The integral cohomology of the symmetric square of quaternionic projective space

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Work in Progress

Thank you for giving me an opportunity to share my work; this is a progress report to my extended mathematical community.

This work has not been completed yet .....

- the last three months have been spent on this topic.
- some results are only 6 weeks old !
- I expect to include this topic in my thesis.

Motivation

Question: Why try to compute the integral cohomology of the symmetric square $H^*(SP^2(\mathbb{HP}^n); \mathbb{Z})$?

Answer: Possible applications to the quaternionic/symplectic cobordism ring MSp_*, which my thesis is about.

Some properties of MSp_* to which the study of $H^*(SP^2(\mathbb{HP}^n); \mathbb{Z})$ may be applied are related to works of

- Victor Buchstaber
- Malkhaz Bakuradze
- Fred Roush.

History

Symmetric squares/products and their (co)homology have a long history. For example,

- Minoru Nakaoka. *Cohomology theory of a complex with a transformation of prime period and its applications.* (1956).
- Ioan James, Emery Thomas, Hiroshi Toda and George W Whitehead. *The symmetric square of a sphere.*(1963).
- R.James Milgram. *The homology of symmetric products.*(1969).
- Stephen Mitchell and Stewart Priddy. *Symmetric product spectra and splittings of classifying spaces.*(1984).
- Sadok Kallel. *Symmetric products, duality and homological dimension of configuration spaces.*(2008).

Symmetric Squares

What is a symmetric square $SP^2(X)$?

- The quotient space $SP^2(X) = X \times X / \sim$, by identifying $(x, y) \sim (y, x)$ where $x, y \in X$. Usually X is a manifold, in particular when $X = \mathbb{HP}^n$ is our interest.
- $SP^2(X)$ can be thought as the set $\{(x, y) \mid x, y \in X\}$ of all unordered pairs of points of X.
- The symmetric square $SP^2(X) = X \times X/\mathbb{Z}/2$ is an example of an **orbifold**, i.e. $\mathbb{Z}/2$ acting on the smooth manifold $X \times X$.

Related to the symmetric square, there is a space called the braid space $Br(X) \subset SP^2(X)$.

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Braid Spaces

The braid space $Br(X) := Br(X, 2)$

The braid space $Br(X)$ can be viewed as the set of all unordered pairs of two **distinct** points on X.

- The cohomology ring $H^*(Br(X); \mathbb{Z}/2)$ where $X = \mathbb{RP}^n$ and $\mathbb{CP}^n$ was described by Samuel Feder in 1967, and also in *The reduced symmetric product of projective spaces and the generalized whitney theorem*(1972)※.
- The integral cohomology ring $H^*(Br(\mathbb{CP}^n); \mathbb{Z})$ is described by Tsutomu Yasui in *The reduced symmetric product of a complex projective space and the embedding problem* (1971)⊛.
- explicit computations for $H^*(Br(\mathbb{RP}^n); \mathbb{Z})$ are given by Jesús González and Peter Landweber in 2012.

$H^*(Br(\mathbb{HP}^n);\mathbb{Z})$

Similar to the real and complex cases described in ⊛ and ⋇, there are fibrations associated to $Br(\mathbb{HP}^n)$.

Example

There exists a fibration

$$Br(\mathbb{HP}^1) \longrightarrow Br(\mathbb{HP}^n) \longrightarrow Gr_{n+1,2}(\mathbb{H})$$

where $Gr_{n+1,2}(\mathbb{H})$ is the Grassmannian, and $Br(\mathbb{HP}^1) \simeq \mathbb{RP}^4$.

The integral cohomology $H^*(Br(\mathbb{HP}^n);\mathbb{Z})$ is used to compute the cohomology of the symmetric square $H^*(SP^2(\mathbb{HP}^n);\mathbb{Z})$ with integer coefficients.

Example : $H^*(SP^2(\mathbb{HP}^2);\mathbb{Z})$

Table: $H^* := H^*(SP^2(\mathbb{HP}^2);\mathbb{Z})$ with generators.

*	0	1	2	3	4	5	6	7	8
H^*	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}$
	1				g			$t_{1,1}$	g^2, h
*	9	10	11	12	13	14	15	16	$*\geq 17$
H^*	0	0	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$	0
			$t_{2,1}$	$\frac{1}{2}gh$	$t_{2,2}$		$t_{2,3}$	$\frac{1}{2}h^2$	

$$g^3 = 3\cdot\tfrac{1}{2}gh,\ g^2h = h^2,\ t_{i,j}t_{k,l} = 0\ \forall i,j,k,l,\ \mid t_{i,j}\mid = 4i+2j+1.$$

The Main Methods

The Mayer-Vietoris sequence

Using the decomposition of $SP^2(\mathbb{HP}^n) = L \cup_A N$, where $A = L \cap N$.

- $N \simeq \mathbb{HP}^n$ is a closed neighbourhood of the diagonal $\triangle := \{(x,x) \mid x \in \mathbb{HP}^n\}$ in $SP^2(\mathbb{HP}^n)$.
- L is the closure of the complement $SP^2(\mathbb{HP}^n) \setminus N$.

The Serre spectral sequence for A and L

- A is the total space $\mathbb{RP}^{4n-1} \to A \to \mathbb{HP}^n$ of the (real)projectivisation $\mathbb{RP}(\tau_{\mathbb{HP}^n})$ of the tangent bundle of $\mathbb{HP}^n$.
- $L \simeq Br(\mathbb{HP}^n)$.

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The Orbifold Cohomology with integer coefficients

Computing the ordinary cohomology of orbifolds with **integer coefficients** is not simple; so the orbifold integral cohomology for the quotient orbifold is often defined to be the ordinary cohomology of the Borel space with integer coefficients[1].

Definition

For the global quotient orbifold $\mathbb{HP}^n \times \mathbb{HP}^n/\mathbb{Z}/2$, the orbifold integral cohomology is defined to be the usual equivariant cohomology,

$$H^*_{orb}(SP^2(\mathbb{HP}^n); \mathbb{Z}) := H^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{HP}^n \times \mathbb{HP}^n); \mathbb{Z}).$$

Note: The integral cohomology $H^*(S^\infty \times_{\mathbb{Z}/2} (\mathbb{HP}^n \times \mathbb{HP}^n); \mathbb{Z})$ can be obtained by applying the Serre spectral sequence for the fibration

$$\mathbb{HP}^n \times \mathbb{HP}^n \longrightarrow S^\infty \times_{\mathbb{Z}/2} (\mathbb{HP}^n \times \mathbb{HP}^n) \longrightarrow \mathbb{RP}^\infty.$$

[1] see *Orbifolds and Stringy topology* by A.Adem, J.Leida and Y.Ruan(2007)

$H^*_{orb}(SP^2(\mathbb{HP}^n);\mathbb{Z}[1/2]) \cong H^*(SP^2(\mathbb{HP}^n);\mathbb{Z}[1/2])$

The homomorphism $H^*_{orb}(SP^2(\mathbb{HP}^n);\mathbb{Z}) \longleftarrow H^*(SP^2(\mathbb{HP}^n);\mathbb{Z})$ has rather nice structure; for instance, with $\mathbb{Z}[1/2]$ coefficients

Proposition

There is an isomorphism

$$H^*_{orb}(SP^2(\mathbb{HP}^n);\mathbb{Z}[1/2]) \cong H^*(SP^2(\mathbb{HP}^n);\mathbb{Z}[1/2]).$$

Example: $n = 2$

$$H^*_{orb}(SP^2(\mathbb{HP}^2);\mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2][x,y]/(x^3 - 3xy,\ x^2y - 2y^2),$$

$$H^*(SP^2(\mathbb{HP}^2);\mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2][g,h]/(g^3 - \frac{3}{2}gh,\ g^2h - h^2).$$

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Further work

1. complete the product structures for $H^*(SP^2(\mathbb{HP}^n); \mathbb{Z})$ (by May 2014)
2. applications to $MSp^*(X)$ (by December 2014)

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Finally, and most importantly **write my thesis !**

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Thank you for listening .