Toric structure of $(2n, k)$-manifolds

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Content of the talk

- Problems
- Basic notions
- Toric and Quasitoric manifolds
- Theory of toric $(2n, k)$ - manifolds
- Complex Grassmann manifolds as $(2n, k)$ - manifolds
- Seminal examples of orbit spaces of $(2n, k)$-manifolds

Problems

The goal of our talk is to present the class of $(2n, k)$ - manifolds. It is the wide class of $2n$ - dimensional manifold with an action of a compact torus T^k. We develop the theory of such manifolds and propose the tools which effectively describe their structure.

This class contains toric and quasitoric manifolds whose effective description uses combinatorics of their orbit spaces. It contains as well the Grassmann manifolds equipped with an action of the maximal torus T^k and in this way we develop the results of Gelfand, MacPherson, Goresky and Serganova concerning the action of algebraic torus $(\mathbb{C}^*)^k$ on Grassmannians.

Any $(2n, k)$-manifold has an almost moment map $\mu : M^{2n} \to \mathbb{R}^k$ whose image is a convex polytope P^k. In general case, including Grassmann manifolds, P^k is not a simple polytope and the orbit space M^{2n}/T^k is not homeomorphic to P^k. Therefore we were needed to introduce the notion of admissible polytopes and to develop the results of Karshon, Ginzburg and Guillemin on abstract moment map.

Basic notions

Assume it is given a smooth action of the compact torus T^k on a smooth manifold M^{2n}.

Let T_x be the stabilizer and $T(x)$ be the orbit of the point $x \in M^{2n}$.

- It is defined the orbit space M^{2n}/T^k;
- It is defined the characteristic function

$$\chi : M^{2n}/T^k \to S(T^k) \text{ by } \chi(T(x)) = T_x,$$

  where $S(T^k)$ is partially ordered by inclusion set of subgroups of T^k.
- The function χ is continuous related to the lower topology on $S(T^k)$.

Toric and Quasitoric manifolds

- M^{2n} - projective toric variety, $T^n \hookrightarrow M^{2n}$;
- $\mu : M^{2n} \to \mathbb{R}^n$ — moment map, T^n - invariant;
- Image$\mu = P$ — simple convex polytope.

μ splits as:

$$M^{2n} \xrightarrow{\pi} M^{2n}/T^n \xrightarrow{\hat{\pi}(\cong)} P$$

Toric and Quasitoric manifolds

- M^{2n} — smooth, closed manifold; $T^n \hookrightarrow M^{2n}$;
- The action is locally standard action $T^n \hookrightarrow \mathbb{C}^n$;
- There exists $\mu : M \to P^n$ – simple convex polytope;
- μ splits as:

$$M^{2n} \xrightarrow{\pi} M^{2n}/T^n \xrightarrow{\hat{\pi}(\cong)} P$$

and

$$\mu^{-1}(x) = T(x) \text{ for any } x \in P.$$

- Then χ is defined on the faces of P and if P_1 is a face of P and P_2 a face of P_1 then

$$\chi(P_1) \subset \chi(P_2).$$

There exists a T^n-equivariant homeomorphism

$$g : (T^n \times P)/_{\approx} \to M^{2n},$$

defined by a section

$$s : P \to M^{2n}, \ \ \mu \circ s = I_d,$$

where $(t_1, x_1) \approx (t_2, x_2)$ iff $x_1 = x_2$ and $t_1 t_2^{-1} \in T_{\hat{\pi}^{-1}(x_1)}$,
where $T_{\hat{\pi}^{-1}(x_1)}$ is the stabilizer of the point $\hat{\pi}^{-1}(x_1)$.

Definition of $(2n, k)$-manifolds

We assume the following to be given:

- a smooth, closed simply connected manifold M^{2n};
- a smooth, effective action θ of the torus T^k on M^{2n}, where $1 \leq k \leq n$, such that the stabilizer of any point is connected;
- an open θ-equivariant map $\mu : M^{2n} \to \mathbb{R}^k$ whose image is a k-dimensional convex polytope, where $\mathbb{R}^k$ is considered with trivial T^k - action.

- μ - we call an *almost moment map*.

- $Im\mu$ we denote by P^k.

Smooth structure

Axiom 1:

There is a smooth atlas $\mathfrak{M} = \{(M_i, \varphi_i)\}_{i \in I}$ with a homeomorphisms $\varphi_i : M_i \to \mathbb{R}^{2n} \approx \mathbb{C}^n$ for the fixed identification $\approx$, such that any chart M_i

- is T^k-invariant,
- contains exactly one fixed point x_i with $\varphi_i(x_i) = (0, \ldots, 0)$,
- the closure of M_i is M^{2n}.

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- is T^k-invariant,
- contains exactly one fixed point x_i with $\varphi_i(x_i) = (0, \ldots, 0)$,
- the closure of M_i is M^{2n}.

Corollary. The action of T^k on M^{2n} has finitely many isolated fixed points.

Denote by m the number of fixed points for T^k-action on M^{2n}.

The charts given by Axiom 1 we enumerate as $(M_1, \varphi_1), \dots, (M_m, \varphi_m)$.

The sets $Y_i = M - M_i$ are closed and T^k-invariant, that is $Y_i = \partial M_i$.

Define the sets W_σ, where $\sigma = \{i_1, \dots, i_l\} \subseteq \{1, \dots, m\}$ as:

$$W_\sigma = M_{i_1} \cap \cdots M_{i_l} \cap Y_{i_{l+1}} \cap \cdots Y_{i_m},$$

where $\{i_{l+1}, \dots, i_m\} = \{1, \dots, m\} - \{i_1, \dots, i_l\}$.

Definition

The non-empty set W_σ is called admissible.

Lemma

The admissible sets W_σ are T^k-invariant, pairwise disjoint and their union is the whole manifold M^{2n}.

$W_{\{1,\dots,m\}}$ is an admissible set which is everywhere dense in M^{2n}.

$W_{\{i\}}$ is an admissible set for any $1 \leq i \leq m$.

The boundary $\partial W_\sigma = \overline{W_\sigma} - W_\sigma$ of an admissible set W_σ is contained in the union of the admissible sets $W_{\tilde\sigma}$ for all subsets $\tilde\sigma \subset \sigma$.

Remark. In the paper of Gel'fand-Serganova it is given the description of the action of T^6 on the Grassmann manifold $G_{7,3}$ from which we deduce the example of our $(2n, k)$-manifold for which

$$\partial W_\sigma \neq \cup_{\tilde\sigma \subset \sigma} W_{\tilde\sigma}.$$

Almost standard action

Consider an action θ of the torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$.

Definition

An action θ is called almost standard if:

1. it is effective,
2. the origin is the only fixed point,
3. the stabilizer of any point $x \in \mathbb{C}^n$ is connected.

Remark. For $k = n$ an almost standard action is isomorphic to the standard one.

- The representation ρ can be written as $\rho = (\rho_1, \dots, \rho_n)$, where $\rho_i : \mathbb{T}^k \to S^1$, $1 \le i \le n$.
- The characters ρ_i can be represented as $\rho_i(t) = e^{2\pi\sqrt{-1}\langle\Lambda_i,\, t\rangle}$, where $\Lambda_i \in \mathbb{Z}^k$ are the weight vectors for the representation ρ.

- Let V be a $(k \times n)$-matrix whose rows are given by the weight vectors Λ_i.
- Denote by $P^J(V)$ the Plücker coordinates of the matrix V, where $J \subseteq \{1, \dots, n\}$ and $|J| = k$.
- The matrix V gives the linear map $\mathbb{R}^k \to \mathbb{R}^n$.
- Furthermore, for any subset $J \subseteq \{1, \dots, n\}$, the matrix V^J defined by the vectors $\Lambda_j, j \in J$ gives the linear map $f_J : \mathbb{R}^k \to \mathbb{R}^J$.

Proposition

If the map $f_J : \mathbb{R}^k \to \mathbb{R}^J$ is induced by an almost standard action of $\mathbb{T}^k$ on $\mathbb{C}^n$ then the image $f_J(\mathbb{Z}^k)$ is a direct summand in $\mathbb{Z}^J$ for any $J \subseteq \{1, \dots, n\}$.

Corollary. The Plücker coordinates $P^J(V) \in \{-1, 0, 1\}$.

Corollary. The weight vectors Λ_i, $1 \leq i \leq n$ are primitive.

The action - locally almost standard

We suggest the axioms which allow us to describe the structure of the admissible sets and the lattice of the set of admissible sets.

Axiom 2:

For any chart (M_i, φ_i) it is given the characteristic homomorphism $\alpha_i : \mathbb{T}^k \to \mathbb{T}^n$ such that the homeomorphism φ_i is α_i - equivariant: $\varphi_i(tx_i) = \alpha_i(t)\varphi_i(x_i)$, $t \in \mathbb{T}^k$, $x_i \in M_i$.

Lemma.

Any characteristic homomorphism $\alpha_i : \mathbb{T}^k \to \mathbb{T}^n$ gives an almost standard action of $\mathbb{T}^k$ on $\mathbb{C}^n$.

Almost standard action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$

Consider an action of the algebraic torus $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$. It induces the action of the compact torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$.

Definition

An action of $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$ we call almost standard action if the induced action of $\mathbb{T}^k$ on $\mathbb{C}^n$ is almost standard.

Lemma

If the weight vectors of the representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ for almost standard action $(\mathbb{C}^*)^k$ on $\mathbb{C}^n$ are pairwise linearly independent then

- any one-dimensional $(\mathbb{C}^*)^k$-orbit is one of the coordinate axis,
- for any codimension one subgroup $H < \mathbb{T}^k$ the fixed point set $(\mathbb{C}^n)^H$ either the origin either it is one of the coordinate axis.

The set of one-dimensional orbits

The following axiom gives the strong requirement on the set of one-dimensional orbits in M^{2n}.

Axiom 3:

For any characteristic homomorphism $\alpha_i : \mathbb{T}^k \to \mathbb{T}^n$, the weight vectors are pairwise linearly independent.

Let H be a codimension one subgroup of T^k. Then

$$(M^{2n})^H = \cup_{1\leq i\leq m} M_i^H, \quad S^1 = T^k/H \text{ acts smoothly on } (M^{2n})^H.$$

Denote by X^H a connected component of $(M^{2n})^H$. Then X^H is a closed submanifold in M^{2n} and S^1 acts smoothly on X^H.

Proposition

- *X^H is either a fixed point or it is homeomorphic to the sphere S^2 equipped with S^1-action with the fixed point $\{x_i, x_j\}$.*
- *For $X^H \cong S^2$ it holds $X^H \subseteq W_{\{i,j\}}$ and it is given by the closure of the preimage of the coordinate axis in the chart M_i as well as the corresponding axis in the chart M_j.*

Corollary

The closure of the set of points in M^{2n} which have one-dimensional orbits is given by the union of $\frac{n \cdot m}{2}$ spheres S^2 such that $S^2 - \{x_i, x_j\} \subseteq W_{\{i,j\}}$ for an admissible set $W_{\{i,j\}}$.

Note: If n is odd then the number of fixed points m must be even.

Orbits of almost standard $(\mathbb{C}^*)^k$-action on $\mathbb{C}^n$

Let $J \subseteq \{1, \dots, n\}$ and let

$$\mathbb{C}^J = \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_j \neq 0 \; j \in J, \;\; z_j = 0, \; j \notin J\}.$$

The coordinates subspaces $\mathbb{C}^J$ are $(\mathbb{C}^*)^k$-invariant and $\mathbb{C}^n = \cup_J \mathbb{C}^J$.
We have the matrix V^J given by the weight vectors Λ_j, $j \in J$ and the linear map $f_J : \mathbb{R}^k \to \mathbb{R}^{|J|}$ such that $f_J(\mathbb{Z}^k)$ is a direct summand in $\mathbb{Z}^J$.
Let $q = rank V^J$.

The points from $\mathbb{C}^J$ have the same stabilizer $(\mathbb{C}^*)^k_J \subseteq (\mathbb{C}^*)^k$, where $(\mathbb{C}^*)^k_J = (\mathbb{C}^*)^{k-q}$, that is $(\mathbb{C}^*)^q$ acts freely on $\mathbb{C}^J$.

1. If $|J| = q$ then $\mathbb{C}^J$ is a $\mathbb{C}^q$-orbit.

1. Let $q < |J|$. Then:

- Consider the lattice L in $\mathbb{Z}^J$ spanned by $f_J(\Lambda_j)$, $j \in J$.
- There exists unique integral lattice $\hat{L}$ in the dual lattice for $\mathbb{Z}^J$ which is orthogonal to L.
- By fixing the basis in $\hat{L}$ we obtain the matrix $\hat{V}^J$ of dimension $|J| \times (|J| - q)$. Denote its elements by $\omega_i^j \in \mathbb{Z}$, $1 \leq i \leq |J|$, $1 \leq j \leq |J| - q$.

Let $J = \{j_1, \ldots, j_l\}$. We obtain the algebraic map $F_J : (\mathbb{C}^*)^J \to (\mathbb{C}^*)^{l-q}$ given by

$$(z_{j_1}, \ldots, z_{j_l}) \to (z_{j_1}^{\omega_1^1} \cdots z_{j_l}^{\omega_1^l}, \ldots, z_{j_1}^{\omega_{l-q}^1} \cdots z_{j_l}^{\omega_{l-q}^l}).$$

- F_J is $(\mathbb{C}^*)^k$-invariant, where $\mathbb{C}^{l-q}$ is with trivial $(\mathbb{C}^*)^k$-action. This follows from the fact that $\hat{L}$ is orthogonal to L.
- The preimage $F_J^{-1}(c)$ of any point $c = (c_1, \ldots, c_{l-q})$ is a $(\mathbb{C}^*)^q$ - orbit, whose normal bundle inherits trivial $(\mathbb{C}^*)^q$-action.

Admissible sets and polytopes

Axiom 4:

The map μ gives the bijection between the set of fixed points and the set of vertices of the polytope P^k.

Let $S(P^k)$ be the family of convex polytopes which are spanned by the vertices of the polytope P^k and $\{W_\sigma\}$ the family of all admissible sets.

Define the map $s : \{W_\sigma\} \to S(P^k)$ by

$$s(W_\sigma) = P_\sigma, \text{ where } \sigma = \{i_1, \dots, i_l\} \text{ and } P_\sigma = convhull(v_{i_1}, \dots, v_{i_l}),$$

and $v_{i_1}, \dots, v_{i_l}$ are the vertices of the polytope P^k determined by

$$v_{i_j} = \mu(x_{i_j}) \text{ for } x_{i_j} \in M_{i_j} - \text{ the fixed point.}$$

Definition

A polytope $P_\sigma \in S(P^k)$ is said to be admissible if it corresponds to an admissible set.

The polytope P^k is an admissible polytope.

Denote by $\widehat{\mu} : M^{2n}/T^k \to P^k$ the map induced by the almost moment map μ.

Axiom 5:

For any admissible set W_σ the almost moment map μ induces the fiber bundle $\widehat{\mu} : W_\sigma/T^k \to \overset{\circ}{P}_\sigma$.

Let X^H be a connected component of M^H, where H is codimension one subgroup in T^k. In the case $X^H \cong S^2$ we have that $X^H - \{x_i, x_j\} \subseteq W_{\{i,j\}}$.

Corollary. $\mu(X^H) = [v_i, v_j]$

Definition

The fiber of the bundle $\widehat{\mu} : W_\sigma / T^k \to \overset{\circ}{P}_\sigma$ we call the set of parameters of an admissible set W_σ and denote it by F_σ.

Since $\overset{\circ}{P}_\sigma$ is contractible we obtain:

Corollary. The fiber bundle $\widehat{\mu} : W_\sigma / T^k \to \overset{\circ}{P}_\sigma$ is isomorphic to the trivial bundle. Hence W_σ / T^k is homeomorphic to $\overset{\circ}{P}_\sigma \times F_\sigma$.

Before to formulate the next axioms let us summarize previous constructions and results. Let us fix a chart (M_i, φ_i).

- We have the weight vectors $\{\Lambda_i^1, \dots, \Lambda_i^n\}$.
- The set of one-dimensional orbits having the same stabilizer is given by the coordinate axis.
- On each coordinate axis $\mathbb{C}_j$ acts S^1 by the representation $\alpha_i^j : T^k \to S^1$ with weight vector Λ_i^j.
- The closure in M^{2n} of the preimage $\varphi_i^{-1}(\mathbb{C}_j)$ of any axis $\mathbb{C}_j$ is homeomorphic to S^2.
- Such S^2 maps by the almost moment map to an admissible interval of P^k.
- One vertex of this interval is $v_i = \mu(x_i)$, where x_i is a unique fixed point in M_i.
- Denote by v_j the other vertex of this interval.

Further tasks

At any fixed point x_i we have:

- the configuration of the weight vectors,
- the configuration of the admissible one-dimensional polytopes with the vertex v_i, where $\mu(x_i) = v_i$,
- the moment map μ.

We have two tasks:

1. to connect the configuration of the weight vectors at x_i with the configuration of admissible one-dimensional polytopes at v_i,
2. to connect the configuration of the weight vectors in different fixed points in terms of the almost moment map μ and combinatorics of admissible polytopes.

We already solved this tasks partially by the construction of the 2-spheres which describe the set of points having one-dimensional orbits.

The cones over the weight vectors and the admissible 1-polytopes must be isomorphic

From the previous there is the map $\widetilde{\psi_i^j} : \{r\Lambda_i^j \mid r \geq 0\} \to [v_i, v_j)$ given by

$$\widetilde{\psi_i^j}(r\Lambda_i^j) = \mu(\varphi_i^{-1}(0, \ldots 0, r, 0, \ldots, 0)).$$

It can be written as

$$\widetilde{\psi_i^j}(r\Lambda_i^j) = \beta_i^j(r)(v_i - v_j),$$

for some function $\beta_i^j : \mathbb{R}_{\geq 0} \to [0, 1)$.

Note that (v_i, v_j) is an admissible polytope which is not in general an edge of the polytope P^k.

On the other hand let $e_l(v_j)$, $j \in J(i)$ be the edge-vectors of P^k having v_i as a vertex. Let

$$C_{P,i} = cone(e_l(v_i),\ l \in J(i)),$$

which is the strictly convex cone in $\mathbb{R}^k$ with the vertex v_i.

From above there is a map

$$\psi_i^j : \{r\Lambda_i^j \mid r \geq 0\} \to \{r(v_i - v_j)\ r \geq 0\} \subseteq C_{P,i}$$

$$\psi_i^j(r\Lambda_i^j) = \frac{1}{1 - \beta_i^j(r)} \widetilde{\psi_i^j}(r\Lambda_i^j).$$

For any vertex v_i of P^k we obtain the *canonical* map $\eta_i : \mathbb{R}^n_{\geq 0} \to C_{P,i}$ given by

$$\eta_i(x_1, \dots, x_n) = \psi_i^1(x_1\Lambda_i^1) + \dots + \psi_i^n(x_n\Lambda_i^n),$$

where we consider the sum of the points of the cone $C_{P,i}$ with the vertex v_i.

Let

$$C_i = cone(\Lambda_i^1, \dots, \Lambda_i^n) \subset \mathbb{R}^k.$$

Consider the other canonical map $\gamma_i : \mathbb{R}^n_{\geq 0} \to C_i$ defined by

$$\gamma_i(x_1, \dots, x_n) = x_1\Lambda_i^1 + \dots + x_n\Lambda_i^n.$$

By the construction it holds $\eta_i = \psi_i^j \circ \gamma_i$ on any coordinate axis of $\mathbb{R}^n$.

We need the existence of the map $\psi_i : C_i \to C_{P,i}$ which is the extension of each ψ_i^j, $1 \leq j \leq n$.

Definition

We say that there exists compatibility between the configuration of the weight vectors at the fixed point x_i and the configuration of the one-dimensional admissible polytopes at the vertex v_i given by the map $\psi_i : C_i \to C_{P,i}$ if

1. $\eta_i = \psi_i \circ \gamma_i$,
2. for any subset $\{j_1, \dots j_s\} \subseteq \{1, \dots n\}$ the induced map

$$\psi_i : C_i(\Lambda_i^{j_1}, \dots \Lambda_i^{j_s}) \to C_{P,i}(\psi_i(\Lambda_i^{j_1}), \dots, \psi_i(\Lambda_i^{j_s}))$$

   is the homeomorphism of the cones.
3. $\psi_i : C_i(\Lambda_i^j) \to C_{P,i}(\psi_i(\Lambda_i^j))$ coincides with the map ψ_i^j, $1 \leq j \leq n$.

In this case we say that fixed point x_i and the vertex $v_i = \mu(x_i)$ are compatible by the map $\psi_i : C_i \to C_{P,i}$.

Remark. The map ψ_i is defined and thus unique if and only if $\gamma_i(x) = \gamma_i(y)$ implies $\eta_i(x) = \eta_i(y)$. We say that the map ψ_i solves the first task.

Consider an action of the torus $\mathbb{T}^k$ on $\mathbb{C}^n$ given by a representation $\rho : \mathbb{T}^k \to \mathbb{T}^n$ and the standard action of $\mathbb{T}^n$ on $\mathbb{C}^n$, with the weight vectors by $\Lambda^1, \ldots, \Lambda^n$.

Consider the canonical map $\gamma : \mathbb{R}^n_{\geq 0} \to C = cone(\Lambda^1, \ldots, \Lambda^n)$ defined by

$$\gamma(x_1, \ldots, x_n) = x_1 \Lambda^1 + \cdots + x_n \Lambda^n$$

The standard way to define the moment map $\mu_f : \mathbb{C}^n \to C$ for this action :

$$\mu_f(z_1, \ldots, z_n) = (\gamma \circ f)(z_1, \ldots, z_n),$$

where $f : \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ is given by $f(z_1, \ldots, z_n) = (|z_1|^2, \ldots, |z_n|^2)$.

The representation $\rho = (\rho_1, \dots, \rho_n) : \mathbb{T}^k \to \mathbb{T}^n$ induces the map $\rho_* : \mathbb{R}^k \to \mathbb{R}^n$. For the standard moment map we have:

- Let $H < \mathbb{T}^k$ be a stationary subgroup of a point in $\mathbb{C}^n$, i. e. $H \in Im\chi(\mathbb{C}^n)$. Then

$$H = H_J = \cap_{j \in J} Ker \rho_j, \text{ for } J = \{j_1, \dots j_l\} \subseteq \{1, \dots, n\}$$

  where J is such that for any $\tilde{J}$, $J \subseteq \tilde{J}$ it holds $H_{\tilde{J}} < H_J$.
- Let $\mathfrak{h} = \mathfrak{L}(\Lambda^{j_1}, \dots, \Lambda^{j_l})$ be the Lie algebra for $\rho(H)$ in $\mathfrak{t}^n$. Then for the fixed point set $(\mathbb{C}^n)^H$ it holds

$$\rho_* \circ \mu_f : (\mathbb{C}^n)^H \to \mathfrak{h}.$$

Definition

We say that the map $\mu : \mathbb{C}^n \to \mathbb{R}^k$ is an almost standard moment map for the given T^k-action on $\mathbb{C}^n$ if

1. μ is T^k-invariant,
2. for any stationary subgroup $H < \mathbb{T}^k$ it holds

$$\mu : (\mathbb{C}^n)^H \to \mathfrak{h},$$

where $\mathfrak{h}$ is the Lie algebra for $\rho(H)$.

It is equivalent to: $\mu = \mu_\xi = \gamma \circ \xi$, for some $\xi : \mathbb{C}^n \to R^n_{\geq 0}$ such that

- ξ is T^k-invariant map,
- $\mu_\xi(z) \in C(\Lambda^{i_1}, \dots, \Lambda^{i_l})$ iff $\mu_f(z) \in C(\Lambda^{i_1}, \dots, \Lambda^{i_l})$, for any $z \in \mathbb{C}^n$ and any $\{\Lambda^{i_1}, \dots, \Lambda^{i_l}\}$ such that $\mathbb{Z}^J = \mathbb{Z}(\Lambda_{j_1}, \dots, \Lambda_{j_l})$, $J = \{j_1, \dots, j_l\}$ is a maximal element in the set of latices $\{\mathbb{Z}^J\}$ partially ordered by inclusion.

From the previous we obtain that any $(\mathbb{C}^*)^k$-orbit of dimension $2q$, $1 \leq q \leq k$ is

- either the whole $\mathbb{C}^J$, $J = \{j_1, \ldots, j_q\}$ and

$$\mu_f(\mathbb{C}^J) = \overset{\circ}{C} (\Lambda^{j_1}, \ldots, \Lambda^{j_q})$$

- either it is algebraic manifold $F_J^{-1}(c_1, \ldots, c_{l-q})$, where $J = \{j_1, \ldots, j_l\}$ which is given by the equation

$$z_{j_1}^{\omega_1^s} \cdots z_{j_l}^{\omega_1^s} = c_s, \ 1 \leq s \leq l-q$$

Using a basis $\Lambda^{j_1}, \ldots, \Lambda^{j_q}$ for $C(\Lambda^{j_1}, \ldots, \Lambda^{j_l})$ we obtain:
let $z \in F_J^{-1}(c_1, \ldots, c_{l-q})$ then

$$\mu_f(z) = \sum_{1 \leq s \leq q} |z_{j_s}|^2 \Lambda^{j_s} + \sum_{q+1 \leq s \leq l} |\tilde{c}_s|^2 |z_{j_1}^{\beta_1^s}|^2 \cdots |z_{j_q}^{\beta_q^s}|^2 \Lambda^{j_s},$$

for some $(\tilde{c_{q+1}}, \ldots, \tilde{c}_l) \in (\mathbb{C}^*)^{l-q}$ and some $\beta_i^j \in \mathbb{Z}$ which are uniquely defined. Consequently:

$$\mu_f(F_J^{-1}(c_1, \ldots, c_{l-q})) = \overset{\circ}{C} (\Lambda^{j_1}, \ldots, \Lambda^{j_l}).$$

The next axiom gives the conditions under which exists the solution of the second task.

Axiom 7:

For any chart (M_i, φ_i) the fixed point x_i and the vertex $v_i = \mu(x_i)$ are compatible by the map $\psi_i : C_i \to C_{P,i}$ and there exists almost standard moment map $\mu_i : \mathbb{C}^n \to C_i$ such that on M_i it holds

$$\mu = \psi_i \circ \mu_i \circ \varphi_i.$$

The first important consequence is:

Theorem

The characteristic function χ is constant on any admissible set.

The direct corollaries are:

- Any fixed point x_{i_l}, $1 \leq l \leq m$ is an admissible set: $W_{i_l} = M_{i_l} \cap Y_{i_1} \cap \cdots \cap \hat{Y}_{i_l} \cap \ldots \cap Y_{i_m}$.
- Let H be a codimension one subgroup of T^k and let some $X^H \cong S^2$. Then X^H without fixed points $\{x_{i_1}, x_{i_2}\}$ is an admissible set: $W_{i_1,i_2} = M_{i_1} \cap M_{i_2} \cap Y_{i_3} \cap \ldots \cap Y_{i_m}$.

- $x \in M^{2n}$ is a fixed point if and only if $\mu(x)$ is a vertex.
- $x \in M^{2n}$ has one-dimensional orbit if and only if $\mu(x) \in (v_i, v_j)$ for some admissible polytope (v_i, v_j)

An admissible set is a stratum

Let $H = \chi(W_\sigma)$ and $T_\sigma = T^k/H$.

- The group T_σ acts freely on W_σ.
- There is the fiber bundle $\widehat{\mu} : W_\sigma/T_\sigma \to \overset{\circ}{P}_\sigma$ with the fiber F_σ. Note: we proved that if $|\sigma| \leq 2$ then F_σ is a point.
- The fiber bundle $\widehat{\mu} : W_\sigma/T_\sigma \to \overset{\circ}{P}_\sigma$ is isomorphic to the trivial bundle. Thus there exists a projection $W_\sigma/T_\sigma \to F_\sigma$ which is the homeomorphism on the fibers of the bundle $\widehat{\mu} : W_\sigma/T_\sigma \to \overset{\circ}{P}_\sigma$.
- Then $W_\sigma = \cup_{c_\sigma \in F_\sigma} W_\sigma^{c_\sigma}$, where $W_\sigma^{c_\sigma}$ is the preimage of $c_\sigma \in F_\sigma$ by the composition $W_\sigma \to W_\sigma/T_\sigma \to F_\sigma$.

Definition

Any admissible set W_σ we call a stratum which consists of the leafs $W_\sigma^{c_\sigma}$.

Note. We use the terms stratum and leaf in more general setting motivated by the situation when T^k-action extends to $(\mathbb{C}^*)^k$-action in which case $W_\sigma^{c_\sigma}$ are $(\mathbb{C}^*)^k$-orbits.

By the construction we obtain:

For any $c_\sigma \in F_\sigma$ the map $\mu : W_\sigma^{c_\sigma} \to \overset{\circ}{P}_\sigma$ is a fiber bundle with the fiber T_σ. Hence $W_\sigma^{c_\sigma}$ is homeomorphic to $\overset{\circ}{P}_\sigma \times T_\sigma$ and $\mu(\overline{W_\sigma^{c_\sigma}}) = P_\sigma$.

We use the description of the admissible sets as the strata consisting of leafs to formulate the properties which allow to describe the gluing of the admissible sets.

Axiom 8:

For any $c_\sigma \in F_\sigma$, the boundary $\partial W_\sigma^{c_\sigma}$ of the leaf $W_\sigma^{c_\sigma}$ of the stratum W_σ is the union of the leafs $W_{\bar\sigma}^{c_{\bar\sigma}}$ for exactly one $c_{\bar\sigma} \in F_{\bar\sigma}$, where $P_{\bar\sigma}$ runs through the admissible faces for P_σ.

Note. Axiom 8 is motivated by the results of Atiyah, Guillemin-Sternberg and Gel'fand-MacPherson about $(\mathbb{C}^*)^k$-action on M^{2n}.

Let Axiom 8 is satisfied. Since $\mu(\overline{W_\sigma^{c_\sigma}}) = P_\sigma$ we obtain:

Lemma

For any $c_\sigma \in F_\sigma$, the boundary $\partial W_\sigma^{c_\sigma}$ of the leaf $W_\sigma^{c_\sigma}$ of the stratum W_σ is the union of the leafs $W_{\bar\sigma}^{c_{\bar\sigma}}$ for exactly one $c_{\bar\sigma} \in F_{\bar\sigma}$, where $P_{\bar\sigma}$ runs through the all faces for P_σ.

Corollary. A face of any admissible polytope is an admissible polytope.

The properties of the set of admissible polytopes $\mathfrak{S}$

- $P^k \in \mathfrak{S}$ and $v \in \mathfrak{S}$ for any vertex v.
- Let $P_\sigma \in \mathfrak{S}$ and $P_{\bar\sigma}$ is a face of P_σ. Then $P_{\bar\sigma} \in \mathfrak{S}$.

If Axiom 6 is satisfied define the function

$$\widehat{\chi} : \mathfrak{S} \to S(T^k) \text{ by } \widehat{\chi}(P_\sigma) = \chi(x), \; x \in W_\sigma.$$

By construction we have:

If $P_{\bar\sigma}$ is a facet of P_σ then $\widehat{\chi}(P_\sigma) \subseteq \widehat{\chi}(P_{\bar\sigma})$.

Moreover it holds:

Proposition

If $P_\sigma \in \mathfrak{S}$ and $P_{\bar\sigma}$ is a facet of P_σ then $\widehat{\chi}(P_\sigma) \subset \widehat{\chi}(P_{\bar\sigma})$.

Theorem

Codim $\chi(W_\sigma) =$ dim T_σ is the same as the dim P_σ for any stratum W_σ.

Corollary. $dim W_\sigma^{c_\sigma} = 2 dim P_\sigma$ for any admissible polytope P_σ and any $c_\sigma \in F_\sigma$

CW complex of admissible polytopes

Let $\mathfrak{S}$ as before be the family of admissible polytopes.
It is defined the operator

$$d : \mathfrak{S} \to S(\mathfrak{S}) \text{ by } dP_\sigma \text{ is disjoint union of the faces of } P_\sigma.$$

We obtain CW complex $CW(M^{2n}, P^k)$: the cells are the admissible polytopes from $\mathfrak{S}$ and we glue them by induction using the operator d.

There is the canonical map $\pi : CW(M^{2n}, P^k) \to P^k$.

For any $\overset{\circ}{P}_\sigma \in \mathfrak{S}$ there is the cell P'_σ in $CW(M^{2n}, P^k)$ such that the map $\pi : P'_\sigma \to \overset{\circ}{P}_\sigma$ is a homeomorphism.

It implies:

$$x \in M^{2n} \Rightarrow \exists! W_\sigma,\ x \in W_\sigma \Rightarrow \mu(x) \in P_\sigma,$$
$$\exists! y \in P'_\sigma \subseteq CW(M^{2n}, P^k),\ \pi(x) = \mu(y).$$

We obtain:

Lemma

There is the canonical map $f : M^{2n} \to CW(M^{2n}, P^k)$ such that $\mu = \pi \circ f$.

Note. In general case operator $\tilde{d}$ defined on the admissible sets by $\tilde{d}(W_\sigma) = \overline{W_\sigma} - W_\sigma$ does not allow to construct the complex of admissible sets which covers $CW(M^{2n}, P^k)$. Although there are important examples of $(2n, k)$-manifolds for which $CW(M^{2n}, P^k)$ is covered by the complex of admissible sets.

$(2n, k)$ - compactification

Proposition

F_σ is either a point either non-compact set for any admissible set W_σ

Consider the main stratum $W = W_{\{1,\dots,m\}} = M_1 \cap \cdots \cap M_m$.

- T^k acts freely on W and $\overline{W/T^k} = M^{2n}/T^k$
- $\widehat{\mu}(W/T^k) = \overset{\circ}{P}{}^k$ and $W/T^k \cong \overset{\circ}{P}{}^k \times F$, where $F = F_{\{1,\dots,m\}}$
- $W = \cup_{c \in F} W^c$.

We obtain:

Corollary. If F is a point then $M^{2n}/T^k \cong P^k$.

Assume that F is not a point $\Longrightarrow$ F is a non-compact set.

Let $v \in \overset{\circ}{P^k}$ and denote by $\mathfrak{S}(v) = \{P_\sigma \in \mathfrak{S} | v \in \overset{\circ}{P_\sigma}\}$. Then

- $F_\sigma(v) = \widehat{\mu}^{-1}(v) \cap W_\sigma/T_\sigma \cong F_\sigma$ for any $P_\sigma \in \mathfrak{S}(v)$,
- $\widehat{\mu}^{-1}(v) = \cup_{P_\sigma \in \mathfrak{S}(v)} F_\sigma(v)$ is a compact set.
- $\overline{F(v)} = \widehat{\mu}^{-1}(v) \cap \overline{W/T^k} = \widehat{\mu}^{-1}(v)$.

Denote by $\mathfrak{S}'(v) = \mathfrak{S}(v) - P^k$. Then

$$\partial F(v) = \cup_{P_\sigma \in \mathfrak{S}'(v)} F_\sigma(v).$$

Complex Grassmann manifolds $G_{k+1,q}$

$G_{k+1,q}$ - q-dimensional subspaces in $\mathbb{C}^{k+1}$;

T^{k+1} acts on $G_{k+1,q}$ by the canonical action on $\mathbb{C}^{k+1}$ which induces an effective action of T^k on $G_{k+1,q}$.

Theorem

$G_{k+1,q}$ has the canonical structure of $(2q(k-q+1), k)$-manifold.

Note: Plücker coordinates give the equivariant embedding of $G_{k+1,q}$ into $\mathbb{C}P^N$, where $N = \binom{k+1}{q} - 1$.

The standard moment map on $\mathbb{C}P^n$ gives the moment map $\mu : G_{k+1,q} \to \mathbb{R}^{k+1}$ (see Gel'fand-MacPherson):

$$\mu(X) = \frac{\sum_J |P^J(X)|^2 \delta_J}{\sum_J |P^J(X)|^2},$$

where $\delta_J \in \mathbb{R}^{k+1}$ is given by

$$(\delta_J)_i = 1, \quad i \in J, \quad (\delta_J)_i = 0, \quad i \notin J.$$

$$\text{Image}\mu = \text{convexhull}(\delta_J) \ - \ \text{hypersimplex} \ \ \Delta_{k+1,q}$$

About hypersimplex

Hypersimplex $\Delta_{k+1,q}$ - convex hull of vectors δ_J in $\mathbb{R}^{k+1}$ having 1 at q places and 0 at the other $k+1-q$ places

- $\Delta_{k+1,q} = I^{k+1} \cap \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} : x_1 + \cdots + x_{k+1} = q\}$, where I^{k+1} is $(k+1)$-dimensional cube.
- It has $\binom{k+1}{q}$ vertices.
- At each vertex it has $q(k-q+1)$ edges.
- It is simple iff $l = 1$ or $q = k$.

Example: $\Delta_{k+1,1}$ – standard simplex.

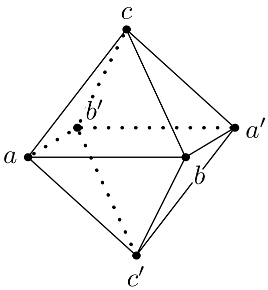
Complex projective spaces $\mathbb{C}P^n$

$G_{k+1,q}$ is an algebraic submanifold in $\mathbb{C}P^n$, where $n = \binom{k+1}{q} - 1$.

Theorem

$(2q(k-q+1), k)$-structure on $G_{k+1,q}$ extends to $(2n, k)$-structure on $\mathbb{C}P^n$.

$\mu(G_{4,2}) = \Delta_{4,2}$ – octahedron



Orbit space $G_{4,2}/T^3$

Theorem

$X = G_{4,2}/T^3$ is homeomorphic to the quotient space

$$(\Delta_{4,2} \times CP^1)/\approx$$

$$\text{where } (x, c) \approx (y, c') \Leftrightarrow x = y \in \partial\Delta_{4,2}.$$

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Corollary

$G_{4,2}/T^3$ is homeomorphic to the join $S^2 * S^2$.

Orbit space $G_{4,2}/T^3$

Theorem

$G_{4,2}/T^3$ is a topological manifold without boundary, and, thus, $G_{4,2}/T^3$ is homeomorphic to the sphere S^5.

$T^4 \hookrightarrow \mathbb{C}P^5$ by the representation $T^4 \to T^6$:

$$(t_1, t_2, t_3, t_4) \to (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4).$$

Theorem

$$\mathbb{C}P^5/T^3 \cong (\Delta_{4,2} \times \mathbb{C}P^2)/\approx$$

$$\text{where } (x, c) \approx (y, c') \Leftrightarrow x = y \in \partial\Delta_{4,2}$$

Corollary

$$\mathbb{C}P^5/T^3 \cong S^2 * \mathbb{C}P^2$$

$G_{k+1,q}$ - q-dimensional subspaces in $\mathbb{C}^{k+1}$;

T^{k+1} acts on $G_{k+1,q}$ by the canonical action on $\mathbb{C}^{k+1}$ which induces an effective action of T^k on $G_{k+1,q}$.

Theorem

$G_{k+1,q}$ has the canonical structure of $(2q(k-q+1), k)$-manifold.

- Any q-dimensional subspace in $\mathbb{C}^{k+1}$, up to choice of a basis in this subspace, is represented by $(k+1)\times q$ matrix A, $rang A = q$;
- For any $J \subset \{1, \dots, k+1\}$, $|J| = q$, denote by A_J the matrix of dimension $q \times q$ given by the rows of A indexed by J.
- The set of all subsets $J = \{j_1 < j_2 \dots < j_q\} \subset \{1, \dots, k+1\}$ is ordered lexicographically.
- Using this ordering it is defined the map

$$P(A) = (P^J(A)) = (\det A_J) \; - \; \text{Plücker coordinates},$$

- Plücker coordinates, up to constant, are uniquely defined;
- Plücker coordinates give the embedding of $G_{k+1,q}$ into $\mathbb{C}P^{\binom{k+1}{q}-1}$.

Atlas for $G_{k+1,l}$: (M_J, φ_J), $J \subset \{1, \dots, k+1\}$, $|J| = l$ – given by

$$M_J = \{X \in G_{k+1,l} \mid P^J(X) \neq 0\}, \quad \varphi_J : M_J \to \mathbb{C}^{l(k+1-l)}.$$

$X \in M_J$ can be represented by the matrix A such that $A_J = I_d$ and

$$\varphi_J(X) = (a_{ij}(X)) \in \mathbb{C}^{l(k+1-l)}, \quad i \notin J$$

Atlas for $G_{k+1,l}$: (M_J, φ_J), $J \subset \{1, \dots, k+1\}$, $|J| = l$ – given by

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$X \in M_J$ can be represented by the matrix A such that $A_J = I_d$ and

$$\varphi_J(X) = (a_{ij}(X)) \in \mathbb{C}^{l(k+1-l)}, \quad i \notin J$$

- The charts M_J are T^{k+1}-invariant,
- The homomorphism $\alpha_J : T^k \to T^{l(k+1-l)}$, for example for $J = \{1, \dots, l\}$ we obtain as:
  1. consider the homomorphism $T^{k+1} \to T^{l(k+1-l)}$ given by the canonical action of T^{k+1} in M_J: $\tau_{ij} = \frac{\bar{t}_i}{\bar{t}_j}$, $1 \leq j \leq l$, $l+1 \leq i \leq k+1$
  2. Put
  $$\alpha_J^{(l+1)q}(t_1, \dots, t_k) = \tau_{(l+1)q} = t_q, \quad 1 \leq q \leq l$$
  $$\alpha_J^{(q+1)l} = \tau_{(q+1)1} = t_q, \quad l+1 \leq q \leq k$$
  3. It implies

$$\alpha_J^{ij} = \tau_{ij} = \frac{t_{i-1} t_j}{t_1}, \quad 2 \leq j \leq l, \ l+2 \leq i \leq k+1$$

- $W_{J_1,\dots,J_l} = \{X \in G_{k+1,l} : P^{J_i} \neq 0$ for $1 \leq i \leq l$ and $P^J = 0$ for $J \neq J_1,\dots,J_l\}$
- $\widehat{\mu}(W_{J_1,\dots,J_l})/T^k = \overset{\circ}{\sigma}$, where $\sigma = convexhull(\delta_{J_1},\dots,\delta_{J_l})$ is a fiber bundle - follows from the representation in a charts.
- σ-admissible $\Rightarrow$ any facet of σ is admissible.
- The cones C_J and $C_{\Delta_{k+1,l},J}$ are generated by the same number of vectors.

Proposition

The stratification of Grassmann manifolds by the strata W_σ coincides with the one of the equivalent stratification defined by Ge'fand-Serganova-Goresky-MacPherson:

- *by "small" Schubert cells - intersection of the Schubert cells from $n!$ Schubert cell decomposition,*
- *by the moment map - the points whose $(\mathbb{C}^*)^{k+1}$-orbits have the same image,*
- *by the function $s : S(\{1,\dots,n\}) \to \mathbb{Z}$ - $\{X \in G_{k+1,l} | dim(X \cap \mathbb{C}^J) = s(J)\}$.*

$G_{4,2}$ - instructive example

We explicitly demonstrate Axiom 7 for $G_{4,2}$

Fix the chart M_{12}:

- Let $V_j = \delta_{pq} - \delta_{12}$, where $\{p, q\} \neq \{1, 2\}$, $1 \leq j \leq 5$. We assume here that $\{p, q\} \subset \{1, \dots, 4\}$, $\{p, q\} \neq \{1, 2\}$ are ordered lexicographically.
- Note that in such ordering: $V_5 = V_2 + V_3 = V_1 + V_4$
- Then $\mu \circ \varphi_{12}^{-1} : \mathbb{C}^4 \to \Delta_{4,2}$ can be written as

$$(z_1, z_2, z_3, z_4) \to \delta_{12} + \frac{\sum_{j=1}^{4} |z_j|^2 V_j + |z_1 z_4 - z_2 z_3|^2 V_5}{1 + \sum_{j=1}^{4} |z_j|^2 + |z_1 z_4 - z_2 z_3|^2}$$

- Consider $\xi_{12} : \mathbb{C}^4 \to \mathbb{R}^4_{\geq 0}$ given by

$$\xi_{12}(\mathbf{z}) = \kappa_0(\mathbf{z}) \cdot (|z_1|, \sqrt{|z_2|^2 + |z_1 z_4 - z_2 z_3|^2}, \sqrt{|z_3|^2 + |z_1 z_4 - z_2 z_3|}, |z_4|),$$

$$\kappa_0(\mathbf{z}) = \frac{1}{\sqrt{1 + \sum_{j=1}^{4} |z_j|^2 + |z_1 z_4 - z_2 z_3|^2}}.$$

It is an admissible map.

- The homomorphism $\alpha_{12} : T^3 \to T^4$ is given by

$$\alpha_{12}(t_1, t_2, t_3) = (t_1, t_2, t_3, \frac{t_2 t_3}{t_1})$$

- The weight vectors are $\Lambda_{12} = \{\Lambda_{12}^1 = (1, 0, 0), \Lambda_{12}^2 = (0, 1, 0), \Lambda_{12}^3 = (0, 0, 1), \Lambda_{12}^4 = (-1, 1, 1)\}$;
- The map $\psi_{12}^j : \{r\Lambda_{12}^j | r \geq 0\} \to C_{\Delta_{4,2}, \delta_{12}}$ is given by

$$\psi_{12}^j(r\Lambda_{12}^j) = \mu(\varphi^{-1}(z_j = r, 0, 0, 0)) = \delta_{12} + r^2 V_j$$

- The canonical map $\eta_{12} : R_{\geq 0}^4 \to C_{\Delta_{4,2}, \delta_{12}}$ is given by

$$\eta_{12}(x_1, \ldots, x_4) = \delta_{12} + \sum_{j=1}^{4} x_j^2 V_j.$$

- Note that $\gamma_{12} : R_{\geq 0}^4 \to C_{12}$ by $\gamma_{12}(x_1, \ldots, x_4) = \sum_{j=1}^4 x_j \Lambda_{12}^j$.
- We obtain the map $\psi_{12} : C_{12} \to C_{\Delta_{4,2}, \delta_{12}}$ defined by $\eta_{12} = \psi_{12} \circ \gamma_{12}$, which is homeomorphism of the cones.
- Therefore we have

$$\mu = \psi_{12} \circ \gamma_{12} \circ \xi_{12} \circ \varphi_{12}.$$

Complex projective spaces $\mathbb{C}P^n$

$G_{k+1,q}$ is an algebraic submanifold in $\mathbb{C}P^n$, where $n = \binom{k+1}{q} - 1$.

Theorem

$(2q(k-q+1), k)$-structure on $G_{k+1,q}$ extends to $(2n, k)$-structure on $\mathbb{C}P^n$.

- consider the action of T^k on $\mathbb{C}P^n$ induced by the representation $T^k \to T^n$ given by the l-th exterior power of the standard representation of T^{k+1}.
- the corresponding polytope is again hypersimplex $\Delta_{k+1,l}$.
- Each cone C_i has $n = \binom{k+1}{l} - 1$ generating vectors, while the number of edges at a vertex for $\Delta_{k+1,l}$ is $l(k-l+1)$.

$\mathbb{C}P^5$ - instructive example

$T^4 \hookrightarrow \mathbb{C}P^5$ by the representation $\alpha : T^4 \to T^6$:

$$(t_1, t_2, t_3, t_4) \to (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4)$$

It induces an effective action T^3 on $\mathbb{C}P^5$.

The standard moment moment map for $\mathbb{C}P^5$ gives:

$$\mu(z_0 : \ldots : z_5) = \frac{1}{1 + \|z\|^2}(|z_0|^2\delta_{12}+|z_1|^2\delta_{13}+|z_2|^2\delta_{14}+|z_3|^2\delta_{23}+|z_3|^2\delta_{24}+|z_5|^2\delta_{34})$$

We consider the Axiom 7.

Fix the chart $M_0 = \{(z_0 : \ldots : z_5) \mid z_0 \neq 0\}$, $\varphi_0 : M_0 \to \mathbb{C}^5$ by $\varphi_0(1 : z_1 : \ldots : z_5) = (z_1, \ldots, z_5)$

- Let $V_j = \delta_{pq} - \delta_{12}$, where $\{p, q\} \neq \{1, 2\}$, $1 \leq j \leq 5$ We assume here that $\{p, q\} \subset \{1, \ldots, 4\}$, $\{p, q\} \neq \{1, 2\}$ are ordered lexicographically.
- Note that in such ordering: $V_5 = V_2 + V_3 = V_1 + V_4$
- Then $\mu \circ \varphi_0^{-1} : \mathbb{C}^5 \to \Delta_{4,2}$ can be written as

$$(z_1, z_2, z_3, z_4, z_5) \to \delta_{12} + \frac{1}{1 + \sum_{j=1}^{5} |z_j|^2} \sum_{j=1}^{5} |z_j|^2 V_j$$

- $C_{\Delta_{4,2},\delta_{12}} = \{\delta_{12} + r_1 V_1 + r_2 V_2 + r_3 V_3 + r_4 V_4,\ r_j \in \mathbb{R}_{\geq 0}\}$
- Consider $\xi_0 : \mathbb{C}^5 \to \mathbb{R}^5_{\geq 0}$ given by

$$\xi_0(z_1, z_2, z_3, z_4, z_5) = \frac{1}{\sqrt{1 + \sum_{j=1}^{5} |z_j|^2}} (|z_1|, |z_2|, |z_3|, |z_4|, |z_5|)$$

- The homomorphism $\alpha_0 : T^3 \to T^5$ is given by

$$\alpha_0(t_1, t_2, t_3) = (t_1, t_2, t_3, \frac{t_2 t_3}{t_1}, t_2 t_3)$$

- The weight vectors are $\Lambda_0 = \{\Lambda_0^1 = (1, 0, 0), \Lambda_0^2 = (0, 1, 0), \Lambda_0^3 = (0, 0, 1), \Lambda_0^4 = (-1, 1, 1), \Lambda_0^5 = (0, 1, 1)\}$;
- The map $\psi_0^j : \{r\Lambda_0^j\} \to C_{\Delta_{4,2},\delta_{12}}$ is defined by

$$\psi_0^j(r\Lambda_0^j) = \delta_{12} + r^2 V_j$$

  is admissible.
- Note: $\psi_0^5(\Lambda_0^5) = \delta_{12} + V_5$ - belongs to the interior of $C_{\Delta_{4,2},\delta_{12}}$ as $V_5 = V_2 + V_3$
- We obtain the map $\psi_0 : C_0 \to C_{\Delta_{4,2},\delta_{12}}$

$$\psi_0(\sum_{j=1}^{4} r_j \Lambda_0^j) = \delta_{12} + \sum_{j=1}^{4} r_j^2(\psi_0(\Lambda_0^j) - \delta_{12})$$

- Therefore,

$$\mu = \psi_0 \circ \gamma_0 \circ \xi_0 \circ \varphi_0$$

In local chart M_{12} the point is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}.$$

The strata with admissible polytope σ:

- $a_i = 0$ for all $i \Rightarrow W_\sigma = M_{12} \cap Y_{ij}$ for $\sigma = \delta_{12}$;
- $a_{i_1} = a_{i_2} = a_{i_3} = 0 \Rightarrow W_\sigma = M_{12} \cap M^{1i_4} \cap Y_{kl}$ for σ a edge on $\partial\Delta_{4,2}$ with a vertex δ_{12} ;
- $a_{i_1} = a_{i_2} = 0 \Rightarrow W_\sigma = M_{12} \cap M_{1i_3} \cap M_{1i_4} \cap Y_{kl}$ for σ a triangle on $\partial\Delta_{4,2}$ or diagonal square with a vertex δ_{12};
- $a_{i_1} = 0 \Rightarrow W_\sigma = M_{kl} \cap Y_{1i_1}$ for σ a pyramid with a vertex δ_{12} at the base;
- $a_1 a_4 = a_2 a_3 \Rightarrow W_\sigma = Y_{34} \cap M_{kl}$ for σ a pyramid wit a top-vertex δ_{12}.
- $a_1 \cdots a_4 \neq 0, \;\; a_1 a_4 \neq a_2 a_3 \Rightarrow W_\sigma = \cap M_{kl}$ for $\sigma = \Delta_{4,2}$

The admissible polytopes are:

1. $\Delta_{4,2}$;
2. any four-sided pyramid;
3. three diagonal squares;
4. any face on the boundary for $\Delta_{4,2}$.

Corollary

The strata on Grassmannian $G_{4,2}$, dimension and the number:

$$\begin{bmatrix} 8 & 6 & 4 & 2 & 0 \\ 1 & 6 & 11 & 12 & 6 \end{bmatrix}.$$

1. *the strata of dimension 8 is an open everywhere dense in $G_{4,2}$;*
2. *each strata of dimension ≤ 6 consists of one orbit.*

Orbit space $G_{4,2}/T^3$

1. $W_{\Delta_{4,2}}/T^3 \cong \overset{\circ}{\Delta}_{4,2} \times F_{\Delta_{4,2}}$
2. $W_{\Delta_{4,2}} = \{(z_1, z_2, z_3, z_4) | \frac{z_1 z_4}{z_2 z_3} = c, c \neq 0, 1\}$
3. the hypersurfaces $W^c_{\Delta_{4,2}} = \frac{z_1 z_4}{z_2 z_3}$ are T^3-invariant and $\mu : W^c_{\Delta_{4,2}} \to \overset{\circ}{P}{}^k$ is a homeomorphism,
4. $W_{\Delta_{4,2}}/T^3 \cong \overset{\circ}{\Delta}_{4,2} \times (\mathbb{C} - \{0, 1\})$.

Using this we continuously parametrize by $c \in \mathbb{C}P^1$ all strata in $G_{4,2}$.

Orbit space $G_{4,2}/T^3$

Proposition

For the 6-dimensional strata with admissible polytope σ_{ij} the set F_σ is:

- *for* $\sigma_{14} = \Delta_{4,2} - \delta_{14}$ *or* $\sigma_{23} = \Delta_{4,2} - \delta_{23} \longrightarrow c = 0$;
- *for* $\sigma_{13} = \Delta_{4,2} - \delta_{13}$ *or* $\sigma_{24} = \Delta_{4,2} - \delta_{24} \longrightarrow c = \infty$;
- *for* $\sigma_{12} = \Delta_{4,2} - \delta_{12}$ *or* $\sigma_{34} = \Delta_{4,2} - \delta_{34} \longrightarrow c = 1$.

Proposition

For the $2l$-dimensional strata, where $l \leq 2$, with admissible polytope σ the set F_σ is:

- *if* $l = 0, 1 \longrightarrow$ *any* $c \in CP^1$;
- *if* $l = 2$ *and* σ *is a triangle* $\longrightarrow$ *any* $c \in CP^1$;
- *if* $l = 2$ *and* σ *is a square:*
  1. $\sigma_{14,23} = \Delta_{4,2} - \{\delta_{14}, \delta_{23}\} \longrightarrow c = 0$,
  2. $\sigma_{13,24} = \Delta_{4,2} - \{\delta_{13}, \delta_{24}\} \longrightarrow c = \infty$,
  3. $\sigma_{12,34} = \Delta_{4,2} - \{\delta_{12}, \delta_{34}\} \longrightarrow c = 1$.

Orbit space $G_{4,2}/T^3$

Corollary

- $\sigma_{14} \cup \sigma_{23} \cup \sigma_{14,23} = \Delta_{4,2} \longrightarrow c = 0,$
- $\sigma_{13} \cup \sigma_{24} \cup \sigma_{13,24} = \Delta_{4,2} \longrightarrow c = \infty,$
- $\sigma_{12} \cup \sigma_{34} \cup \sigma_{12,34} = \Delta_{4,2} \longrightarrow c = 1.$

This leads to:

Theorem

$X = G_{4,2}/T^3$ is homeomorphic to the quotient space

$$(\Delta_{4,2} \times CP^1)/\approx \ \textit{where}\ (x, c) \approx (y, c') \Leftrightarrow x = y \in \partial\Delta_{4,2}.$$

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Corollary

$G_{4,2}/T^3$ is homeomorphic to the join $S^2 * S^2$.

Orbit space $G_{4,2}/T^3$

Theorem

$G_{4,2}/T^3$ is a topological manifold without boundary, and, thus, $G_{4,2}/T^3$ is homeomorphic to the sphere S^5.

We have:

$$f : G_{4,2}/T^3 \cong \partial\Delta_{4,2} * S^2 \longrightarrow \Delta_{4,2} - - - \text{projection}$$

$$\mu = \pi \circ f.$$

Orbit space $\mathbb{C}P^5/T^3$

$T^4 \hookrightarrow CP^5$ by the representation $T^4 \to T^6$:

$$(t_1, t_2, t_3, t_4) \to (t_1 t_2, t_1 t_3, t_1 t_4, t_2 t_3, t_2 t_4, t_3 t_4).$$

We prove:

$$P = \Delta_{4,2}$$

$$CP^5/T^4 \cong \partial\Delta_{4,2} * CP^2,$$

$$f : \partial\Delta_{4,2} * CP^2 \to \Delta_{4,2} \quad --- \quad \text{projection}$$

Any polytope spanned by some subset of vertices for $\Delta_{4,2}$ is admissible polytope;

$M_0 = \{z_0 \neq 0\}$ – chart on CP^5 – coordinates $(z_1, \dots, z_5)$;

$$T^4 \cdot (a_1, \dots, a_5) = (\frac{t_3}{t_2} a_1, \frac{t_4}{t_2} a_2, \frac{t_3}{t_1} a_3, \frac{t_4}{t_1} a_4, \frac{t_3 t_4}{t_1 t_2} a_5) =$$

$$(\bar{t}_1 a_1, \bar{t}_2 a_2, \bar{t}_3 a_3, \frac{\bar{t}_2 \bar{t}_3}{\bar{t}_1} a_4, \bar{t}_2 \bar{t}_3 a_5).$$

The strata with admissible polytopes:

1. $(0, 0, 0, 0, 0)$ — point δ_{12};
2. $a_{i_j} = 0,\ 1 \leq j \leq 4$ — edges having vertex δ_{12} ;
3. $a_{i_j} = 0,\ 1 \leq j \leq 3$ — triangles having vertex δ_{12};
4. $a_1 = a_4 = 0$ or $a_2 = a_3 = 0$ — squares having vertex δ_{12};
5. $a_{i_1} = a_{i_2} = 0$ for $\{i_1, i_2\} \neq \{1, 4\}, \{2, 3\}$ — tetrahedra having vertex δ_{12};

1. surfaces: $a_i = 0$
— four-sided pyramids having vertex δ_{12};
2. $a_1 \cdots a_5 \neq 0$ — $\Delta_{4,2}$

Orbit space CP^5/T^3

Let

$$W_{\Delta_{4,2}}^{(c_1,c_2)} = \{\frac{z_2 z_3}{z_5} = c_1 \ \wedge \ \frac{z_1 z_4}{z_5} = c_2\},$$

where $c_1 c_2 \neq 0$. Then $\mu : W_{\Delta_{4,2}}^{(c_1,c_2)} \to \overset{\circ}{\Delta}_{4,2}$ is a homeomorphism.

$$W_{\Delta_{4,2}}/T^3 \cong \overset{\circ}{\Delta}_{4,2} \times F_{\Delta_{4,2}},$$

$F_{\Delta_{4,2}} = \{(c_1 : c_2 : 1), c_1, c_2 \neq 0\} \subseteq \mathbb{C}P^2$.

Theorem

Using the parametrization of the main stratum, each stratum which does not map to $\partial\Delta_{4,2}$

- *can be parametrized by $(0 : c_2 : 1)$, $c_1 \neq 0$ or $(c_1 : 0 : 1)$, $c_2 \neq 0$ or $(0 : 0 : 1)$ or $(c_1 : c_2 : 0)$, $c_1, c_2 \in C$, $(c_1, c_2) \neq (0, 0)$.*
- *They can be divided into four groups such that:*
  1. *All strata from the same group are equally parametrized;*
  2. *The admissible polytopes for the strata from the same groups glue together to give $\Delta_{4,2}$.*

The strata on $\partial\Delta_{4,2}$ can be parametrized by CP^2.

Orbit space CP^5/T^3

This leads:

Theorem

$$CP^5/T^3 \cong (\Delta_{4,2} \times CP^2)/\approx \ \ where\ (x,c) \approx (y,c') \Leftrightarrow x = y \in \partial\Delta_{4,2}$$

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$$CP^5/T^3 \cong S^2 * CP^2$$

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Corollary

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Remark: Embedding $G_{4,2} \subset CP^5$ by the Plücker coordinates is equivariant for T^3-action $\Longrightarrow G_{4,2}/T^3 \subset CP^5/T^3$.
In homogeneous coordinates:

$$G_{4,2} \subset CP^5 : \quad z_1z_6 + z_3z_4 = z_2z_5$$

$$G_{4,2}/T^3 \subset CP^5/T^3 : \quad S^2 * CP^1 \subseteq S^2 * CP^2, \text{ where } CP^1 \subset CP^2$$

$$(c,1) \to (c:1:(1-c)), \;\; (1,0) \to (0,0,1).$$