Toric structure of $(2n, k)$-manifolds.

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The toric structure of a quasitoric manifold M^{2n} is completely described in terms of a simple polytope P^n and a characteristic function which is in this case given by the canonical mapping from the orbit space M^{2n}/T^n to the set of subgroups of T^n. The description of toric structure on such manifold M^{2n} essentially uses the fact that the orbit space M^{2n}/T^n is homeomorphic to P^n.

There are many papers devoted to the canonical action of the algebraic torus $(\mathbb{C}^*)^m$ on the complex Grassmann manifolds $G_{m,q}$, $m > q$, and other complex generalized flag manifolds. On the other hand the well known and important problem to describe the orbit space for the induced action of the compact torus $T^m \subset (\mathbb{C}^*)^m$ on the Grassmann manifold $G_{m,q}$ is far from being solved. The action of $(\mathbb{C}^*)^m$ allows to define the moment map from $G_{m,q}$ to $\mathbb{R}^{m-1}$ whose image is a convex polytope P^{m-1}, which is not simple in general. The orbit space $G_{m,q}/T^m$ is essentially different from P^{m-1} as well.

We consider an action of a compact torus T^k on a smooth, closed manifold M^{2n}. The initial goal of our research on this topic was, appealing to the results on quasitoric and Grassmann manifolds, to extract the structural common properties which are essential for the description of their orbit space M^{2n}/T^n.

The first property we require is the existence of an analogue of the moment map which is a map from M^{2n} to $\mathbb{R}^k$ constant on T^k-orbits and whose image is a convex polytope P^k. We call it an almost moment map. The existence of such map is not any more enough for the description of the orbits space structure. The reason is that it may happen that some convex polytopes over a vertices of P^k which are not the faces of P^k are of the structural importance for the description of the orbit space as well. This leads us to distinguish the set of polytopes which consist of some polytopes over the vertices of P^k and to call them admissible polytopes.

The next property uses the canonical mapping from M^{2n}/T^k to the set of subgroups of T^k, which we call characteristic function. We require the existence of a function from the set of admissible polytopes to the set of subgroups of T^k which is naturally related to an almost moment map and the characteristic function.

We propose the description of these structural properties through the set of axioms. The manifolds which satisfy these axioms we call $(2n, k)$-manifolds. We provide the verification of these axioms in the case of quasitoric $M^{2n},\ k = n$, and Grassmann manifolds $G_{m,q},\ n = q(m-q),\ k = m-1$. The study of $(2n, k)$-structure on Grassmann manifolds brings us to study $(2N, k)$-structure on complex projective spaces $\mathbb{C}P^N$ for the case of T^k-equivariant embedding of $G_{m,q}$ in $\mathbb{C}P^N$. We also present the general results we obtained about the toric structure of $(2n, k)$-manifolds and, as a corollary, the results on orbit space of Grassmann manifolds and complex projective spaces. In our approach we use, and develop the methods and results of algebraic geometry of homogeneous spaces and toric topology.

The talk is based on the recent results obtained jointly with Svjetlana Terzić.