Torus Manifolds in Equivariant Complex Bordism

Alastair Darby

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Stably Complex Torus Manifolds

Definition (Torus Manifold)

A *torus manifold* is a $2n$-dimensional smooth compact manifold M with an effective smooth T^n-action whose fixed point set is non-empty.

Note that our fixed point set is finite; we only have isolated fixed points.

Definition (Stably Complex Torus Manifold)

A *stably complex torus manifold* is torus manifold with a complex T^n-structure on

$$\tau(M) \oplus \mathbb{R}^{2k},$$

for some large k.

The Tangential Representation

Let M^{2n} be a stably complex torus manifold and $p \in M^{T^n}$. We have a complex T^n-structure on

$$(\tau(M) \oplus \mathbb{R}^{2k})|_p \cong T_pM \oplus \mathbb{R}^{2k} \cong \tau(p) \oplus \nu_p^M \oplus \mathbb{R}^{2k}.$$

So we can write

$$T_pM = V_1 \oplus \cdots \oplus V_n, \quad \text{where } V_i \in \mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n.$$

Since the T^n-action is effective we have:

Lemma

The irreducible T^n-representations $V_1, \ldots, V_n$ form a basis of $\mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n$.

The Tangential Representation

Notice T_pM has two orientations:

- one from its complex structure
- one from the canonical orientation of M.

Definition (Sign of p)

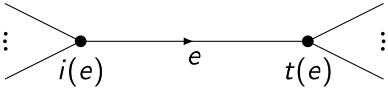
For each isolated fixed point p of a stably complex torus manifold, the *sign of p* is given by

$$\sigma(p) := \begin{cases} +1, & \text{if the two orientations coincide;} \\ -1, & \text{if the two orientations differ.} \end{cases}$$

Torus Graphs (Maeda, Masuda & Panov)

Let

- Γ be an n-valent connected graph with $n \geq 1$.
- $\mathcal{V}(\Gamma)$ denote the set of vertices.
- $\mathcal{E}(\Gamma)$ denote the set of *oriented* edges.



For $p \in \mathcal{V}(\Gamma)$, define

$$\mathcal{E}(\Gamma)_p := \{e \in \mathcal{E}(\Gamma) \mid i(e) = p\}.$$

Torus Graphs (Maeda, Masuda & Panov)

Definition (Torus Axial Function)

A *torus axial function* is a map

$$\alpha \colon \mathcal{E}(\Gamma) \longrightarrow \mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n,$$

satisfying the following conditions:

1. $\alpha(\bar{e}) = \pm\alpha(e)$;
2. elements of $\alpha(\mathcal{E}(\Gamma)_p)$ form a basis of $\mathbb{Z}^n$;
3. $\alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)}) \bmod \alpha(e)$, for any $e \in \mathcal{E}(\Gamma)$.

Definition (Torus Graph)

A *torus graph* is a pair (Γ, α) consisting of an n-valent graph Γ with a torus axial function α.

Example

Example (Torus Manifold)

Let M^{2n} be a torus manifold. Define an n-valent graph Γ_M where

- $\mathcal{V}(\Gamma_M) = M^{T^n}$;
- $\mathcal{E}(\Gamma_M) = \{\text{2-dim submanifolds of } M \text{ fixed by a } T^{n-1} \leq T^n\}$.

Every $S \in \mathcal{E}(\Gamma_M)$ is diffeomorphic to a sphere and contains exactly two T^n-fixed points. The summands $T_pM = V_1(p) \oplus \cdots \oplus V_n(p)$ correspond to the edges $\mathcal{E}(\Gamma_M)_p$. We assign each $e \in \mathcal{E}(\Gamma_M)_p$ to its corresponding $V_i(p)$. This gives a function

$$\alpha_M \colon \mathcal{E}(\Gamma_M) \longrightarrow \mathrm{Hom}(T^n, S^1),$$

which satisfies the three conditions of being a torus axial function and we get a torus graph (Γ_M, α_M).

Oriented Torus Graphs

Definition

An *orientation* of a torus graph (Γ, α) is an assignment

$$\sigma\colon \mathcal{V}(\Gamma) \longrightarrow \{\pm 1\},$$

satisfying

$$\sigma(i(e))\alpha(e) = -\sigma(i(\bar{e}))\alpha(\bar{e}), \quad \text{for every } e \in \mathcal{E}(\Gamma).$$

Example (Stably Complex Torus Manifold)

Set $\sigma(p)$, for $p \in M^{T^n} = \mathcal{V}(\Gamma_M)$, to agree with the definition of the sign of an isolated fixed point of a stably complex torus manifold.

Free Exterior Algebra

Definition

Let J_n denote the set of non-trivial elements of $\mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n$.

Consider the free exterior $\mathbb{Z}$-algebra on the set J_n:

$$\Lambda(J_n),$$

e.g. $V \wedge V = 0$ and $V \wedge W = -W \wedge V$.

Definition (Faithful Polynomials)

We call an exterior polynomial in $\Lambda^n(J_n)$ *faithful* if the indeterminates from each monomial form a basis of $\mathbb{Z}^n$.

Torus Polynomials

Suppose (Γ, α, σ) is an oriented torus graph. For a vertex p, order the basis elements $\alpha(\mathcal{E}(\Gamma)_p) = \{\alpha(e_1), \ldots, \alpha(e_n)\}$ so that

$$\det[\alpha(e_1) \cdots \alpha(e_n)] = \sigma(p).$$

This defines a faithful exterior monomial

$$\mu_p = \alpha(e_1) \wedge \cdots \wedge \alpha(e_n) \in \Lambda^n(J_n), \quad \forall p \in \mathcal{V}(\Gamma).$$

Definition (Torus Polynomial)

The *torus polynomial* of an oriented torus graph (Γ, α, σ) is the faithful exterior polynomial

$$g(\Gamma, \alpha, \sigma) := \sum_{p \in \mathcal{V}(\Gamma)} \mu_p \in \Lambda^n(J_n).$$

Definition

Definition

Define J_n^* to be the set of non-trivial elements of $\mathrm{Hom}(S^1, T^n)$.

For each faithful exterior polynomial $h \in \Lambda^n(J_n)$ we can obtain a *dual polynomial* $h^* \in \Lambda^n(J_n^*)$.
We now define a chain complex $(\Lambda^k(J_n^*), d_k)$ as follows: for each monomial $s_1 \wedge \cdots \wedge s_k \in \Lambda^k(J_n^*)$, with all $s_i \in J_n^*$,

$$d_k(s_1 \wedge \cdots \wedge s_k) := \begin{cases} \sum_{i=1}^{k} (-1)^{i+1} s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_k, & \text{if } k > 1; \\ 1 & \text{if } k = 1. \end{cases}$$

and $d_0(1) = 0$. It is easy to see that $d^2 = 0$.

Theorem

Theorem (D.)

Let $h \in \Lambda^n(J_n)$ be a faithful polynomial. Then $h = g(\Gamma, \alpha, \sigma)$ is the torus polynomial of an oriented torus graph if and only if $d(h^*) = 0$.

Let K_n denote the abelian group of all faithful exterior polynomials $h \in \Lambda^n(J_n)$ such that $d(h^*) = 0$.

Definitions

Let

$$\Omega_m^{U:T^n}$$

denote the geometric equivariant complex bordism groups of m-dimensional stably complex T^n-manifolds. We have a commutative ring

$$\Omega_*^{U:T^n} := \bigoplus_{m \geq 0} \Omega_m^{U:T^n}$$

via the diagonal T^n-action on the cartesian product of two T^n-manifolds.

Definition

Let $\mathcal{Z}_*^{U:T^n} \subset \Omega_*^{U:T^n}$ denote the subring given by elements that can be represented by a stably complex T^n-manifold where the T^n-action is effective.

Restriction to Fixed Point Data

We have a monomorphism by ‘restriction to fixed point data’:

$$\varphi\colon \mathcal{Z}_*^{U:T^n} \longrightarrow \mathbb{Z}[J_n]$$

$$[M] \longmapsto \sum_{p\in M^{T^n}} \sigma(p)\prod_{i=1}^{m} V_i(p),$$

where $T_pM = V_1(p)\oplus\cdots\oplus V_m$. When $*=2n$ we obtain the commutative diagram of abelian groups

$$\begin{array}{ccc} & \mathcal{Z}_{2n}^{U:T^n} & \\ {}^{g}\swarrow & & \searrow^{\varphi} \\ K_n & \xrightarrow{\quad f \quad} & \mathbb{Z}[J_n] \end{array}$$

where $f(s_1\wedge\cdots\wedge s_n) = \det[s_1\cdots s_n]s_1\cdots s_n$ for a faithful monomial $s_1\wedge\cdots\wedge s_n\in\Lambda^n(J_n)$.

Equivariant K-theory Characteristic Numbers

Let $S \subset K^*(BT^n_+)$ denote the multiplicative subset generated by the K-theory Euler classes $\lambda_{-1}(V) = \sum_{i\geq 0}(-1)^i\lambda^i(V)$ of the bundles $ET^n \times_{T^n} V \to BT^n$, for $V \in J_n$.

Theorem (Hattori '74)

There is a commutative pullback square with all maps injective

$$\begin{array}{ccc} \mathcal{Z}_*^{U:T^n} & \xrightarrow{\Psi} & K^*(BT^n_+)[\![\mathbf{t}]\!] \\ \varphi\downarrow & & \downarrow\lambda \\ \mathbb{Z}[J_n] & \xrightarrow{S^{-1}\Psi} & S^{-1}K^*(BT^n_+)[\![\mathbf{t}]\!] \end{array}$$

where $\mathbf{t} = (t_1, t_2, \dots)$ *is a sequence of indeterminates.*

Equivariant K-theory Characteristic Numbers

The coefficients of $\Psi[M]$ are known as *equivariant K-theory characteristic numbers* for M. Again, when $* = 2n$ we have

$$\begin{array}{ccccc} & & \mathcal{Z}_{2n}^{U:T^n} & \xrightarrow{\Psi} & K^*(BT^n_+)[\![\mathbf{t}]\!] \\ & \swarrow^{g} & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\lambda} \\ K_n & & & & \\ & \searrow^{f} & & & \\ & & \mathbb{Z}[J_n] & \xrightarrow{S^{-1}\Psi} & S^{-1}K^*(BT^n_+)[\![\mathbf{t}]\!] \end{array}$$

Theorem

$$
\begin{array}{ccc}
\mathcal{Z}_{2n}^{U:T^n} & \xrightarrow{\quad\Psi\quad} & K^*(BT^n_+)[\![\mathbf{t}]\!] \\
\downarrow \varphi & & \downarrow \lambda \\
\mathbb{Z}[J_n] & \xrightarrow{\quad S^{-1}\Psi\quad} & S^{-1}K^*(BT^n_+)[\![\mathbf{t}]\!]
\end{array}
\qquad g: \mathcal{Z}_{2n}^{U:T^n} \to K_n, \quad f: K_n \to \mathbb{Z}[J_n]
$$

Theorem (D.)

Every polynomial $h \in K_n$ satisfies

$$(S^{-1}\Psi \circ f)(h) \in \lambda(K^*(BT^n_+)[\![\mathbf{t}]\!]).$$

Corollaries

Corollary

We have an isomorphism of abelian groups

$$\mathcal{Z}_{2n}^{U:T^n} \cong K_n.$$

Define the graded rings

$$\Xi_* := \bigoplus_{n \geq 0} \mathcal{Z}_{2n}^{U:T^n} \qquad \cong \qquad K_* := \bigoplus_{n \geq 0} K_n.$$

Warning

These are non-commutative rings.

Corollaries

Suppose M^{2n} is a non-bounding stably complex torus manifold. Then $g[M] \in K_n$ is a non-zero faithful polynomial in $\Lambda^n(J_n)$ such that $d(g[M]^*) = 0$. Any such exterior polynomial must have at least $n+1$ monomials.

Corollary

As a strict lower bound, $n+1$ is the minimum number of fixed points of a non-bounding stably complex torus manifold.

Definition

A *quasitoric manifold* is an even-dimensional smooth closed manifold M^{2n} with a locally standard smooth T^n-action such that the orbit space is a simple polytope P.

Definition

A *quasitoric pair* (P, λ) consists of a combinatorial oriented simple n-polytope P and a map

$$\lambda\colon \mathcal{F}(P) \longrightarrow \mathrm{Hom}(S^1, T^n) \cong \mathbb{Z}^n$$

that satisfies:

$$\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_n})\} \text{ forms a basis of } \mathrm{Hom}(S^1, T^n) \text{ whenever } F_{i_1} \cap \cdots \cap F_{i_n} \text{ is a vertex of } P. \qquad (\star)$$

Quasitoric Pairs

We have a bijection

$$\{\text{Quasitoric manifolds with a stably complex } T^n\text{-structure}\}$$
$$\updownarrow$$
$$\{\text{Quasitoric pairs}\}$$

We define a product on quasitoric pairs

$$(P_1, \lambda_1) \times (P_2, \lambda_2) := (P_1 \times P_2, \lambda_1 \times \lambda_2),$$

where the characteristic map is defined as

$$(\lambda_1 \times \lambda_2)(F_i \times P_2) = (\lambda_1(F_i), 0, \ldots, 0) \quad \text{and}$$
$$(\lambda_1 \times \lambda_2)(P_1 \times F_i') = (0, \ldots, 0, \lambda_2(F_i')).$$

Ring of Quasitoric Pairs

Definition (Ring of Quasitoric Pairs)

Denote the free abelian group generated by all quasitoric pairs by $\mathcal{Q}_*$, where we may interpret $+$ as disjoint union and grade $\mathcal{Q}_*$ by the dimension of the polytope.

The multiplication depends on the ordering of $P_1 \times P_2$ so $\mathcal{Q}_*$ forms a graded non-commutative ring. We have a homomorphism of non-commutative graded rings

$$\mathcal{M} \colon \mathcal{Q}_* \longrightarrow \Xi_*,$$

by constructing the omnioriented quasitoric manifold associated to a quasitoric pair.

Conjecture

Conjecture

The homomorphism $\mathcal{M}\colon \mathcal{Q}_* \longrightarrow \Xi_*$ is surjective, that is, every class is Ξ_* contains an omnioriented quasitoric manifold.

True for $n = 1, 2$.

Thank you!