X = \{x_1, \ldots, x_m\} \subset \mathbb{S}^r \sqcup \{0\}$ (repetitions are allowed) consider corresponding sets $\{H(x_i)\}$.

Definition

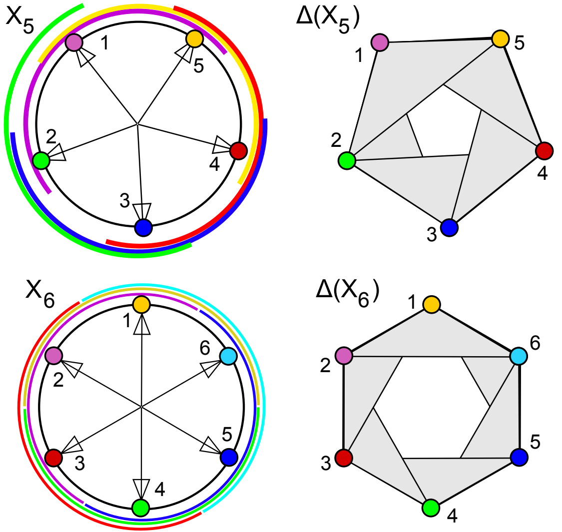
The nerve $\Delta(X)$ of the covering $\bigcup H(x_i)$ is called the **constellation complex** of X. In other words, $\{i_1, \ldots, i_k\}$ is a simplex of $\Delta(X)$ if and only if $H(x_{i_1}), \ldots, H(x_{i_k})$ intersect $\Leftrightarrow$ points $x_{i_1}, \ldots, x_{i_k}$ **lie in a common open hemisphere**.



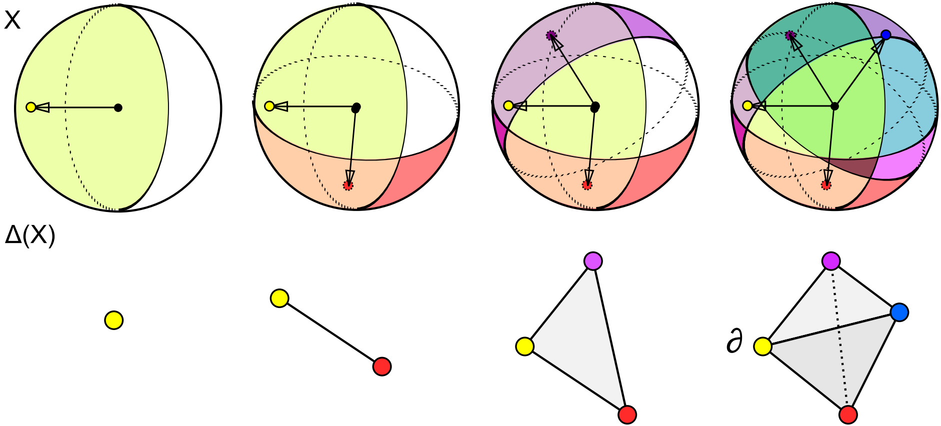
Remark

It is natural to call a set of stars on a celestial sphere a **constellation**, if they can be observed simultaneously from one point on Earth (= lie in a common hemisphere). Therefore the simplices of $\Delta(X)$ are "constellations", which explains the name "constellation complex".

Example



Example



Answer to the problem: constellation complexes are dual to nerve-complexes

Claim

1. For any polytope P^d with m vertices, there exists a configuration of m points $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ such that $K(P)^{\wedge} = \Delta(X)$.
2. If $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ is a configuration of points, for which hemispheres $H(x_i)$ cover the sphere $\mathbb{S}^{m-d-2}$ at least twice, then there exists a polytope P, such that $K(P)^{\wedge} = \Delta(X)$.

Answer to the problem: constellation complexes are dual to nerve-complexes

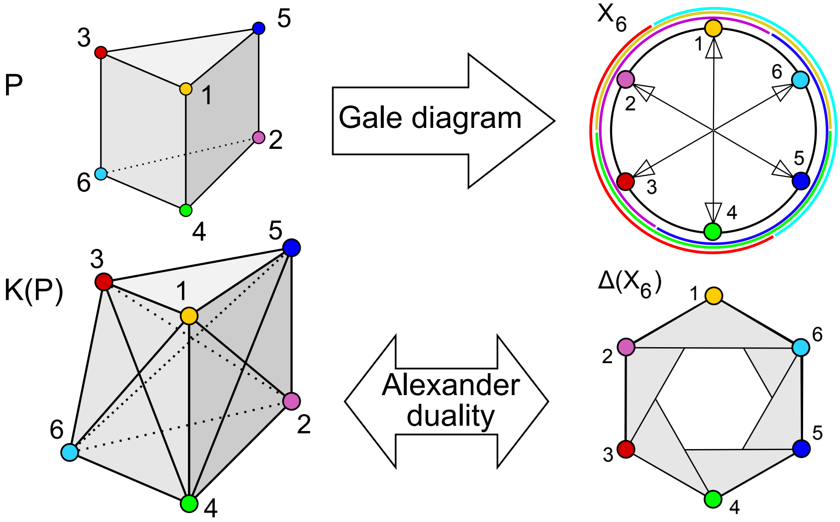
Claim

1. For any polytope P^d with m vertices, there exists a configuration of m points $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ such that $K(P)^{\wedge} = \Delta(X)$.
2. If $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ is a configuration of points, for which hemispheres $H(x_i)$ cover the sphere $\mathbb{S}^{m-d-2}$ at least twice, then there exists a polytope P, such that $K(P)^{\wedge} = \Delta(X)$.

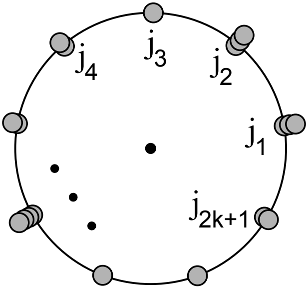
Remark

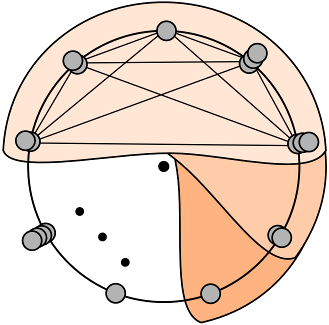
Such configuration X can be built as an **affine Gale diagram** of P. The claim follows from the theory of Gale duality.

Example


P
3
5
1
2
6
4
Gale diagram
X6
1
6
2
5
3
4
K(P)
5
3
1
6
2
4
Alexander
duality
Δ(X6)
1
6
2
5
3
4

Polytopes with $m - d = 3$





P — a simplicial polytope with m vertices, $\dim P = d$, $m - d = 3$. Then $X \subset \mathbb{S}^1$. All Gale diagrams in this dimension are classified: they look like the left picture. The constellation complex of such configuration is shown on the right. This complex has a simple description, and it is Alexander dual to ∂P.

Properties of polytopes in terms of constellation complexes

A polytope P is called ***k*-neighborly** if any k of its vertices lie in a common proper face. A simplicial polytope P is called **flag** if any set of pairwise adjacent vertices is a face of P. In other words, P is flag if $|J| = 2$ for any $J \in N(\partial P)$.

Properties of polytopes in terms of constellation complexes

A polytope P is called **k-neighborly** if any k of its vertices lie in a common proper face. A simplicial polytope P is called **flag** if any set of pairwise adjacent vertices is a face of P. In other words, P is flag if $|J| = 2$ for any $J \in N(\partial P)$.

Proposition

Let P^d be a polytope with m vertices and $X \subseteq \mathbb{S}^{m-d-2} \sqcup \{0\}$ — its Gale diagram (i.e. $K(P)^{\wedge} = \Delta(X)$). Then

1. P is a pyramid $\Leftrightarrow \{0\} \in X \Leftrightarrow \Delta(X)$ has ghost vertices.
2. P is k-neighborly $\Leftrightarrow \dim \Delta(X) \leqslant m - k - 2$.
3. If P is simplicial, then P is flag $\Leftrightarrow$ every maximal simplex $I \in \Delta(X)$ has $m - 2$ vertices.

Stanley–Reisner algebra

- $\Bbbk$ is a ground field;

Stanley–Reisner algebra

- $\Bbbk$ is a ground field;
- $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ — the polynomial algebra, $\deg v_i = 2$;

Stanley–Reisner algebra

- $\Bbbk$ is a ground field;
- $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ — the polynomial algebra, $\deg v_i = 2$;
- The **Stanley–Reisner algebra** $\Bbbk[K]$ of a simplicial complex K is the quotient algebra

$$\Bbbk[K] = \Bbbk[m]/(v_{i_1} \cdot \ldots \cdot v_{i_k}; \{i_1, \ldots, i_k\} \notin K);$$

Stanley–Reisner algebra

- $\Bbbk$ is a ground field;
- $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ — the polynomial algebra, $\deg v_i = 2$;
- The **Stanley–Reisner algebra** $\Bbbk[K]$ of a simplicial complex K is the quotient algebra

$$\Bbbk[K] = \Bbbk[m]/(v_{i_1} \cdot \ldots \cdot v_{i_k}; \{i_1, \ldots, i_k\} \notin K);$$

- $\Bbbk$ and $\Bbbk[K]$ are graded modules over $\Bbbk[m]$;

Stanley–Reisner algebra

- $\mathbb{k}$ is a ground field;
- $\mathbb{k}[m] = \mathbb{k}[v_1, \ldots, v_m]$ — the polynomial algebra, $\deg v_i = 2$;
- The **Stanley–Reisner algebra** $\mathbb{k}[K]$ of a simplicial complex K is the quotient algebra

$$\mathbb{k}[K] = \mathbb{k}[m]/(v_{i_1} \cdot \ldots \cdot v_{i_k}; \{i_1, \ldots, i_k\} \notin K);$$

- $\mathbb{k}$ and $\mathbb{k}[K]$ are graded modules over $\mathbb{k}[m]$;
- $\mathrm{Tor}^{*,*}_{\mathbb{k}[m]}(\mathbb{k}[K], \mathbb{k}) = \bigoplus_{i,j} \mathrm{Tor}^{-i,2j}_{\mathbb{k}[m]}(\mathbb{k}[K], \mathbb{k})$ is a Tor-functor of modules $\mathbb{k}[K], \mathbb{k}$.

Stanley–Reisner algebra

- $\Bbbk$ is a ground field;
- $\Bbbk[m] = \Bbbk[v_1, \dots, v_m]$ — the polynomial algebra, $\deg v_i = 2$;
- The **Stanley–Reisner algebra** $\Bbbk[K]$ of a simplicial complex K is the quotient algebra

$$\Bbbk[K] = \Bbbk[m]/(v_{i_1} \cdot \ldots \cdot v_{i_k}; \{i_1, \dots, i_k\} \notin K);$$

- $\Bbbk$ and $\Bbbk[K]$ are graded modules over $\Bbbk[m]$;
- $\operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K], \Bbbk) = \bigoplus_{i,j} \operatorname{Tor}^{-i,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$
  is a Tor-functor of modules $\Bbbk[K], \Bbbk$.
- **Bigraded Betti numbers** $\beta^{-i,2j}(K) = \dim_{\Bbbk} \operatorname{Tor}^{-i,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk)$

Bigraded Betti numbers

Theorem (Hochster formula)

For any simplicial complex

$$\beta^{-i,2j}(K) = \sum_{J\subseteq[m],|J|=j} \dim_{\Bbbk} \widetilde{H}^{j-i-1}(K_J;\Bbbk).$$

Bigraded Betti numbers

Theorem (Hochster formula)

For any simplicial complex

$$\beta^{-i,2j}(K) = \sum_{J\subseteq[m],|J|=j} \dim_{\Bbbk} \widetilde{H}^{j-i-1}(K_J;\Bbbk).$$

We have for $i > 0$:

$$\beta^{-i,2j}(K) = \sum_{|J|=j} \dim \widetilde{H}^{j-i-1}((\mathrm{link}_{\widehat{K}}([m]\setminus J))^{\wedge};\Bbbk) =$$

$$= \sum_{|J|=j,[m]\setminus J\in\widehat{K}} \dim \widetilde{H}_{i-2}(\mathrm{link}_{\widehat{K}}([m]\setminus J);\Bbbk).$$

Bigraded Betti numbers of polytopes can be computed from constellation complexes

Proposition

Let P be a polytope with m vertices of dimension n. Let $X \subset \mathbb{S}^{m-n-2}$ be its Gale diagram. Then for $i > 0$ we have

$$\beta^{-i,2j}(K(P)) = \sum_{J \in \Delta(X), |J| = m-j} \dim \widetilde{H}_{i-2}(\mathrm{link}_{\Delta(X)} J).$$

Bigraded Betti numbers of polytopes can be computed from constellation complexes

Proposition

Let P be a polytope with m vertices of dimension n. Let $X \subset \mathbb{S}^{m-n-2}$ be its Gale diagram. Then for $i > 0$ we have

$$\beta^{-i,2j}(K(P)) = \sum_{J \in \Delta(X), |J| = m-j} \dim \widetilde{H}_{i-2}(\operatorname{link}_{\Delta(X)} J).$$

This method allows to compute $\beta^{-i,2j}(\partial P)$ for simplicial polytopes P with $m - d = 3$. In this case $X \subset \mathbb{S}^1$ and all Gale diagrams are classified, as I mentioned before. This computation was done previously by Nickolai Erokhovets using different approach.

Bigraded Betti numbers of constellation complexes can be computed from polytopes

Proposition

Let P be a simplicial d-polytope with m vertices and $X \subset \mathbb{S}^{m-d-2}$ — its Gale diagram. Then for $i > 0$ we have

- $\beta^{-i,2(m-d-1+i)}(\Delta(X)) = f_{d-i}(P)$ — the number of $(d-i)$-dimensional simplices of ∂P.
- $\beta^{-i,2j}(\Delta(X)) = 0$ if $j \neq m-d-1+i$.

Bigraded Betti numbers of constellation complexes can be computed from polytopes

Proposition

Let P be a simplicial d-polytope with m vertices and $X \subset \mathbb{S}^{m-d-2}$ — its Gale diagram. Then for $i > 0$ we have

- $\beta^{-i,2(m-d-1+i)}(\Delta(X)) = f_{d-i}(P)$ — the number of $(d-i)$-dimensional simplices of ∂P.
- $\beta^{-i,2j}(\Delta(X)) = 0$ if $j \neq m-d-1+i$.

Indeed,

$$\beta^{-i,2j}(\Delta(X)) = \sum_{J \in \partial P, |J| = m-j} \dim \widetilde{H}_{i-2}(\mathrm{link}_{\partial P} J). \qquad (*)$$

Since ∂P is a sphere, $\mathrm{link}_{\partial P} J \cong S^{d-1-|J|}$. Therefore each $(d-i)$-dimensional simplex contributes 1 to the sum at the right-hand side of (*).

Bigraded Betti numbers of constellation complexes can be computed from polytopes

Consider an arbitrary polytope P. Let $f_{n,k}(P)$ be the number of n-dimensional proper faces of P with exactly k vertices. Formally set $f_{-1,0}(P) = 1$ — this corresponds to the "empty face" of P.

Obviously, if P is simplicial, we have $f_{n,n+1}(P) = f_n(P)$ and $f_{n,k}(P) = 0$ for $k \neq n+1$.

Generally, the numbers $f_{n,k}(P)$ provide more detailed information than the ordinary f-vector.

Bigraded Betti numbers of constellation complexes can be computed from polytopes

Proposition

P — d-polytope with m vertices, $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ — its Gale diagram. For $i > 0$ we have $\beta^{-i,2j}(\Delta(X)) = f_{d-i,m-j}(P)$.

Bigraded Betti numbers of constellation complexes can be computed from polytopes

Proposition

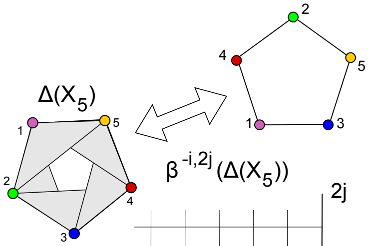
P — d-polytope with m vertices, $X \subset \mathbb{S}^{m-d-2} \sqcup \{0\}$ — its Gale diagram. For $i > 0$ we have $\beta^{-i,2j}(\Delta(X)) = f_{d-i,m-j}(P)$.

Indeed,

$$\beta^{-i,2j}(\Delta(X)) = \sum_{J \in K(P), |J| = m-j} \dim \widetilde{H}_{i-2}(\mathrm{link}_{K(P)} J).$$

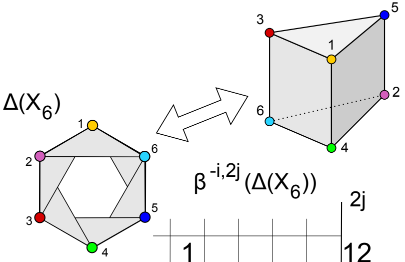
The proposition follows from the properties of nerve-complexes, since $\mathrm{link}_{K(P)} J$ is either homotopy equivalent to a sphere or contractible.

Example



$\beta^{-i,2j}(\Delta(X_5))$

2j \ -i	-3	-2	-1	0
10	1			
8		5		
6			5	
4				
2				
0				1



$\beta^{-i,2j}(\Delta(X_6))$

2j \ -i	-4	-3	-2	-1	0
12	1				
10		6			
8			9		
6				2	
4				3	
2					
0					1

Moment-angle complex

Definition

Let K be a simplicial complex on m vertices. The **moment-angle complex** is the subspace:

$$\mathcal{Z}_K = \bigcup_{I \in K} (D^2, S^1)^I \subseteq (D^2)^m,$$

where $(D^2, S^1)^I = \prod_{i \in I} D^2 \times \prod_{i \in [m] \setminus I} S^1$.
We have the coordinatewise action of a compact torus $T^m \curvearrowright \mathcal{Z}_K$. This action is not free for $K \neq \varnothing$.

Moment-angle complex

Definition

Let K be a simplicial complex on m vertices. The **moment-angle complex** is the subspace:

$$\mathcal{Z}_K = \bigcup_{I \in K} (D^2, S^1)^I \subseteq (D^2)^m,$$

where $(D^2, S^1)^I = \prod_{i \in I} D^2 \times \prod_{i \in [m] \setminus I} S^1$.
We have the coordinatewise action of a compact torus $T^m \curvearrowright \mathcal{Z}_K$. This action is not free for $K \neq \varnothing$.

Definition

The **Buchstaber invariant** $s(K)$ is the maximal rank of subgroups $G \subseteq T^m$ acting freely on $\mathcal{Z}_K$. For a polytope P define $s(P) := s(K(P^*))$.

Characterization in terms of minimal non-simplices

Theorem (Nickolai Erokhovets'12)

$s(K) \geqslant 2$ if and only if there exist $J_1, J_2, J_3 \in N(K)$ such that $J_1 \cap J_2 \cap J_3 = \varnothing$ (subsets may coincide).

This condition is equivalent to "there exist $I_1, I_2, I_3 \in K^{\wedge}$ such that $I_1 \cup I_2 \cup I_3 = [m]$". Just take $I_i = [m] \setminus J_i$.

Characterization in terms of minimal non-simplices

Theorem (Nickolai Erokhovets'12)

$s(K) \geqslant 2$ if and only if there exist $J_1, J_2, J_3 \in N(K)$ such that $J_1 \cap J_2 \cap J_3 = \varnothing$ (subsets may coincide).

This condition is equivalent to "there exist $I_1, I_2, I_3 \in K^{\wedge}$ such that $I_1 \cup I_2 \cup I_3 = [m]$". Just take $I_i = [m] \setminus J_i$.

Using this result and Gale diagrams we can prove

Theorem

$s(P) = 1 \Leftrightarrow P$ is a pyramid.

For simple polytopes this was known: if P is simple, then $s(P) = 1 \Leftrightarrow P$ is a simplex. Simplex is the only pyramid which is simple.

Proof: P is a pyramid $\Rightarrow s(P) = 1$

P is a pyramid;

Proof: P is a pyramid $\Rightarrow s(P) = 1$

P is a pyramid;

$\Rightarrow P^*$ is a pyramid;

Proof: P is a pyramid $\Rightarrow s(P) = 1$

P is a pyramid;

$\Rightarrow P^*$ is a pyramid;

$\Rightarrow$ the Gale diagram X of P^* contains 0 and $\Delta(X) = K(P^*)^\wedge$ has a ghost vertex i. Vertex i is not contained in any simplex;

Proof: P is a pyramid $\Rightarrow s(P) = 1$

P is a pyramid;

$\Rightarrow P^*$ is a pyramid;

$\Rightarrow$ the Gale diagram X of P^* contains 0 and $\Delta(X) = K(P^*)^{\wedge}$ has a ghost vertex i. Vertex i is not contained in any simplex;

$\Rightarrow I_1 \cup I_2 \cup I_3 \neq [m]$ for any $I_1, I_2, I_3 \in K(P^*)^{\wedge}$;

Proof: P is a pyramid $\Rightarrow s(P) = 1$

P is a pyramid;

$\Rightarrow P^*$ is a pyramid;

$\Rightarrow$ the Gale diagram X of P^* contains 0 and $\Delta(X) = K(P^*)^\wedge$ has a ghost vertex i. Vertex i is not contained in any simplex;

$\Rightarrow I_1 \cup I_2 \cup I_3 \neq [m]$ for any $I_1, I_2, I_3 \in K(P^*)^\wedge$;

$\Rightarrow s(P) = s(K(P^*)) = 1$ by Erokhovets result.

Proof: $s(P) = 1 \Rightarrow P$ is a pyramid

Suppose P is not a pyramid;

Proof: $s(P) = 1 \Rightarrow P$ is a pyramid

Suppose P is not a pyramid;

$\Rightarrow P^*$ is not a pyramid;

Proof: $s(P) = 1 \Rightarrow P$ is a pyramid

Suppose P is not a pyramid;

$\Rightarrow P^*$ is not a pyramid;

$\Rightarrow$ the Gale diagram X of P^* does not contain $\{0\}$. This means $X \subset \mathbb{S}^{m-d-2}$.

Proof: $s(P) = 1 \Rightarrow P$ is a pyramid

Suppose P is not a pyramid;

$\Rightarrow P^*$ is not a pyramid;

$\Rightarrow$ the Gale diagram X of P^* does not contain $\{0\}$. This means $X \subset \mathbb{S}^{m-d-2}$.

Choose a vector $y \in \mathbb{S}^{m-d-2}$ not orthogonal to any $x_i \in X$. Consider two opposite open hemispheres $H(y)$ and $H(-y)$.

Both subsets $I_+ = H(Y) \cap X$ and $I_- = H(-y) \cap X$ are simplices of $\Delta(X) = K(P)^\wedge$, and $I_+ \cup I_- = [m]$.

Proof: $s(P) = 1 \Rightarrow P$ is a pyramid

Suppose P is not a pyramid;

$\Rightarrow P^*$ is not a pyramid;

$\Rightarrow$ the Gale diagram X of P^* does not contain $\{0\}$. This means $X \subset \mathbb{S}^{m-d-2}$.

Choose a vector $y \in \mathbb{S}^{m-d-2}$ not orthogonal to any $x_i \in X$. Consider two opposite open hemispheres $H(y)$ and $H(-y)$.

Both subsets $I_+ = H(Y) \cap X$ and $I_- = H(-y) \cap X$ are simplices of $\Delta(X) = K(P)^\wedge$, and $I_+ \cup I_- = [m]$.

$\Rightarrow$ Erokhovets theorem implies $s(P) = s(K(P^*)) \geqslant 2$.

Final remarks

It is known that

$$H^*(\mathcal{Z}_K) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K];\Bbbk),$$

so bigraded Betti numbers $\beta^{-i,2j}(K)$ represent a lot of information about moment-angle complex.

Final remarks

It is known that

$$H^*(\mathcal{Z}_K) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K]; \Bbbk),$$

so bigraded Betti numbers $\beta^{-i,2j}(K)$ represent a lot of information about moment-angle complex.

On the other hand, Buchstaber invariant also represents the information about the action $T^m \curvearrowright \mathcal{Z}_K$.

Final remarks

It is known that

$$H^*(\mathcal{Z}_K) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K]; \Bbbk),$$

so bigraded Betti numbers $\beta^{-i,2j}(K)$ represent a lot of information about moment-angle complex.

On the other hand, Buchstaber invariant also represents the information about the action $T^m \curvearrowright \mathcal{Z}_K$.

Problem

Does $\forall i, j$ $\beta^{-i,2j}(K_1) = \beta^{-i,2j}(K_2)$ imply $s(K_1) = s(K_2)$?

Final remarks

It is known that

$$H^*(\mathcal{Z}_K) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K]; \Bbbk),$$

so bigraded Betti numbers $\beta^{-i,2j}(K)$ represent a lot of information about moment-angle complex.

On the other hand, Buchstaber invariant also represents the information about the action $T^m \curvearrowright \mathcal{Z}_K$.

Problem

Does $\forall i, j\ \beta^{-i,2j}(K_1) = \beta^{-i,2j}(K_2)$ imply $s(K_1) = s(K_2)$?

The answer to this problem is negative.

Final remarks

The most interesting case is when K is the boundary of a simplicial polytope. In this case $\mathcal{Z}_K$ is a manifold. The question remains:

Final remarks

The most interesting case is when K is the boundary of a simplicial polytope. In this case $\mathcal{Z}_K$ is a manifold. The question remains:

Refined problem

Let P_1 and P_2 be simplicial polytopes. Does $\forall i, j$ $\beta^{-i,2j}(\partial P_1) = \beta^{-i,2j}(\partial P_2)$ imply $s(\partial P_1) = s(\partial P_2)$?

Nickolai Erokhovets verified this implication for a restricted class of polytopes (polytopes with $m - d \leqslant 3$ or $d \leqslant 3$).

Final remarks

The most interesting case is when K is the boundary of a simplicial polytope. In this case $\mathcal{Z}_K$ is a manifold. The question remains:

Refined problem

Let P_1 and P_2 be simplicial polytopes. Does $\forall i, j$ $\beta^{-i,2j}(\partial P_1) = \beta^{-i,2j}(\partial P_2)$ imply $s(\partial P_1) = s(\partial P_2)$?

Nickolai Erokhovets verified this implication for a restricted class of polytopes (polytopes with $m - d \leqslant 3$ or $d \leqslant 3$).

I believe that this does not hold in general, but it is difficult to find a counterexample in a class of polytopes. Possibly, constellation complexes can play a certain role in constructing nontrivial counterexamples.

Final remarks

Problem

I do not know how to describe the multiplication in $H^*(\mathcal{Z}_P) \cong \operatorname{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K(P^*)], \Bbbk)$ in terms of Gale diagram X and its constellation complex $\Delta(X)$.

Final remarks

Problem

I do not know how to describe the multiplication in $H^*(\mathcal{Z}_P) \cong \mathrm{Tor}^{*,*}_{\Bbbk[m]}(\Bbbk[K(P^*)], \Bbbk)$ in terms of Gale diagram X and its constellation complex $\Delta(X)$.

Problem

Maybe we can find non-polytopal spheres by analyzing their dual complexes?

Thank you!

- Anton Ayzenberg, *Simplicial complexes Alexander dual to boundaries of polytopes*, arXiv:1310.5487 [math.CO]

- A. A. Ayzenberg, V. M. Buchstaber, *Moment-angle spaces and nerve-complexes of convex polytopes*, Proceedings of the Steklov Institute of Mathematics, V.275, 2011.

- V. M. Buchstaber and T. E. Panov, Torus Actions and Their Applications in Topology and Combinatorics // 2002.

- Nickolai Erokhovets, *Criterion for the Buchstaber invariant of simplicial complexes to be equal to two*, arXiv:1212.3970 [math.AT]

- Branko Grünbaum, Convex Polytopes // 2nd ed., Graduate Texts in Mathematics Vol. 221, 2003.